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CONSTRUCTIVE DERIVATION OF THE DECOMPOSITION-FIELD OF A POLYNOMIAL

BY H. S. VANDIVER

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1. In attempting to set up a constructive theory of commutative algebra, that is, a theory in which every quantity or concept employed may be built up in a finite number of steps in terms of quantities which we agree in advance *exist*, one is faced with several fundamental problems at the beginning. Kronecker[1] gave a finite algorithm of the above mentioned character for determining if a given polynomial $f(x)$ with coefficients in the rational field F is reducible or irreducible in the field F. At the same time he gave without proof a finite algorithm of a quite different character for determining if a polynomial $\phi(x, \alpha)$ with coefficients in a field $F[\alpha]$, where α is an algebraic number, is irreducible or reducible in $F[\alpha]$. Van der Waerden[2] gave an argument to prove that Kronecker's algorithm actually yields the information mentioned, but in the course of it he assumes[3] that $\phi(x, \alpha)$ has a decomposition into irreducible factors in $F[\alpha]$.

This does not satisfy the requirement that in order for us to speak of a decomposition of this type we must be able to show in advance how to obtain it in a finite number of steps. The method employed by Kronecker also assumes that we know the decomposition-field (Zerlegungskörper) of $f(x)$, where $f(\alpha) = 0$, so that we can speak of the conjugates of α. This prevents us from applying the algorithm to the constructive derivation of the decomposition-field of $f(x)$, that is, if we write α_1, for α, we have

$$f(x) = (x - \alpha_1)f_1(x, \alpha_1),$$

and in order to carry out a scheme similar to the usual one for obtaining the factor $x - \alpha_2$, where α_2 determines an algebraic field, we must be able to find the irreducible factors of $f_1(x, \alpha_1)$ in $F[\alpha_1]$.

Kronecker[4] obtained a theorem, using constructive methods, and independent of the method mentioned above, from which we may infer the existence of the decomposition-field of a polynomial. However, it does not permit us to adjoin one root at a time, in the usual way, rather we must take the whole decomposition field *at once*. Briefly, he proceeded as follows: If $x, x_1, \cdots, x_n$ are

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[1] *Werke*, Bd. 2, pp. 257–260, 411–416.
[2] *Moderne Algebra*, erster Teil, pp. 130–1.
[3] This was pointed out to me by Dr. A. E. Ross.
[4] *Werke*, Bd. 3, Erster Halbband pp. 209–240.

indeterminates and f_k is the elementary symmetric function of $x_1, x_2, \cdots, x_n$ of degree k, then

$$(x - x_1)(x - x_2) \cdots (x - x_n) = x^n - f_1 x^{n-1} + \cdots \pm f_n$$

and if

$$f(x) = x^n - c_1 x^{n-1} + \cdots \pm c_n$$

where the c's belong to F, then

$$f(x) \equiv (x - x_1)(x - x_2) \cdots (x - x_n) \tag{1}$$

with respect to the polynomial ideal (modular system)

$$(f_1 - c_1, f_2 - c_2, \cdots, f_n - c_n).$$

In general, however, this ideal is not prime, i.e., the ring of its residue classes may contain zero divisors. Kronecker was able, however, to replace this by a prime polynomial ideal in (1) of the form

$$f_1 - c_1, \cdots, f_n - c_n, g' - c', g'' - c'', \cdots$$

where the c's are rational and the g's are polynomials in $x_1, x_2, \cdots, x_n$ with rational coefficients. The method by which he obtained this is very complicated, however, requiring, for the argument, twenty-nine pages of his collected works and involving a number of results proved in other papers published by him.

In the present paper we obtain, very simply, the decomposition field of $f(x)$ by first giving a finite algorithm for obtaining the irreducible factors of $\phi(x, \alpha)$ in $F[\alpha]$ without assuming the existence of any decomposition field or considering any decomposition of $\phi(x, \alpha)$ into irreducible factors in $F[\alpha]$ before setting up the algorithm.

From the point of view of polynomial ideals the result gives

$$f(x) \equiv (x - x_1)(x - x_2) \cdots (x - x_n)$$

with respect to a prime ideal which has a different form from the one employed by Kronecker.

The writer wishes to state here his deep conviction, that the method of polynomial ideals (modular systems involving indeterminates) is fundamental for the foundations of commutative algebra just as it has been found fundamental in other parts of algebra and geometry. (This idea is not exactly new; in fact, it goes back to Kronecker). *If we avoid the use of it entirely in our algebraic theories, then something is definitely lost at times.*

To illustrate, we start with an algebraic field defined by α, where α satisfies $f(x) = 0$, with coefficients in F and irreducible in F. Any integral relation involving only polynomials in α, say,

$$g(\alpha) = 0 \tag{2}$$

is such that if we take $g(x)$ and divide by $f(x)$, then we get the remainder zero since $f(x)$ is irreducible; hence, in place of (2) we have the following identity in x

$$g(x) = f(x)\, A(x) \tag{2a}$$

where $A(x)$ is a polynomial in x with coefficients in F. We may obtain many results from (2a) which we cannot obtain directly from (2); for example, we may differentiate (2a) formally and obtain in general many other identities. Hence, if we fail to carry out analogies of this notion in connection with other commutative algebraic systems, we may expect our theories to be incomplete. *At the same time it is important to develop the theories based on symbolic adjunction of new elements, using residue classes, as these involve very convenient abbreviations of the formulae based on polynomial ideals.* Both points of view will be kept in the foreground in this paper.

Set

$$f(x) = (x - \alpha)\, \phi(x, \alpha), \tag{3}$$

where $f(x)$ is irreducible in F with coefficients in F and leading coefficient unity. We now consider residue classes formed by polynomials in x with coefficients in $F[\alpha]$ modulo $\phi(x, \alpha)$, the latter possibly being reducible in $F[\alpha]$. These residue classes form a *ring*, isomorphic with a ring obtained by assuming a β such that $\phi(\beta, \alpha) = 0$. Because of this isomorphism we shall say that β *exists*. Hence, in the ring $F[\alpha, \beta]$ we obtain

$$\phi(x, \alpha) = (x - \beta)\, \varphi_1(x, \alpha, \beta),$$

and

$$f(x) = (x - \alpha)\, (x - \beta)\, \varphi_1(x, \alpha, \beta).$$

Proceeding in this way we find a ring $F' = F[\beta_1, \beta_2, \cdots, \beta_n]$ such that in F' we have the decomposition, if $\alpha = \beta_1$, $\beta = \beta_2$,

$$f(x) = (x - \beta_1)\, (x - \beta_2) \cdots (x - \beta_n), \tag{4}$$

where β_k is adjoined immediately after β_{k-1} and therefore β_k satisfies an equation of degree $(n - k + 1)$ with coefficients in the ring $F[\beta_1, \beta_2, \cdots, \beta_{k-1}]$, but no equation of lower degree with coefficients in the same ring. We have from (4) by formal differentiation,

$$f'(x) = \sum_{i=1}^{n} \frac{f(x)}{x - \beta_i}.$$

Setting $x = \beta_1$, we have

$$f'(\beta_1) = (\beta_1 - \beta_2)\, (\beta_1 - \beta_3) \cdots (\beta_1 - \beta_n), \tag{5}$$

and this is not zero, since the degree of $f'(x)$ is less than n and not all the coefficients are zero.

We also may show that if we have an integral expression involving only polynomials in β_1 with rational coefficients, say

$$B(\beta_1) = 0,$$

then this expression also vanishes for $\beta_2 \cdots \beta_n$ each in lieu of β_1. For if we divide $B(x)$ by $f(x)$ we obtain

$$B(x) = f(x)\, Q(x) + R(x),$$

where the degree of $R(x)$ is less than n. Setting $x = \beta_1$ we have

$$R(\beta_1) = 0.$$

But this is impossible unless $R(x) = 0$, whence

$$B(x) = f(x)\, Q(x),$$

and hence $B(x)$ vanishes for $x = \beta_2, \beta_3, \cdots, \beta_n$.

2. If $G(\beta_1, x, y)$ is any polynomial in the indeterminates x, y with coefficients in $F[\beta_1]$, set

$$G(\beta_1, x, y)\, G(\beta_2, x, y) \cdots G(\beta_n, x, y) = N(G(\beta_1, x\, y)\,) = N(G).$$

The coefficients of $N(G)$ are in F, since the expression on the left is symmetric in $\beta_1, \beta_2, \cdots, \beta_n$. We shall call $N(G)$ the norm of G.

Let β generate a field and let $\phi(\beta, z)$ be a polynomial in z with coefficients in $F[\beta]$, and of degree n. Then we may set

$$N\, \phi(\beta, z - u\, \beta) = G(z, u) = G_1(z, u)\, G_2(z, u) \cdots G_r(z, u),$$

where the G's in the right hand member are irreducible in F and u is an indeterminate. They may be found in a finite number of steps by a known method. Now assume that all polynomials in z with coefficients in $F[\beta]$ of degree $< n$ may be decomposed into irreducible factors in $F[\beta]$ in a finite number of steps. This is obviously true if $n = 1$. We shall then proceed by induction and set up a method to decompose $\phi(\beta, z)$ in a finite number of steps. Suppose that

$$\phi(\beta, z) = \phi_1(\beta, z)\, \phi_2(\beta, z) \cdots \phi_k(\beta, z);\ k > 1,$$

where the ϕ's on the right are irreducible in $F[\beta]$. Setting $z - u\beta$ for z in this relation and taking the norm of each member we obtain, because of the result just preceding Article 2, applied to each coefficient in the polynomial in z and u,

$$N\phi(\beta, z - u\beta) = G(z, u) = N\phi_1(\beta, z - u\beta)\, N\phi_2(\beta, z - u\beta) \cdots N\phi_k(\beta, z - u\beta).$$

We shall now show that the greatest common divisor of $\phi(\beta, z - u\beta)$ and each of the polynomials $G_1(z, u), G_2(z, u), \cdots, G_r(z, u)$ will give for $u = 0$

all the irreducible factors of $\phi(\beta, z - u\beta)$ in $F[\beta]$. The greatest common divisor referred to may be found in a finite number of steps by known methods.

The decomposition of $\phi(\beta, z - u\beta)$ into irreducible factors in $F[\beta]$ is known to be unique aside from elements in $F[\beta]$. Now $\phi(\beta, z - u\beta)$ and $N\phi_1(\beta, z - u\beta)$ have only the common irreducible factor $\phi_1(\beta, z - u\beta)$ in common, (it must be irreducible, otherwise $\phi_1(\beta, z)$ is reducible) since

$$\frac{\phi(\beta, z - u\beta)}{\phi_1(\beta, z - u\beta)} = \phi_2(\beta, z - u\beta) \cdots \phi_k(\beta, z - u\beta)$$

and

$$\frac{N\phi_1(\beta, z - u\beta)}{\phi_1(\beta, z - u\beta)} = \prod{}' \phi_1(\beta_\nu, z - u\beta_\nu)$$

where the product extends over all the β's aside from $\beta = \beta_1$ and if they have, for example, the irreducible factor $\phi_2(\beta, z - u\beta)$ in common, then, noting that the right hand member of the last equation is symmetric in $\beta_2, \beta_3, \cdots, \beta_n$ and therefore has coefficients in $F[\beta]$,

$$\prod{}' \phi_1(\beta_\nu, z - u\beta_\nu) = g(\beta, z, u)\,\phi_2(\beta, z - u\beta).$$

Equating the terms of highest degree in z and u, we have positive integers m_1 and m_2 such that

$$\prod{}' (z - u\beta_\nu)^{m_1} = h(\beta, z, u)\,(z - u\beta)^{m_2}$$

whence,

$$\prod{}' u^{m_1}(\beta - \beta_\nu)^{m_1} = h_1(\beta, z, u)\,(z - u\beta),$$

and by (5), $h_1 \neq 0$. But this is impossible since the right hand member must contain the indeterminate z. This gives the result for ϕ's of degree n, and the induction is complete.

In case we have $\phi(\beta, \gamma, z)$ in which the coefficients are in the algebraic field $F[\beta, \gamma]$ we may proceed as before with $F[\beta]$ taking the place of F, and so on for any finite number of algebraic adjunctions.

3. We shall now apply the above to our decomposition problem. Consider again the relation (3). We may now set $\alpha = \alpha_1$ and find by the method given above an irreducible factor of $\phi(x, \alpha_1)$ in $F[\alpha_1]$. Let it be $\theta(x, \alpha_1)$. Consider the residue classes formed by polynomials in x, with coefficients in $F[\alpha_1]$, modulo $\theta(x, \alpha_1)$. These classes form a field, isomorphic with a field obtained by assuming an α_2 such that $\theta(\alpha_2, \alpha_1) = 0$. Hence, adjoining α_2 to $F[\alpha_1]$, we have in $F[\alpha_1, \alpha_2]$

$$\theta(x, \alpha_1) = (x - \alpha_2)\,\theta_1(x, \alpha_1, \alpha_2)$$

and $f(x) = (x - \alpha_1)\,(x - \alpha_2)\,w(x, \alpha_1, \alpha_2)$, and proceeding in this way we have finally

$$f(x) = (x - \alpha_1)\,(x - \alpha_2) \cdots (x - \alpha_n)$$

where each α defines a field. The field $F[\alpha_1, \alpha_2, \cdots, \alpha_n]$ is said to be the *decomposition field* of $f(x)$.

4. Following the same method but employing indeterminates in place of symbolic adjunctions, we may set up a prime polynomial ideal such that with respect to this ideal we have

$$f(x) \equiv (x - x_1)(x - x_2) \cdots (x - x_n),$$

where $x_1, x_2, \cdots, x_n$ are indeterminates. This ideal is defined by means of the irreducible polynomials obtained during the process employed in the last paragraph with $x_1, x_2, \cdots, x_n$ replacing $\alpha_1, \alpha_2, \cdots, \alpha_n$.

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ON THE ORDERING OF REAL ALGEBRAIC NUMBERS BY CONSTRUCTIVE METHODS

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1. In the present article we shall consider the ordering of real algebraic numbers by constructive methods, that is, methods which may be carried out in a finite number of steps in terms of elements which we agree exist at the beginning. The discussion is quite elementary, and little is presupposed. We assume, first, the decomposition field of a polynomial $F(\alpha_1, \alpha_2, \cdots, \alpha_n)$, so that

$$f(x) = (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n), \tag{1}$$

where $f(x)$ is a polynomial of degree n with coefficients in an algebraic field F, with leading coefficient unity. In another paper[1] the writer has set up constructive methods for obtaining this decomposition field.

We also employ "formal" differentiation. That is, for a polynomial $g(x)$ with coefficients in a ring we define $g'(x)$ (the derivative of $g(x)$) as the polynomial obtained from $g(x)$ by taking[2]

$$g(x + h) - g(x),$$

expanding by the binomial theorem, dividing by h, and setting in the result $h = 0$. Hence, we obtain without any use of limits or the methods of analysis Taylor's Theorem for polynomials

$$g(x + h) = g(x) + hg'(x) + \cdots + h^n \frac{g^{(n)}(x)}{n!}, \tag{1a}$$

where $g^{(k)}(x)$ is the k^{th} derivative of $g(x)$, the letter being defined by induction as the derivative of the $(k - 1)^{\text{th}}$ derivative of $g(x)$.

As the methods are constructive, we make no use of any notion of continuity, the Dedekind cut, the Zermelo postulate, or the set of all real numbers. The methods employed here depend on isolating a real algebraic field as corresponding to finding rational values a and b, such that the $f(x)$ in (1), say, is such that $f(a)f(b) < 0$. Further, we show how to make the interval (a, b) (called a *root interval*) so small that it corresponds, expressed in the usual language, to just one root of $f(x) = 0$.

A constructive method for carrying this out in the case where F is the rational

[1] "Constructive derivation of the decomposition field of a polynomial," these *Annals*, vol. 37 (2), 1936, pp. 1–6.

[2] Van der Waerden, *Moderne Algebra*, Erster Teil, pp. 67–8.

field was first given by Kronecker.[3] As to the ordering of the elements of fields, other methods than that of regarding them as included in the set of all real numbers or than that given here, have been proposed. Veblen[4] gave a method based on dividing the elements into squares and not-squares, the real field being characterized by properties equivalent to the property that the sum of n squares cannot be equal to (-1). Artin and Schreier[5] gave the same method.

This scheme cannot be employed here because of our restriction to constructive proofs. For one thing, the above definition of a real field cannot be applied in a finite number of steps to an arbitrary algebraic field, and, from the constructive standpoint, we cannot say that such a field is either real or not real.

Artin and Schreier further employ an argument to show that each real closed field can be ordered in one and but one way. In the course of this, they assume that a given element in the field is either a square or a not-square. Since no method is proposed for determining in a finite number of steps if an element is a square or not-square in the real closed field, the argument is not constructive, and the use of the law of excluded middle in this manner is questionable, particularly in view of the fact that the assumption as to squares and not-squares may be equivalent to the statement in the theorem.

Schatunovsky[6] attempted to order algebraic numbers by constructive methods, employing root intervals. He does not employ irreducible polynomials at the outset in his work and this fact helps to make his argument long and complicated, occupying, as it does, some thirty pages of his book.

Also, in carrying out his argument he falls into errors, the details of which are discussed later in the present paper.

As to isolating root-intervals, we employ mainly the methods used by Kronecker.[7]

As to the general question of algebraic order, this is partly included in the problem of determining certain types of semi-rings, the latter system being defined as in other papers by the author, as a semi-group under addition, a semi-group under multiplication, and in which the right and left distributive laws hold. Also a semi-group is a set of elements closed under an associative operation.

For, considering the usual definition of an ordered field as being a field in which we may divide elements into three classes, positive, negative, or zero, with the properties that if p_1 and p_2 are positive, then $p_1 + p_2$ and $p_1 p_2$ are positive and if p is positive then $-p$ is negative, then we see that the set of positive elements form a semi-ring which is maximal and contains no zero-

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[3] *Werke*, Bd. 3, erster Halbband, pp. 262–72.

[4] *Trans. Amer. Math. Soc.:* 7, 197–99 (1906).

[5] *Abh. Math. Sem.*, Hamburg, 5, 83–115 (1926).

[6] *Algebra as a study of congruences with respect to functional moduli*, (Russian), Odessa, (1913?), Chap. 8.

I am indebted to Dr. A. E. Ross for an English translation of this.

[7] l.c., and also, H. B. Fine, *Bull. Amer. Math. Soc.*, 20, pp. 341–2, (1914).

element. For want of a better name, I shall call such a system an *order-semi-ring*. We may then consider the problem of finding all the order-semi-rings in a given field. But, as is known, the problem of determining the semi-rings containing only real elements is not completely algebraic. Thus, the fields defined by $(\sqrt{2})$ and $(-\sqrt{2})$ are automorphic, but this gives two distinct isomorphic semi-rings, one containing only what we usually call positive elements, but the other containing some positive and some negative elements.

Although the methods of this paper enable us to carry out a process which is usually called approximation to a real root of an algebraic equation, this amounts to finding a succession of intervals each of which is included in the preceding, and in no case do we pass to a limit. The roots of the equations are in all cases purely symbolic in the sense of Steinitz.

After the real algebraic fields are ordered, we may arrive at the result (stated by Schatunovsky (l. c.)) that it is possible, in a finite number of steps, to indicate a finite number of real algebraic fields such that if each is adjoined in turn to F (defined in (1)), then $f(x)$ decomposes, in the enlarged field, into factors of degree < 3.

The details of the proof of this result, which takes the place in the constructive theory that the so-called "fundamental theorem of algebra" takes in the usual algebraic theory, will be given in another paper, so that attention will not be distracted from the much more general question of algebraic order.

If the term to *determine* is used in this paper, it means to find in a finite number of steps in terms of quantities we agree exist in advance. If the term *exist* is used in connection with elements or quantities not agreed upon as existing in advance, then it means that they may be determined in terms of the latter type of elements. The quantities or elements which we agree on in advance as *existing* are the algebraic numbers, that is, those α's obtained from (1) by taking initially F as the rational field, and any field obtained from such a one by adjunction of a finite number of other algebraic numbers. We assume some of the elementary properties of algebraic numbers. We also assume the known order relations for the rational field.

2. An algebraic field F is said to be ordered if the following postulates hold:

POSTULATE I: *Certain of the elements a of F have a property, called positive and symbolized by >0, such that for any element a of F we may determine that just one of the following relations hold:*

$$a = 0, \qquad a > 0, \qquad -a > 0.$$

POSTULATE II: *If $a > 0$ and $b > 0$, then*

$$a + b > 0 \text{ and } ab > 0.$$

POSTULATE III: (ARCHIMEDEAN POSTULATE) *For any element a of F there exists a rational integer $c > 0$ such that*

$$c > a.$$

If p is positive, then $(-p)$ is said to be *negative.*

If a and b are in F, we write $a > b$ or $b < a$ if $a - b > 0$. From these postulates and definitions[8] follow the usual inequality relations such as, from $a > b$ follows $a + c > b + c$ and if $c > 0$, $ac > bc$. We may employ Postulate I and introduce the symbol $| a |$, meaning the one of the two numbers $\pm a$ which is not negative, and call this the absolute value of a. Then we find for any a and b in F

$$| ab | = | a | | b |, \qquad | a + b | \leqq | a | + | b |. \tag{1b}$$

If θ ($\neq 0$) is an algebraic number, then $1/| \theta |$ is likewise, and by Postulate III there is a natural number c such that $c > 1/| \theta |$, whence $| \theta | > 1/c$, so that there exist rational numbers r and s such that

$$r < | \theta | < s; r > 0. \tag{2}$$

Also, there is a natural number $d > -\theta$, whence

$$-d < \theta. \tag{2a}$$

3. To accomplish our ultimate purpose we need to prove a set of lemmas. Lemmas 1, 3 and 4 are variations of well known results; Lemma 2 takes the place in this theory that Rolle's theorem does in the usual theory of polynomials.

LEMMA 1. *If*

$$\phi(x) = a_0 + a_1 x + \cdots + a_s x^s$$

has coefficients in an ordered algebraic field F, then a rational $h > 0$ may be determined such that $a_0 \phi(h_1) > 0$; $0 < h_1 \leqq h$, and h_1 is in F.

By Postulate III we have a rational $d_i > | a_i |$; $i = 0, 1, \cdots, s$. Let g be the greatest of these; then if $h > 0$, by (1b)

$$\phi(h) - a_0 \leqq \frac{gh(1 - h^s)}{1 - h} < \frac{gh}{1 - h}.$$

By (2) we have $| a_0 | > c > 0$; c rational. Then for $h = c/(g + c)$ we find $| \phi(h) - a_0 | < | a_0 |$, whence the result.

If $f(x)$ in (1) is such that $f(a)f(b) < 0$ for a and b in F and $a < b$, we call (a, b) a *root interval* of $f(x)$, abbreviated *R.I.* If (a, b) and (a_1, b_1) are *R.I.*'s and $a_1 \geqq b$, then they are said to be *isolated R.I.*'s. Referring to the numbers in a *R.I.* (a, b) and in F, we include a and b, unless otherwise stipulated. We now prove

LEMMA 2. *Let a polynomial $f(x)$, of degree $n > 0$, have coefficients in the ordered algebraic field F, and let (a, b); (a_1, b_1) be R.I.'s of $f(x)$; $b < a_1$; $f(a)f(b_1) > 0$.*

[8] *Van der Waerden.* Moderne Algebra, Erster Teil, pp. 209–10.

Then if for numbers x_1 in F and in either of these intervals none of the derivatives $f^{(i)}(x_1)$; $i = 1, 2, \cdots, n-1$, change sign, then $f'(b)f'|(a_1) < 0$.

Unless otherwise specified, every quantity employed in the proof of this is in F. We prove first that under the conditions of the theorem, with $f(x) = f^{(0)}(x)$; sgn for "sign of;" $i = 0, 1, \cdots, n-1$;

$$\text{sgn}\,[f^{(i)}(b) - f^{(i)}(a)] = \text{sgn}\, f^{(i+1)}(\xi)\,. \tag{3}$$

where ξ is any element in (a, b). We employ induction. Assume (3) true for $i = n-1, n-2, \cdots, n-k$; we shall show it holds for $i = n-k-1$. Set $g(x)$ for $f^{(n-k)}(x)$, then by hypothesis

$$\text{sgn}\,(g(b) - g(a)) = \text{sgn}\, g'(\xi)\,.$$

If $g'(\xi)\, g(a) > 0$, write $x_1(\neq a)$ for any element in (a, b), then $g(x_1) - g(a) = d$, say, and $g(x_1) = g(a) + d$ and since $dg(a) > 0$,

$$|\, g(x_1)\,| > |\, g(a)\,|\,. \tag{4}$$

Similarly, if $g'(\xi)\, g(a) < 0$, then for an $x_2\ (\neq b)$ in (a, b) we find

$$|\, g(x_2)\,| > |\, g(b)\,|\,. \tag{5}$$

In view of (2), (4) and (5) we then have a fixed rational c such that

$$|\, g(x)\,| > c > 0\,, \tag{6}$$

for any x in (a, b). If we set $f^{(n-k-1)}(x) = \phi(x)$, we have by (1a)

$$\frac{\phi(x_1 + h) - \phi(x_1)}{h} = g(x_1) + \frac{hg'(x_1)}{2!} + \cdots + \frac{h^k g^{(k)}(x_1)}{(k+1)!}\,.$$

By Lemma 1 we may select a rational h so that the left hand member has the same sign as $g'(x_1)$. For any x_1 in (a, b) we use the same h by taking the c in (6).

Now divide the interval (a, b) into δ equal parts so that each interval $< h$ in extent, that is, take a rational integer $\delta > (b-a)/h$, using Postulate III. Let these intervals be (x_i, x_{i+1}); $i = 1, 2, \cdots, \delta$; $x_1 = a$, $x_{\delta+1} = b$. Then $x_{i+1} - x_i < h$ and if $\phi(x_{i+1}) - \phi(x_i) = \rho_i$ the sign of ρ is the same for each i, since $g(x)$ does not change sign by hypothesis; hence, adding these equations for each i we have

$$\text{sgn}\,(\phi(b) - \phi(a)) = \text{sgn}\,(\rho_i) = \text{sgn}\, g(\xi)\,,$$

where ξ is any number in (a, b). This is (3) with $i = n-k-1$ in lieu of $n-k$. To complete the induction we note that (3) obviously holds for $i = n-1$. Hence, (1) holds for $i = 0$, that is,

$$\text{sgn}\,[f(b) - f(a)] = \text{sgn}\, f'(\xi) = \text{sgn}\, f'(b) = -\,\text{sgn}\, f(a)\,,$$

since $f(a)f(b) < 0$. Similarly,

$$\text{sgn}\,[f(b_1) - f(a_1)] = \text{sgn}\, f(b_1) = \text{sgn}\, f'(a_1)\,,$$

and since by hypothesis $f(a)f(b_1) > 0$, we have

$$f'(b)f'(a_1) < 0\,,$$

which proves Lemma 2.

We now consider

LEMMA 3: *Let $f(x)$ be a polynomial with coefficients in an ordered algebraic field F. Let a be in F and $f(a) \neq 0$, then rational numbers $r < a$ and $s > a$ may be determined so that for any x_1 in the interval (r, s) and in F, $f(x_1)f(a) > 0$.*

For proof, we note by (1a) and Lemma 1 that a rational $h > 0$ may be determined so that $f(a + h)$ has the same sign as $f(a)$, and, similarly, for $f(a - h)$, and this is true also for any h_1 in F; $0 < h_1 < h$. By Postulate III there is a rational $d > a + h = a_1$ and also a rational $d_1 < a$ by (2a). The theorem is obvious for a rational. If a is not rational, divide the interval $d - d_1$ into δ equal parts, where δ is a rational integer > 0, such that $\delta > (d - d_1)/h$, whence $h > (d - d_1)/\delta$. Let these intervals be in order, (x_i, x_{i+1}), $i = 1, 2, \cdots, \delta$. Test each difference $x_i - a$ in order; we must find an e such that $x_e < a < x_{e+1}$ and an m; $x_m < a_1 < x_{m+1}$. Now $e \neq m$, otherwise, $h < (d - d_1)/\delta$. Then $x_{e+1} = s$ is a rational number such that $f(s)f(a) > 0$; $a_1 > s > a$. Similarly, we find a rational r such that $f(r)f(a) > 0$; $a - h < r < a$, which proves Lemma 3.

We may now show that if we have two isolated *R.I.*'s of a polynomial, we may find from them two isolated *R.I.*'s of the type (a, b) and (a_1, b_1) mentioned in Lemma 2. For if (α, β), (α_1, β_1); $\alpha_1 > \beta$; are *R.I.*'s of $f(x)$ with $f(\alpha)\, f(\beta_1) < 0$, then (β, α_1) is a *R.I.* of $f(x)$, as is also (c, α_1) where c is determined from Lemma 3 so that $f(\beta)f(c) > 0$; $c > \beta$. Hence, we have the isolated *R.I.*'s (α, β), (c, α_1) of $f(x)$ and such that $f(\alpha)f(\alpha_1) > 0$. From the two isolated intervals (α, β) and (β, β_1) we may obtain a c using Lemma 3 such that $f(\beta)f(c) > 0$ and then (α, β) and (c, β_1) are of the type desired.

We now prove

LEMMA 4: *If $f(x)$ is a polynomial with coefficients in an ordered algebraic field F, then we may determine an interval $(-r, r)$, r rational, such that there are no R.I.'s of $f(x)$ of type (a, b), $a > r$; or (a_1, b_1), $b_1 < -r$; a, b, a_1, b_1, all in F.*

Write $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$; $a_n \neq 0$, whence,

$$f(x) = x^n (a_n + \phi)\,;\ \phi = \frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \cdots + \frac{a_0}{x^n}\,.$$

By Lemma 1 we may determine a rational $r > 0$ so large that $r^n (a_n + \phi(r)) = f(r)$ has the same sign as $r^n a_n$. Similarly, we see that $f(-r)$ has the same sign as $(-1)^n r^n a_n$. The same conclusions hold for any $r_1 > r$ or any $r_2 < -r$, whence the lemma, if r_1, and r_2 are in F.

4. We now proceed to the ordering of a real algebraic field defined by

$$f(x) = x^n + c_{n+1} x^{n-1} + \cdots + c_0\,,$$

where the c's are elements of an ordered algebraic field F and $f(x)$ is irreducible in F. If it is possible to determine a *R.I.* of $f(x)$ such that there are no *R.I.*'s of $f'(x)$ included in the interval, we call such a *R.I.* a *simple R.I.* of $f(x)$, abbreviated *S.R.I.* By Lemma 2 and the remarks following the proof of Lemma 3, it follows that we cannot find two isolated *R.I.*'s in a *S.R.I.* Two *S.R.I.*'s (a, b) and (c, d) of $f(x)$ where $a \leqq c$ are said to be *equivalent* if $c < b$ and (c, b) is a root interval of $f(x)$. Two *S.R.I.*'s which are not equivalent will be termed *isolable.* By definition, two isolated *S.R.I.*'s are isolable. Evidently from two isolable *S.R.I.*'s we may determine two isolated *R.I.*'s while this is not the case with two equivalent *S.R.I.*'s.

In the decomposition field of $f(x)$ suppose we have the relation (1). If a *S.R.I.* (a, b) has been determined of $f(x)$, then we associate one of the symbols $\alpha_1, \alpha_2, \cdots, \alpha_n$ with the interval (a, b), say α_1; we say that α_1 generates a *real field,* α_1 is said to be a *real algebraic number* and a *real root* of $f(x) = 0$, and similarly for each isolable *S.R.I.*

For the determination of the *R.I.*'s in the form (a, b) with a and b rational we note that if $n = 1$ in (1), there is just one isolable *S.R.I.* because of Lemma 2, as $f'(x)$ cannot vanish, and F by hypothesis is ordered. We then proceed by induction and make the following assumptions:

1. *If $i < n$ it is possible to determine all the isolable S.R.I.'s, which do not exceed i in number, of a polynomial of degree i with coefficients in an ordered algebraic F.*
2. *Any real field defined by $\phi(x) = 0$ of degree i, $i < n$, and coefficients in F, or any field obtained from it by a finite number of adjunctions of real fields defined by polynomials of degree $\leqq i$ is ordered.*

We shall show that these assumptions hold also for $i = n$, having noted that they hold for $i = 1$. By Lemma 4 we take an interval $(-r, r)$ in which all *R.I.*'s of $f(x)$ lie, and we assume that $f(x)$ has a *R.I.* Consider the distinct real zeroes of the derivatives $f^{(s)}(x)$; $s = 1, 2, \cdots, n$. By the second assumption the field obtained by the adjunction of all these is ordered. Let

$$\theta_1, \theta_2, \cdots, \theta_k$$

be those included in the interval $(-r, r)$ with $\theta_1 < \theta_2 < \cdots < \theta_k$. Consider the intervals in the set $(\theta_{t-1}\ \theta_t)$; $t = 2, 3, \cdots k$. We cannot have $f(\theta_i) = 0$, since $f(x)$ is irreducible in F. We test each interval to ascertain if it is a *R.I.* of $f(x)$. By Lemma 4 and hypothesis we must find at least one *R.I.* If (θ_v, θ_{v+1}) is a *R.I.*, then by Lemma 3 we may replace this by a *R.I.* (a_v, a_{v+1}); $\theta_v < a_v < a_{v+1} < \theta_{v+1}$, where the a's are rational; $f(\theta_v)f(a_v) > 0$; $f(\theta_{v+1})f(a_{v+1}) > 0$; (θ_v, a_v) contains no *R.I.* of (x) and similarly for (a_{v+1}, θ_{v+1}). By proper choice of notation we may write as *R.I.*'s of $f(x)$,

$$(a_1, a_2)\ (a_2, a_3) \cdots (a_{h-1}, a_h)\ ,$$

which are such that no *R.I.* of $f(x)$ lies outside of them.

Now each of these is simple since if (a_t, a_{t+1}) contains two isolated *R.I.*'s

(b_t, b_{t+1}), (d_t, d_{t+1}), then we may find two *R.I.*'s of $f(x)$ of the type mentioned in Lemma 2, and therefore a real root of $f'(x) = 0$ lying between the two θ's, which contradicts the hypothesis. There cannot be more than n isolable *S.R.I.*'s because of Lemma 2 and the assumption that $f'(x)$ has no more than $(n-1)$ isolable *S.R.I.*'s. Hence, assumption 1 holds for $f(x)$ of degree n.

Let (a, b), a and b rational, be a *S.R.I.* of $f(x)$ and associated with $\alpha = \alpha_1$; then we shall show that $F[\alpha]$ is ordered. Let $\phi(\alpha)$ be any number in $F[\alpha] \neq 0$. Then we may write, if the d's are in F,

$$\phi(\alpha) = d_0 + d_1\alpha + \cdots + d_{n-1}\alpha^{n-1},$$

where not all the d's are zero. This representation is unique since $f(x)$ is irreducible in F. Let

$$\phi(x) = d_0 + d_1x + \cdots + d_{n-1}x^{n-1};$$

hence $\phi(\alpha) \neq 0$.

We now select a *R.I.* of $f(x)$ in (a, b) such that it contains no *R.I.* of $\phi(x)$. Since the degree of $\phi(x)$ is $< n$, this may be done by adjoining to F the real fields corresponding to the real roots of $\phi(x) = 0$, and proceeding as in the determination of a *S.R.I.* of $f(x)$. In this way we obtain a sub-interval of (a, b), say (a_1, b_1) with a_1 and b_1 rational, such that $f(a_1)f(b_1) < 0$, but $\phi(a_1)\phi(b_1) > 0$. Two isolated *R.I.*'s of this type cannot be found since (a, b) is a *S.R.I.* of $f(x)$. *We then order* $\phi(\alpha)$ *among the elements of* $F[\alpha]$ *by adopting the conventions:* $\phi(\alpha) > 0$ *if* $\phi(a_1) > 0$ *and* $\phi(\alpha) < 0$ *if* $\phi(a_1) < 0$. Hence, an element of $F[\alpha]$ is zero, (when all the d's are zero), positive or negative, and Postulate I is satisfied. If

$$\phi_1(x) = d_0' + d_1'x + \cdots + d_{n-1}'x^{n-1}$$

with the d''s in F, then write

$$\psi(x) = \sum_{i=0}^{n-1} (d_i + d_i')x^i = \phi(x) + \phi_1(x).$$

If $\phi(\alpha) > 0$, $\phi_1(\alpha) > 0$, then $\psi(\alpha) > 0$, since we may take a *R.I.* of (x), say (a_2, b_2), lying in (a, b), which is not a *R.I.* of any of the functions $\phi(x)$, $\phi_1(x)$ or $\psi(x)$. Then we have $\phi(a_2) > 0$, $\phi_1(a_2) > 0$ and $\psi(a_2) = \phi(a_2) + \phi_1(a_2) > 0$.

We now examine the product of $\phi(\alpha)\,\phi_1(\alpha)$ and consider the identity

$$\phi(x)\,\phi_1(x) = r(x) + f(x)\,A(x), \tag{7}$$

where $A(x)$ is a polynomial with coefficients in F and

$$r(x) = e_0 + e_1x + \cdots + e_{n-1}x^{n-1}$$

with the e's in F. Take the (a_2, b_2) above and select a sub-interval (a_3, b_3) which is a *R.I.* of $f(x)$ so that $r(x)$ and $A(x)$, each of degree $< n$, do not change sign therein. We then have from (7) for $x = a_3$ and b_3

$$r(a_3) + f(a_3)A(a_3) > 0$$
$$r(b_3) + f(b_3)A(b_3) > 0\,,$$

which gives, since $f(a_3)f(b_3) < 0$, $r(a_3) > 0$, $r(b_3) > 0$, hence, $r(\alpha) > 0$ and Postulate II holds.

With (a, b) having the same meaning as before we shall show that

$$a < \alpha < b\,. \tag{8}$$

Consider $\phi(x) = x - a$. We use the above method of ordering and will show that $\alpha - a > 0$. In the interval (a, b), $\phi(x)$ changes sign since $\phi(a) = 0$, but this is the only root of $\phi(x)$. Also, $f(a) \neq 0$ and we may select a c, using Lemma 3, so that $f(a)f(c) > 0$, $c > a$, with (a, c) not containing a *R.I.* of $f(x)$. Then (c, b) is a *R.I.* of $f(x)$, but not of $\phi(x)$. But $b - a > 0$, hence $\alpha - a > 0$, or $\alpha > a$. Similarly, we find $\alpha < b$.

Let $\psi(\alpha)$ be any number in $F[\alpha]$ so that

$$\psi(\alpha) = h_0 + h_1\alpha^2 + \cdots + h_{n-1}\alpha^{n-1}\,,$$

the h's in F. Then by Postulate III each (h_i) in $\psi(\alpha) < c_i$, where c_i is a rational integer. Also, by (8) either $b > |\alpha|$ or $-a > |\alpha|$ where b is rational, hence, $|\alpha| < r$ where r is some natural number, so that by (1b)

$$\psi(\alpha) < \sum_{i=0}^{n-1} c_i r^i\,,$$

which shows that Postulate III holds in $F[\alpha]$. Hence, the field $F[\alpha]$ is ordered.

If we adjoin a real root β of $g(x) = 0$ (of degree $\leqq n$) with coefficients in F, we find[9] by constructive methods the decomposition

$$g(x) = g_1(x, \alpha)\, g_2(x, \alpha) \cdots g_s(x, \alpha)$$

in the field $F[\alpha]$, where the g_i's are irreducible in $F[\alpha]$. One of these must change sign in the *S.R.I.* of $g(x)$ corresponding to β, let it be g_1. Then we order the elements of $F[\alpha, \beta]$ by using g_1 in lieu of f in the above discussion, $F[\alpha]$ taking the place of F. Proceeding in this way we may order the elements of any field obtained by a finite number of adjunctions of real numbers defined by polynomials of degree $\leqq n$. Hence, the second assumption of our induction holds, and the induction is complete, as we have already noted that it holds for $n = 1$.

Initially we postulated but one ordered field, the rational. Starting with this as our F we may set up an ordered field by taking a real root of a polynomial with rational coefficients, so that ordered fields other than the rational field exist.

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[9] See footnote 1.

5. Schatunovsky, in ordering elements in $F[\alpha]$, began as has the writer by considering $\phi(\alpha)$, a number in $F[\alpha]$ in the form of a polynomial in α with coefficients in F, and defined positive and negative numbers in the same way, but used a different method in isolating a *R.I.* of $f(x)$ which is not a *R.I.* of $\phi(x)$. However, in connection with his conclusion that $\phi(\alpha)\phi_1(\alpha) > 0$ if $\phi(\alpha) > 0$ and $\phi_1(\alpha) > 0$ he merely states that this follows because in the *R.I.* (a, b) which is a *R.I.* of $f(x)$ but not of $\phi(x)$ or $\phi_1(x)$ we have $\phi(\alpha)\phi_1(\alpha) > 0$ because $\phi(a)\phi_1(a) > 0$. This is not sufficient as we show by the following example. If $f(x) = x^2 - 2$; $\alpha^2 - 2 = 0$; $\phi(x) = 1 + x$; $\phi_1(x) = 3 - 2x$; then $\phi(\alpha)\phi_1(\alpha) = -1 + \alpha = r(\alpha)$ if we set $r(x) = x - 1$. Then the *R.I.* $(-1/2, 1.42)$ is a *R.I.* of $f(x)$, not $\phi(x)$ or $\phi_1(x)$, *but is a R.I. of* $r(x)$. Also, in connection with a definition of an ordered field he refers to a pamphlet[10] of his which is not accessible to the writer. It is not clear, however, that he explicitly employs our Postulate III that in an ordered algebraic field we always have for any element a therein a natural number c such that

$$c > a .$$

This assumption is necessary to prove the following statement in his proof of his theorem 2: "Let us now sub-divide the root interval (a, b) into a finite number of intervals in such a way that the length of each one of them should be less than $\epsilon/E + \epsilon$." Here, ϵ and E are quantities in an ordered field not necessarily the rational; hence the above statements imply the existence of a rational integer δ such that $\delta > (b - a)/k$, where $k = \epsilon/E + \epsilon$. If he is employing the postulate here without stating it, then he does not show later that when he orders the algebraic field it satisfies this postulate; in fact, he finds his simple root interval in the form (a, b), where a and b are not necessarily rational.

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[10] "On postulates forming the basis of the notion of magnitude."

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ON THE REDUCTION OF SPACE GROUPS

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Introduction. The connection between the theories of groups and quantum mechanics pointed out by Wigner[1] and developed extensively by Wigner, von Neuman and Weyl has given the work of Frobenius and Schur on the irreducible representations of groups an outstanding position in the conventional mathematical discipline of the physicist. In many of the earlier applications of group theory, such as to the theory of atomic spectra, it is true that the results obtained were already familiar to still earlier workers and this application served principally to define clearly the link between different features of the problem, but with the extension of quantum mechanical methods to other fields, group theoretical developments have proved to be more and more indispensable. In particular, the progress of certain aspects of the theory of the solid state has been directly dependent upon the group theoretical knowledge.[2] For this reason, it may safely be said that a discussion of the representation theory of space groups will prove to be of considerable service in future development of the quantum theory of the solid state, and it is with this in mind that the following developments are presented.

1. **The theory of space groups.** For the purpose of the following investigation, it will be of advantage to outline briefly the basis and important results of the theory of space groups.

The starting point of this development is the assumption that every crystal is invariant under a discrete, three-parameter translation group of the form

$$\Gamma_{n_1 n_2 n_3} : n_1 t_1 + n_2 t_2 + n_3 t_3 , \tag{1}$$

in which t_1, t_2, t_3 are three primitive translation vectors of the form

$$\begin{pmatrix} t_{11} \\ t_{12} \\ t_{13} \end{pmatrix} \quad \begin{pmatrix} t_{21} \\ t_{22} \\ t_{23} \end{pmatrix} \quad \begin{pmatrix} t_{31} \\ t_{32} \\ t_{33} \end{pmatrix} \tag{2}$$

[1] E. P. Wigner, *Gruppentheorie*, Vieweg und Sohn, 1931.
H. Weyl, *The Theory of Groups and Quantum Mechanics* (English translation by H. P. Robertson). Dutton and Co., 1931.

[2] See for example: J. H. Van Vleck, *The Theory of Electric and Magnetic Susceptibilities*. Oxford, 1932.

respectively, for which the determinant

$$\begin{vmatrix} t_{11} & t_{21} & t_{31} \\ t_{12} & t_{22} & t_{32} \\ t_{13} & t_{23} & t_{33} \end{vmatrix}$$

does not vanish, and n_1, n_2, n_3 may take on all integer values. This translation group is then to be viewed as an invariant subgroup of all groups of higher order under which the crystal may be invariant, and the problem of developing the space groups is that of finding groups of transformations of the type

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + t_1 \\ x_2' &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + t_2 \\ x_3' &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + t_3 \end{aligned} \tag{3}$$

which admit a group of the type (1) as an invariant subgroup. In the following discussion, we shall employ a notation in which (3) is written in the form

$$x' = \alpha\, x + t\,, \tag{4}$$

where x', x, and t are the column matrices

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix}, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix},$$

respectively, and α is the matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

From an operational standpoint, (4) may be written

$$x' = \{\alpha \mid t\}\; x = \mathfrak{A}\, x\,, \tag{5}$$

where $\{\alpha \mid t\}$ is a linear operator which acts upon a vector x in the manner defined by (4) and combines with another operator $\{\beta \mid x'\} = \mathfrak{B}$ of the same type in the manner

$$\{\alpha \mid t\}\; \{\beta \mid t'\} = \{\alpha\beta \mid \alpha t' + t\}\,, \tag{6}$$

where in the new product operator the product matrix $\alpha\beta$ plays the part of the *rotational matrix* α of (4) and the vector $\alpha\, t' + t$ plays the part of the *translational matrix* or *vector* t in the same equation. This scheme of space operators

$\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{C}$, constitutes a generalization of the usual rotational or point matrix operators and reduces to the latter in cases in which the translational vector is zero. The unit element is given by $\{\epsilon \mid 0\}$ where ϵ is a unit matrix and 0 is the column vector of zero components, while the inverse to $\{\alpha \mid t\}$ is obviously $\{\alpha^{-1} \mid -\alpha^{-1}t\}$. In this scheme the translation group elements (1) are of the form

$$\{\epsilon \mid n_1t_1 + n_2t_2 + n_3t_3\} = \{\epsilon \mid n_1t_1\}\ \{\epsilon \mid n_2t_2\}\ \{\epsilon \mid n_3t_3\}. \tag{7}$$

From this standpoint, the theory of space groups deals with the investigation of groups of operators (5) which possess an invariant subgroup of type (7). This restriction has a two-fold influence upon the nature of the resulting groups: first, it limits the nature of the rotational elements which may appear, and second, it induces corresponding limitations upon the forms of t_1, t_2, and t_3 (in (1)). These two are closely linked together in the sense that the allowed values of t_1, t_2, and t_3 depend upon the forms of the rotational elements and conversely. A complete discussion of the development of this problem is given elsewhere,[3] but the conclusions and theorems that may be drawn are as follows.

I. The only allowed rotational matrices may be reduced to the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$$

in which $\varphi = 2\pi m/n$, where n is restricted to the integer values 1, 2, 3, 4 and 6, and m is an arbitrary integer less than or equal to n in absolute value. There are thirty-two rotation groups that may be formed of elements possessing only such matrices—the so-called crystal classes—and these are just the groups which describe the macroscopic symmetry properties of crystals; that is, these and only these are met with in nature.

II. In any given space group the rotational matrices constitute one of these thirty-two groups, so that each space group may be associated with a definite crystal class.

III. The restrictions to be placed on the translation groups that appear as invariant subgroups of the space groups going with a given class are such that they depend only upon the class and not upon the space group (i.e. only upon the rotational elements of the space operators). These are such that only fourteen types of translation groups are met with, varying from one in which all the components of t_1, t_2 and t_3 are arbitrary, for the classes C_1 and S_1 of the triclinic system, to three in which only one component of the three vectors is arbitrary, for the groups T. T^d T^h, 0, and 0^h of the cubic system.

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[3] F. Seitz, *Zeits. für Krist.* 88, 433, 1934, et seq.

IV. There is at least one space group corresponding to each class and each translation group (i.e. each *subclass*) and there are 230 space groups on the whole. The characteristic properties possessed by the elements of these groups in general are as follows:

(a) In each space group all operators possessing a given rotational element α may be written in the form $\{\alpha \mid v(\alpha)\}$ $\{\alpha \mid t\}$ where t is an element of the translation group (space operators will appear in the group corresponding to all allowed values of t) and $v(\alpha)$ is a fixed vector for a given group. In other words, the factor group of the space group formed with respect to the translation group is simply isomorphic with the rotation group of the crystal class.

(b) Since a given space group is uniquely characterized by the subclass to which it belongs and the vector $v(\alpha)$ going with each α, a further classification may be afforded on the basis of the following divisions concerning properties of these vectors.

i) Space groups possessing elements such that all $v(\alpha)$ may be taken to be zero.

ii) Space groups possessing elements for which all $v(\alpha)$ may be selected so that each has no component in the direction going with the characteristic value 1 of the matrix α. In this case, of which i) is a special one, a coördinate system may be chosen so that any $v(\alpha)$ may be taken to be zero.

iii) Space groups for which at least one $v(\alpha)$ has a component in the direction going with the characteristic value 1 of α. From the geometric standpoint this means that there is at least one type of symmetry operation present which may not be viewed as being simply a rotation or reflection followed by a translation of the group.

(c) If n is the order of the group of a given crystal class and we form the $n - 1$ cosets of a space group associated with it, we see from (a) that all of the elements in any one of these possess the same rotational matrix and may be viewed as being generated from a given $\{\alpha \mid v(\alpha)\}$ by multiplying this with all elements of the translation group. If we add $\{\epsilon \mid 0\}$ to these $n - 1$ coset representatives, we have a set of n elements which stand in one to one correspondence with the elements of the class, and in the case i) above, this set may be taken to be identical with the group.

V. Every space group may be regarded as being a subgroup of a group of type i) in IV-b. This statement rests upon the fact that a coördinated system may be selected in which the translational component $v(\alpha)$ of all space operators may be written in the form

$$v(\alpha) = a_1 t_1 + a_2 t_2 + a_3 t_3$$

where t_1, t_2 and t_3 are the primitive translations of the translation group and a_1, a_2 and a_3 are fractions. If these fractions are expressed in terms of a lowest common denominator, say n, we may regard the space group as a subgroup of the space group of type i) in which $t_1' = t_1/n$ etc. In actual cases the largest value of n which occurs is 8.

VI. Every crystallographic point group C_1 is solvable[4]; that is, it is possible to form a composition series of the type

$$C_1, C_2, \cdots, C_n = \epsilon$$

such that the factor group C_i/C_{i+1} is an Abelian group of prime order.

It is clear from the previous facts concerning the connection between the point groups and the space groups that a similar statement will be true for the latter, the difference being that C_i will be replaced by a space group whose rotational components constitute this point group, so that C_n will be replaced by the translation group. In the following development, we shall be concerned only with those representations of the translation group that are of finite order, so that for our purposes all space groups may be regarded as being solvable.

2. **A theorem concerning the reduction of finite solvable groups.** The property of solvability constitutes an essential characterization of space groups for the purposes of the present discussion, and forms the basis for a scheme of complete reduction. The fundamental theorem of interest in this connection, namely theorem I of this section, was first given by Schur,[5] and will be presented from a slightly different viewpoint in the following paragraphs.

If we have a finite solvable group G_1 for which the following composition series obtains

$$G_1, G_2, \cdots G_m, \tag{8}$$

G_m being the unit element, the irreducible representations of G_h may be obtained from those of G_{h+1} by means of the following procedure.

Let us assume that the elements of G_{h+1} are arranged in the order

$$\mathfrak{A}_0 = \mathfrak{E}; \mathfrak{A}_{11}, \cdots, \mathfrak{A}_{1f_1}; \cdots; \mathfrak{A}_{q1}, \cdots, \mathfrak{A}_{qf_q}, \tag{9}$$

where elements $\mathfrak{A}_{ij}$ of given i undergo a cyclic permutation under similarity transformation with $\mathfrak{B}$, the element by means of which G_h may be derived from G_{h+1} by augmenting the latter. Since G_h is solvable, the smallest r such that $\mathfrak{B}^r$ is an element of G_{h+1} is a prime number.

Let σ^0 be an irreducible subspace, belonging to an irreducible representation R_0 of G_{h+1}, in the Hilbert space in which the group G_1 is represented, and let $(A^{(0)}_{ij,\mu\nu})$ be the representative of $\mathfrak{A}_{ij}$ in this subspace. Then under $\mathfrak{B}^m$ $(0 < m < r)$, σ^0 is sent into another subspace, which will be assumed to be independent of σ^0 and all spaces generated by other powers of $\mathfrak{B}$. That this assumption of independence does not affect the generality of the following will be shown presently.

Since G_{h+1} is an invariant subgroup of G_h, σ^m will be an irreducible subspace

[4] Solvable groups have been made the object of considerable investigation by G. Frobenius, *Sitz. der. Kön. Preuss. Akad.*, 337, 1893, and following papers.

[5] I. Schur, *Sitz. Preuss. Akad.* 164, 1906. I wish to express my indebtedness to Dr. R. Brauer for calling my attention to this work at the time the present manuscript was submitted for publication.

of the former, and the irreducible representation of G_{h+1} which occurs in it is such that the matrix $(a^{(m)}_{i\,j,\mu\nu})$ representing $\mathfrak{A}_{ij}$ is that corresponding to $\mathfrak{A}_{ij_m} = \mathfrak{B}^{-m}\mathfrak{A}_{ij}\mathfrak{B}^m$ in σ^0. For if ψ^0_μ is a function belonging to σ^0, we have

$$\mathfrak{A}_{ij_m}\psi^0_\mu = \sum_\nu a^{(0)}_{i\,j_m,\mu\nu}\psi^0_\nu, \tag{10}$$

so that

$$\mathfrak{B}^m\mathfrak{A}_{ij_m}\mathfrak{B}^{-m}\mathfrak{B}^m\psi^0_\mu = \mathfrak{A}_{ij}\mathfrak{B}^m\psi^0_\mu = \sum_\nu a^{(0)}_{i\,j_m,\mu\nu}\mathfrak{B}^m\psi^0_\nu \tag{11}$$

or

$$\mathfrak{A}_{ij}\psi^m_\mu = \sum_\nu a^{(0)}_{i\,j_m,\mu\nu}\psi^m_\nu. \tag{12}$$

Now if k is the smallest integer such that R_k is equivalent to R_0, we can say at once that R_{k+p} is equivalent to R_p. For if we order the operators $\mathfrak{A}_{ij}$ and their irreducible representations as in (8), the representation R_l is obtained by performing an l-fold cyclic permutation of the latter in each of the classes formed with respect to $\mathfrak{B}$, this permutation being in a direction opposite to that which the $\mathfrak{A}_{ij}$ (fixed i) undergo under $\mathfrak{B}^l$. Hence if R_k is equivalent to R_0, R_{k+p} will be equivalent to the representation obtained by performing a p-fold cyclic permutation on the order R_0.

By similar reasoning, it follows that R_i may not be equivalent to R_t unless R_m is equivalent to R_0 for $i \leqq m \leqq t$. This means, however, that k must be a factor of r, which is possible only if $k = 1$ or r, since r is prime. Hence:

The representations R_i $(i = 0, \cdots, r-1)$ *are either entirely equivalent or entirely inequivalent.*

If we designate the space spanned by the subspaces σ^i $(i = 0, 1, \cdots, r-1)$ by $\sum$ we know that $\sum$ is irreducible when the R_i are entirely inequivalent. For by Schur's Lemma, the only matrix which commutes with the representation of G_{h+1} in $\sum$ is diagonal with scalar sections in each of the σ^i. But each of these subspaces undergoes a transformation such that $\sigma^i \to \sigma^{i+1} \pmod r$ under $\mathfrak{B}$, so that the constant sections must be equal and $\sum$ is irreducible.

In case the R_i are entirely equivalent, there will be a matrix $\mathfrak{M}$ which transforms R_{i+1} into R_i, so that $\mathfrak{M}^n$ transforms R_n into R_0, and we shall assume that coördinate vectors have been chosen so that in each σ^i R_i has the same form as R_0. These vectors will be designated by φ^i_ν so that

$$\varphi^p_\mu = \sum_\tau \mathfrak{M}^p_{\mu\tau}\psi^p_\tau. \tag{13}$$

For from (10), (11) and (12), we have if

$$\mathfrak{A}_{ij_p} = \mathfrak{B}^{-p}\mathfrak{A}_{ij}\mathfrak{B}^p$$

that

$$\mathfrak{A}_{ij}\psi^p_\mu = \sum_\nu a^{(0)}_{i\,j_p,\mu\nu}\mu^p_\nu,$$

or

$$\mathfrak{A}_{ij}\varphi_\sigma^p = \sum_{\eta,\xi,\tau} \mathfrak{M}_{\sigma\eta}^p a_{i\,j\,p,\eta\xi}^{(0)} \mathfrak{M}_{\xi\tau}^{-p} \varphi_\tau^p = \sum_\tau a_{i\,j,\sigma\tau}^{(0)} \varphi_\tau^p .$$

In addition we have

$$\mathfrak{B}^p \psi_\mu^0 = \psi_\mu^p ,$$

so that

$$\mathfrak{B}^p \varphi_\mu^0 = \mathfrak{M}_{\mu\tau}^{-p} \varphi_\tau^p . \tag{14}$$

Hence it is clear that the representation of $\mathfrak{B}$ in $\sum$ is a matrix possessing zero components everywhere except in sections connecting columns going with σ^i and rows going with σ^{i+1}, and that the matrix appearing in this section is just $\mathfrak{M}^{-1}$. We shall choose coördinates in σ^0 so that $\mathfrak{M}$ is a diagonal matrix, in which case the diagonal elements will all be p^{th} roots of unity if $\mathfrak{B}^p = \mathfrak{E}$ $(p = lr)$, and let these coördinates be designated by φ_ν^i as above. Let us now form functions

$$\chi_\mu^i = \sum b_{ik} \varphi_\mu^k , \tag{15}$$

so that χ_1^i is an eigenfunction of $\mathfrak{B}$. Since the eigenvalue going with this must correspond to one of the irreducible representations of $\mathfrak{B}$, say that in which ω_b corresponds to $\mathfrak{B}$, the coefficient b_{ik} must be of the form

$$\sum_{n=0}^{l-1} (\omega_0^{-1}\omega_1^{-1})^{k+nr} , \tag{16}$$

where generally ω_ν is the matrix component of $\mathfrak{M}$ going with φ_ν^i and (16) will vanish unless $(\omega_0\omega_1)^{-1}$ is an r^{th} root of unity. Let these roots be designated by θ_i $(i = 1, 2, \cdots, r)$, so that (15) becomes

$$\chi_\mu^i = \sum_{n=0}^{r\,l-1} \theta_i^n \varphi_\mu^n . \tag{17}$$

From this it follows at once that χ_μ^i is generally an eigenfunction of $\mathfrak{B}$ associated with the eigenvalue $\theta_i^{-1}\omega_\mu^{-1}$. But the set of these functions going with fixed i belong to the same irreducible representation of G_{h+1} as the set φ_κ^i (fixed κ), so that the former set span an irreducible space under G_h. Hence we have:

THEOREM I. *If each of the subspaces σ^i $(i = 0, 1, \cdots, r-1)$ of $\sum$ belong to different representations of G_{h+1}, $\sum$ is irreducible under G_h. If all of the representations are equivalent, then $\sum$ is reducible to r irreducible subspaces of equal dimensionality. If the representatives of G_{h+1} are taken to be the same in each subspace, then the representative of $\mathfrak{B}$ is $\mathfrak{M}$ times an r^{th} root or unity.*

The case in which some σ^i is not independent of σ^0 can clearly occur only when R_i is equivalent to R_0, so that unless $i = r$ we must have $\sigma' \equiv \sigma^0$, etc., so that $\sum$ is identical with σ^0 and the representative of $\mathfrak{B}$ is simply $\mathfrak{M}^{-1}$.

Now G_{m-3} in (8) will be the translation group of G_1 which may always be reduced by choosing a coördinate system of functions of the type

$$\eta_\kappa e^{i\kappa\cdot r} \tag{18}$$

where η_κ is a function that is invariant under the translation group, and κ is a constant vector. From the physical standpoint it is always convenient to restrict κ to such values that $N_i\kappa\cdot t_i = 2\pi$ for each of the primitive translation vectors t_1, t_2, t_3, where the N_i are integers. In other words κ is restricted to be a lattice vector of the inverse lattice of the crystal. From the present standpoint this has the important consequence that it restricts all representations of the translation group that are considered to be finite ones whose order is less than or equal to $N_1N_2N_3$, so that the preceding theorem is valid and affords a simple, direct method of attaining complete reduction for any space group.

In the next section, we shall present a development which will generally simplify the process of reducing G_1 by allowing us to pass from the irreducible form of a particular subgroup of G_1 to that of G_1. This subgroup will also be solvable, of course, so that it may always be reduced by means of the preceding theorem, though in certain cases it may be reduced at once by direct use of some previous results on the reduction of the macroscopic crystallographic groups.

3. **Additional simplifying developments.** If the factor group G_1/Γ, where G_1 is a given space group and Γ is the translation group, is of order n, we have seen in §1 that there will be n coset representatives, $\mathfrak{A}_1, \cdots, \mathfrak{A}_n$ ($\mathfrak{A}_1 = \mathfrak{E}$), such that any element of G_1 is equal to one of these times an element of the translation group. If we now select a coördinate system in Hilbert space so that all functions are of the form (18), $n - 1$ other functions of this type may be generated from any given $\psi_1 = \eta_{\kappa_1} e^{i\kappa_1\cdot r}$ of (18) by operating on it with these n coset representatives. These n functions

$$\mathfrak{A}_1\psi_1 = \psi_1, \qquad \mathfrak{A}_2\psi_1 = \psi_2, \cdots, \qquad \mathfrak{A}_n\psi_1 = \psi_n$$

will be assumed to be linearly independent so that they span an n dimensional space $\sum_n$. This assumption of linear independence will not be valid in cases when κ is invariant under certain coset representatives and the η's generated by these are not linearly independent. It will not be necessary to make exception to these cases at present, however, because the effect of this is simply to reduce the dimensionality of the representations with which we shall be concerned in such a way as to exclude entire irreducible parts that would be present if the linear dependence did not exist.

Since the product of any two coset representatives is equal to another times an element of the translation group, it is clear that the same set of ψ's will be generated if any one of the set is taken to replace ψ_1; that is, $\sum_n$ is invariant under the entire space group. This means that we have effected a partial

reduction of the whole space into n dimensional subspaces so that we may restrict further discussion to these.

LEMMA I. *Functions ψ_ν^1 ($\nu = 1, \cdots, l$) of $\sum_n$ which belong to a given representation T_1 of the translation group are sent into functions which belong to a representation T_2 by any element of the group. T_2 may or may not be equivalent to T_1.*

If $\mathfrak{A}$ is an arbitrary space operator and $\mathfrak{T}$ and $\mathfrak{T}'$ are translation operators such that $\mathfrak{T}$ is arbitrary and

$$\mathfrak{T}' = \mathfrak{A}^{-1}\,\mathfrak{T}\mathfrak{A}$$

we have

$$\mathfrak{T}\,\psi_\nu^1 = \omega_\iota \psi_\nu^2, \qquad \mathfrak{T}'\psi_\nu^1 = \omega_\iota' \psi_\nu^1 .$$

But

$$\mathfrak{A}\mathfrak{T}'\psi_\nu^1 = \omega_\iota'\,\mathfrak{A}\psi_\nu^1$$

so that

$$\mathfrak{T}\mathfrak{A}\psi_\nu' = \omega_\iota'\mathfrak{A}\psi_\nu^1$$

and the l functions $\mathfrak{A}\psi_\nu^1$ belong to the same representation. Thus the space σ_l^1 of $\sum_n$ which belongs to a given representation of the translation group transforms either entirely into itself or entirely into another space, σ_l^i, belonging to a different representation, under elements of G_1.

Since the l functions ψ_ν^1 may be selected to be the first l of the n introduced previously by properly arranging the latter, it follows that it is possible to arrange the entire n into sets

$$\begin{array}{ll} \sigma_l^1 : \psi_1^1, \cdots, \psi_l^1 & \\ \cdots\cdots\cdots\cdots & h = n/l \\ \sigma_l^h : \psi_1^h, \cdots, \psi_l^h & \end{array} \tag{19}$$

each of which belongs to a different representation of the translation group. Moreover any set may be sent into any other by use of an appropriate transformation of G_n.

LEMMA II. *The operators which send any space σ_l^i into itself constitute a group G_l^i which contains the entire translation group.*

That the translation group leaves the space σ_l^i invariant is obvious from its diagonal form. Similarly, it is clear that $\mathfrak{B}\mathfrak{A}$ will leave it invariant if $\mathfrak{B}$ and $\mathfrak{A}$ do.

Since an integer m always exists such that $\mathfrak{A}^m = \mathfrak{T}$, where $\mathfrak{T}$ is an element of the translation group which depends upon $\mathfrak{A}$, we have

$$\mathfrak{A}^{-1} = \mathfrak{A}^{m-1}\mathfrak{T}^{-1}$$

so that the left hand side must leave σ_l^i invariant because the right does. Hence all such elements constitute a group. It follows at once that the groups G_l^i are equivalent and may be transformed into one another by the elements of G_n.

LEMMA III. *If each of the subspaces σ_l^i are completely reduced into p subspaces $\sigma_{l,j}^i$ $(j = 1, \cdots p)$ for the group G_l^i which leaves each invariant, then all elements of G_n not in G_l^i send each $\sigma_{l,j}^i$ into a space $\sigma_{l,j}^k$ which is equivalent with regard to G_l^k.*

Since the subgroups G_l^i are equivalent, it follows that each may be reduced in such a manner that equivalent elements are reduced in the same form. Now if $\psi_{j,\mu}^i$ is a function belonging to $\sigma_{l,j}^i$ and $\mathfrak{A}_n$ is any element of G_l^i we have

$$\mathfrak{A}_n \psi_{j,\mu}^i = \sum_\nu a_{n,\mu\nu} \psi_{j,\nu}^i .$$

Hence if $\mathfrak{A}'$ is an element of G_n not in G_l^i, we have

$$\mathfrak{A}'\mathfrak{A}_n\mathfrak{A}'^{-1}\mathfrak{A}\psi_{j,\mu}^i = \mathfrak{A}_{n'}\mathfrak{A}'\psi_{j,\mu}^i = \sum_\nu a_{n,\mu\nu}\mathfrak{A}'\psi_{j,\nu}^i = \sum_\nu a_{n,\mu\nu}\psi_{j,\nu}^k$$

so that $\psi_{j,\mu}^i$ is sent into a function $\psi_{j,\mu}^k$ which transforms under the same representation of the equivalent group that leaves σ_l^k invariant.

If we gather together those spaces

$$\sigma_{l,k}^1 \cdots \sigma_{l,m}^1 ; \sigma_{l,k}^2 \cdots \sigma_{l,m}^2, \cdots ; \sigma_{l,k}^h \cdots \sigma_{l,m}^h \tag{20}$$

which belong to the same irreducible representation of the equivalent groups $G_l^1, G_l^2, \cdots, G_l^h$, and designate the resulting space by $\sum^{k\cdots m}$ we shall have effected an additional reduction of the entire space group, because there are no matrix components between the different spaces $\sum^{k\cdots m}$ which as a whole span $\sum_n$.

If a given representation R_s^i of G_l^i occurs but once in σ_l^i, the corresponding subspace, $\sum^s$, will be irreducible. For by Schur's lemma any matrix which commutes with the space group has no components connecting the σ_l^i because each of these corresponds to a different irreducible representation of the translation group. In any σ_l^i there are no matrix components of this commuting matrix connecting any $\sigma_{l,h}^i$ and the other spaces. But each $\sigma_{l,h}^i$ may be sent into any other $\sigma_{l,h}^j$ by some element of G_n so that the matrix is constant in $\sum^h$. Thus $\sum^h$ is irreducible.

In case the representation R_s^i appears more than once in σ_l^i say in the spaces $\sigma_{l,k}^i \cdots \sigma_{l,m}^i$, then $\sum^{h\cdots m}$ will not be irreducible, but it is easily seen that $\sum^k, \cdots \sum^m$ are each irreducible. For by hypothesis, $\sigma_{l,k}^i \cdots \sigma_{l,m}^i$, with fixed i, are irreducible under G_l^i so that no element of this group sends them into one another. If an element $\mathfrak{A}$ not in G_l^i sends these into $\sigma_{l,k}^h \cdots , \sigma_{l,m}^h$ respectively, then any other element $\mathfrak{B}$ which sends σ_l^i into σ_l^h must do the same. For otherwise $\mathfrak{A}^{-1}\mathfrak{B}$ would connect the spaces $\sigma_{l,k}^i \cdots \sigma_{l,m}^i$ of σ_l^i, contrary to assumption. From this point on the proof follows that of the preceding paragraph. Thus when the irreducible representations of G_l^i in σ_l^i are completely known, the space group is essentially reduced, and we need only be concerned with the examination of representations of the space groups for which the matrices representing the translation elements are constants, that is, we need only be concerned with the σ_l^i and the associated R_l^i.

From this discussion we may immediately draw the conclusion:

THEOREM II. *If σ_l^i is one dimensional, $\sum_n$ is irreducible.*

In the event that σ_l^i is not one-dimensional, we may find the irreducible representations for a special class of space groups by a very simple means. These cases are those in which G_l^i, the subgroup of G_n which leaves σ_l^i invariant, is a space group of type i) discussed in IV-b of §2. This clearly includes all cases in which G_n is of type i) and may also include special cases of ii) and iii). For in this case the l coset representatives may be selected to be elements of the form $\{\alpha|0\}$ which constitute a point group of order l are represented by rotational matrices in σ_l^i. The process of reducing G_l^i is then simply one of reducing this subgroup since the only matrix which commutes with R_l^i must commute with this and vice versa. The problem of reducing the crystallographic point groups has been solved by Bethe,[6] however, so that space groups of the special type mentioned above may be reduced by means of his results in a very simple way.

In the event that G_l^i is not a group of type i), the result of §2 may be employed to reduce it in the manner presented there. Since this group will generally be more elementary than G_1, the use of the results of this section will prove to be of considerable practical value.

4. **Concerning physical applications.** The previous development affords a basis for an investigation of the eigenfunctions of the operators met with in physics in cases in which the latter possess space group symmetry and the so-called Born and Kármán[7] boundary conditions are employed, for in all such cases the eigenfunctions will transform under one of the irreducible representations of the corresponding space group. Thus the normal coördinates employed in treating the harmonic vibrations of crystals, which are simply eigenfunctions of a quadratic potential form that possesses the symmetry of a space group, may readily be derived by use of algebraic representation theory on the basis of the preceding work. This procedure avoids the more cumbersome methods developed by Brester.[8]

In a quantum theoretic treatment of the behavior of electrons in solids, an extremely useful starting approximation is that in which electron interactions are taken into account by means of a self-consistent field,[9] so that each electron moves in an effective field resulting from nuclei and other electrons. This field may be taken to be the same for all electrons, and since it possesses the symmetry of the lattice the electron wave functions will belong to irreducible representations of the symmetry group. The form of these functions will always be that of (18). Since functions that are partners in the same irreducible representation possess the same eigenvalue of the energy operator, we

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[6] H. Bethe, *Ann. der Phys.* 3, 133, 1929.
E. P. Wigner, *Göttinger Nachr.*, 133, 1930.

[7] M. Born and T. v. Kármán, *Phys. Zs.* 13, 297, 1912.

[8] C. J. Brester, *Kristallsymmetrie und Reststrahlen*, Utrecht, 1923.

[9] E. P. Wigner and F. Seitz, *Phys. Rev.* 43, 804 (1933); 46, 509 (1934).
F Seitz, *Phys. Rev.* 47, 400, 1935.

know from theorem II that the number of degenerate states will be equal to the number of coset representatives unless the function (18) is sent into one possessing the same κ value by one of the coset representatives, aside from the unit element, when cosets are formed using the translation group as invariant subgroup. In all other cases there is the possibility of more than one energy value being associated with a given value of κ. If we regard the eigenvalue of energy as a function of κ, $E(\kappa)$, this many-valuedness of E, for certain values of κ, corresponds to discontinuities in $E(\kappa)$ for which there is abundant experimental evidence. The natural basis for the discussion of this subject is afforded by the preceding work, and has been given in part in one of the physical journals,[10] and will be extended later.

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[10] R. B. Barnes, R. R. Brattain, F. Seitz, *Phys. Rev.* 48, 582 (1935).

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LINEAR DIFFERENTIAL EQUATIONS WITH ALMOST PERIODIC COEFFICIENTS

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It is the purpose of this paper to continue the investigations of Bochner[2] on "Homogeneous Systems of Differential Equations with Almost Periodic Coefficients," and to point out that the conclusions obtained both by Bochner and myself can also be applied to non-homogeneous equations.

1. We shall deal with vectors y having the n real or complex components $y_1, \cdots, y_n$ and the norm $\| y \| = | y_1 | + \cdots + | y_n |$. These vectors will usually be functions $y(x)$ of the real variable x, and will ordinarily denote solutions of

$$\frac{dy_\mu(x)}{dx} = a_{\mu 1}(x)y_1(x) + \cdots + a_{\mu n}(x)y_n(x) \qquad (\mu = 1, \cdots, n) \tag{1}$$

or of

$$\frac{dy_\mu(x)}{dx} = a_{\mu 1}(x)y_1(x) + \cdots + a_{\mu n}(x)y_n(x) + b_\mu(x) \qquad (\mu = 1, \cdots, n). \tag{2}$$

If the coefficients $a_{\mu\nu}(x)$ are all real for all values of x, the solutions $y(x)$ may if desired be considered to be restricted to real numbers unless otherwise stated. In Theorem II whose proof depends upon the existence of solutions of an algebraic equation, it will be necessary to consider both real and complex solutions, even though the coefficients happen to be real.

Addition of vectors and multiplication by complex numbers has its usual meaning. Limits are to be understood in terms of the norm given above. Thus the statement that $\lim_{i\to\infty} y^{(i)} = y$ is equivalent to the statement that $\lim_{i\to\infty} \| y^{(i)} - y \| = 0$. The symbol $R\,[y(x)]$ will denote the closure of the set of values of the vector $y(x)$ as x takes on all real values. In this paper this closed set will be called the range of $y(x)$. If x is restricted to take on only positive values, the closure of the set of values of $y(x)$ will be called the positive range and denoted by $R^+\,[y(x)]$. $R^{(+)}$ will be used for R or R^+ as $\pm$ is used for $+$ or $-$.

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[1] National Research Fellow.

[2] *Journal of the London Mathematical Society*, Vol. 8, 1933, pp. 283–288. This paper will hereafter be referred to as Bochner I.

DEFINITION 1. A function $y(x)$ will be called *symmetric* if

$$R\,[y(x)] = R\,[cy(x)]$$

for some constant c such that $c \neq 1$ and $|\,c\,| = 1$. If the function satisfies $R^+\,[y(x)] = R^+\,[cy(x)]$ for some such c, it will be called *positive symmetric.*

Note. The symbol $\Re$ will be used to denote the real part of an imaginary number.

2. In terms of the above notation we can now state

BOCHNER'S THEOREM.[3] *Let $a_{\mu\nu}(x)$ be almost periodic ($\mu, \nu = 1, \cdots, n$), and let M be their module. Then all of the solutions of* (1) *are almost periodic if the following hypotheses are satisfied:—*

I. $\int_0^x \Re\,[a_{11}(\xi) + \cdots + a_{nn}(\xi)]\,d\xi$ *is bounded.*

II. *All the solutions of* (1) *are bounded.*

III. *There exists a set of linearly independent solutions $y^{(1)}(x), \cdots, y^{(n)}(x)$ having the property that if $y(x)$ is any solution of* (1), *then it is not true that $R\,[y(x)] \subset R\,[y^r(x)]$, unless $y(x) \equiv y^r(x)$.*

Moreover if these conditions are satisfied, the module of the set of solutions of (1) *is M.*

One of the aims of this paper is to present some alternative conditions to take the place of III. Two such conditions are

IVa. *There exists a set of linearly independent solutions $y^{(1)}(x), \cdots, y^{(n)}(x)$ of* (1) *having the property that if $y(x)$ is any solution of* (1), *then it is not true that $R\,[y(x)] = R\,[y^{(\nu)}(x)]$,* unless *$y(x) \equiv y^{(\nu)}(x)$.*

Va. *There are no symmetric solutions of* (1); (*neither real nor complex ones*).

As special cases of other theorems, it will be proved in this paper that if the $a_{\mu\nu}(x)$ are almost periodic with the module M, then conditions I, II, and IVa, or I, II, and Va, are sufficient to insure that all the solutions of (1) are almost periodic with the module M. Since IVa must obviously be satisfied whenever III is satisfied it follows that the sufficiency of I, II, and IVa implies Bochner's theorem.

3. While I and II are necessary conditions, this is not true of III, IVa or Va, so that none of the three theorems given so far provides necessary and sufficient conditions. However, by postulating that all of the solutions having certain very specialized properties should be almost periodic, it is possible to obtain necessary and sufficient conditions that all the solutions be almost periodic. Consider for instance the following conditions:

IV. *There exists a set of linearly independent solutions $y^{(1)}(x), \cdots, y^{(n)}(x)$ of* (1) *having the property that if $y(x)$ is any solution of* (1), *then it is not true that $R\,[y(x)] = R\,[y^{(\nu)}(x)]$, unless $y(x) \equiv y^{(\nu)}(x)$ or $y(x)$ is almost periodic.*

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[3] Bochner I.

V. *There are no symmetric solutions of* (1), *(neither real nor complex ones), except almost periodic ones.*

Since IVa implies IV and Va implies V, it follows that the two theorems already given are respectively special cases of the following theorems:

THEOREM I. *Let* $a_{\mu\nu}(x)$ *be almost periodic* $(\mu, \nu = 1, \cdots, n)$. *Then all the solutions of* (1) *are almost periodic if and only if conditions* I, II, *and* IV *are satisfied. Moreover if these conditions are satisfied the module of all the solutions of* (1) *is the same as the module of the set consisting of all the* $a_{\mu\nu}(x)$ *together with all the solutions of* (1) *which have the same range as one of the* $y^{(\nu)}(x)$ *but differ from it.*

THEOREM II. *Let* $a_{\mu\nu}(x)$ *be almost periodic* $(\mu, \nu = 1, \cdots, n)$. *Then all the solutions of* (1) *are almost periodic if and only if conditions* I, II, *and* V *are satisfied. Moreover if these conditions are satisfied, the module of all the solutions of* (1) *is the same as the module of the set consisting of all of the* $a_{\mu\nu}(x)$ *together with all the symmetric solutions of* (1); *(both real and complex).*

4. Another aim of this paper is to obtain sufficient conditions that a particular solution of (1) be almost periodic, even when not all the solutions of (1) are almost periodic. Such conditions are furnished by the following theorem, of which Theorem I is an immediate consequence:

THEOREM III. *Let* $a_{\mu\nu}(x)$ *be almost periodic* $(\mu, \nu = 1, \cdots, n)$. *Then if conditions* I *and* II, *are satisfied, and if* $y(x)$ *is a solution of* (1) *such that no solution* $z(x)$ *can have the same range as* $y(x)$ *and be distinct from* $y(x)$ *unless it is almost periodic, it follows that* $y(x)$ *is almost periodic. Moreover the module of* $y(x)$ *is contained in the module of the set consisting of all the* $a_{\mu\nu}(x)$ *together with all the solutions* $z(x)$ *of* (1) *other than* $y(x)$ *which satisfy*

$$R\,[z(x)] = R\,[y(x)].$$

In this paper it is also shown that all these conditions need be stated merely for positive values of x. Thus boundedness in conditions I or II may be replaced by boundedness for positive x, and R may everywhere be replaced by R^+.

Finally, it will be shown that all these theorems may be restated so as to apply to the non-homogeneous equation (2).

5. Before proving the above theorems it will be necessary to give some definitions and lemmas. The work will all be based on the following theorem of Bochner:

THEOREM. *A continuous function*[4] *or vector*[5] *function* $f(x)$ *is almost periodic if and only if to every sequence* $h_1, h_2, \cdots$ *of real numbers there corresponds a subsequence* $h_{p_1}, h_{p_2}, \cdots$ *such that* $\lim_{i\to\infty} f(x + h_{p_i})$ *exists uniformly for all* x.

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[4] *Mathematische Annalen*, Vol. 96 (1926), pp. 129–147, esp. p. 143.

[5] "Abstrakte Fastperiodische Funktionen," *Acta Mathematica*, vol. 61, pp. 149–184, especially page 153.

The method of proof used in this paper was first used by Favard[6] and later by Bochner.[7]

DEFINITION 2. A sequence of real numbers $h_1, h_2, \cdots$ will be said to *converge with respect to a certain module* M if uniformly for all x the sequence $f(x + h_1), f(x + h_2), \cdots$ converges for each almost periodic function $f(x)$ having the module M. Moreover the sequence $k_1, k_2, \cdots$, will be said to *converge to* $l_1, l_2, \cdots$ *with respect to* M if the sequence $k_1, l_1, k_2, l_2, \cdots$ converges with respect to M.

LEMMA 1. *A necessary and sufficient condition that a sequence* $h_1, h_2, \cdots$ *converge with respect to a module* M *is that there exist an almost periodic function* $f(x)$ *having the module* M *such that the sequence* $f(x + h_1), f(x + h_2), \cdots$ *converges uniformly for all* x.

This lemma follows from a theorem of Favard.[8]

LEMMA 2. *The necessary and sufficient condition that a sequence* $h_1, h_2, \cdots$ *be convergent with respect to a module* M *is that to every* $\delta > 0$ *and every finite set* $p_1, p_2, \cdots, p_r$ *of elements of* M *there should correspond an index* N *such that when* $i > N$ and $j > N$ *then* $| p_\sigma(h_i - h_j) | < \delta \pmod{2\pi}$ $(\sigma = 1, 2, \cdots, r)$.

The necessary and sufficient condition that $h_1, h_2, \cdots$ *converge to* $k_1, k_2, \cdots$ *with respect to* M *is that to every* $\delta > 0$ *and every finite set* $p_1, \cdots, p_r$ *of elements of* M *there should correspond an index* N *such that when* $i > N$ *and* $j > N$, *then*

$$| p_\sigma(h_i - k_j) | < \delta \pmod{2\pi} \quad (\sigma = 1, \cdots, r).$$

This lemma can readily be seen to follow from the classical theory.

LEMMA 3. *If* M *is a given countable module and* $h_1, h_2, \cdots$ *is any sequence of real numbers, there exists a sub-sequence of* $h_1, h_2, \cdots$ *which converges with respect to* M. *If* $k_1, k_2, \cdots$ *converges with respect to* M, *a sequence of positive numbers* $l_1, l_2, \cdots$ *can be found which converges to* $k_1, k_2, \cdots$ *with respect to* M.

The first statement of this lemma follows from the above theorem of Bochner and Lemma 1. To verify the second statement, arrange the elements of M in countable order $p_1, p_2, \cdots$, and choose corresponding to each positive integer j a positive number l_j such that

$$| p_\sigma(k_j - l_j) | < \frac{1}{j} \pmod{2\pi} \quad (\sigma = 1, 2, \cdots, j).$$

Then the sequence $l_1, l_2, \cdots$ converges to $k_1, k_2, \cdots$ with respect to M.

LEMMA 4. *If* M *is given countable module and* $f(x)$ *is a continuous function such that uniformly for all* x $\lim_{i\to\infty} f(x + h_i)$ *exists for every sequence* $h_1, h_2, \cdots$ *of positive numbers which converges to* 0, 0, 0, $\cdots$ *with respect to* M, *then* $f(x)$ *is almost periodic and has its module contained in* M.

For suppose that there exists a sequence $k_1, k_2, \cdots$ of real numbers which

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[6] *Acta Math.* vol. 51, pp. 31–81. This paper will be referred to as Favard I.

[7] Bochner I.

[8] Favard I, p. 41.

converges with respect to M and for which $\lim_{i\to\infty} f(x + k_i)$ does not exist uniformly for all x. Then there exist a number $\epsilon > 0$, two subsequences $l_1, l_2, \cdots$ and $l_1', l_2', \cdots$ of $k_1, k_2, \cdots$ and a sequence of real numbers $x_1, x_2, \cdots$ such that for all positive integers i,

$$| f(x_i + l_i) - f(x_i + l_i') | \geqq \epsilon. \tag{3}$$

Moreover the subsequences can be chosen so that for each i, $l_i > l_i'$. But since $k_1, k_2, \cdots$ converges with respect to M, the sequence $l_1 - l_1', l_2 - l_2', \cdots$ converges to $0, 0, \cdots$ with respect to M. Hence $\lim_{i\to\infty} f(x + l_i - l_i') = f(x)$ uniformly for all x; and on replacing x by $x_i + l_i'$ we obtain a contradiction to (3). Thus our assumption was false and it follows from the above mentioned theorem of Favard[9] that $f(x)$ is almost periodic and has a module contained in M.

DEFINITION 3. A vector function $y(x)$ will be called *stationary* with respect to a module M if for all x

$$\lim_{j\to\infty} \lim_{i\to\infty} y(x + h_i + k_j) = y(x)$$

whenever $h_1, h_2, \cdots$ and $k_1, k_2, \cdots$ are sequences such that the above iterated limit exists for all x and such that $h_1, h_2, \cdots$ converges to $-k_1, -k_2, \cdots$ with respect to M. If the sequences $h_1, h_2, \cdots ; k_1, k_2, \cdots$ are restricted to positive numbers, $y(x)$ is called *positive stationary*.

DEFINITION 4. A vector function $y(x)$ will be called *twistable* with respect to a certain module M if there exists a constant $c \neq 1$ whose absolute value is unity such that for all x

$$\lim_{j\to\infty} \lim_{i\to\infty} y(x + h_i + k_j) = cy(x) \tag{4}$$

for some pair of sequences $h_1, h_2, \cdots$ and $k_1, k_2, \cdots$ such that the above iterated limit exists for all x, and such that $h_1, h_2, \cdots$ converges to $-k_1, -k_2, \cdots$ with respect to M. If the sequences can be chosen so as to have all their terms positive the function is called *positive twistable*.

LEMMA 5. *A vector function which is (positive) twistable with respect to one arbitrary module is (positive) symmetric.*

For (4) implies that $R[y(x)] \supset R[cy(x)]$ and hence $R[y(x)] \supset R[cy(x)] \supset R[c^2y(x)] \supset \cdots \supset R[c^my(x)] \supset \cdots$. Since for a given x, $y(x)$ is contained in the closure of the set $c^my(x)$ for $(m = 2, 3, \cdots)$, it follows that $R[y(x)] = R[cy(x)]$; moreover R can everywhere be replaced by R^+ if $h_1, h_2, \cdots$ and $k_1, k_2, \cdots$ are positive.

LEMMA 6. *A necessary and sufficient condition that a vector function $y(x)$ be almost periodic and have its module contained in a given countable module M is that it have the following properties:*

A. *The function $y(x)$ is defined for all x and bounded and continuous for positive x.*

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[9] Favard I, p. 41.

B. *It is positive stationary with respect to* M.

C. *If the sequence of positive numbers* $h_1, h_2, \cdots$ *converges with respect to* M *and* $\lim_{i\to\infty} y(h_i)$ *exists, then* $\lim_{i\to\infty} y(x + h_i)$ *exists for each* x.

D. *If the sequences of positive numbers* $h_1, h_2, \cdots$ *and* $k_1, k_2, \cdots$ *both converge with respect to* M *and* $\lim_{j\to\infty} \lim_{i\to\infty} y(h_i + k_j)$ *exists, then for each* x, $\lim_{j\to\infty} \lim_{i\to\infty} y(x + h_i + k_j)$ *exists.*

E. *If the sequences of positive numbers* $h_1, h_2, \cdots$; $k_1, k_2, \cdots$; $l_1, l_2, \cdots$ *all converge with respect to* M, *and if* $h_1, h_2, \cdots$ *converges to* $k_1, k_2, \cdots$ *with respect to* M, *and if the following iterated limits exist and are equal for all* x:

$$\lim_{j\to\infty} \lim_{i\to\infty} y(x + h_i + l_j); \qquad \lim_{j\to\infty} \lim_{i\to\infty} y(x + k_i + l_j);$$

then for all x

$$\lim_{i\to\infty} y(x + h_i) = \lim_{i\to\infty} y(x + k_i).$$

(*Note.* Although D implies C, it is convenient to have C stated explicitly.)

PROOF. The necessity of the condition is obvious. To prove that it is sufficient assume that it holds and let $h_1, h_2, \cdots$ be any sequence of positive numbers which converges with respect to M. Suppose that $\lim_{i\to\infty} y(x + h_i)$ does not exist uniformly in x for all positive x. Then there exist $\epsilon > 0$, two sub-sequences $k_1, k_2, \cdots$ and $l_1, l_2, \cdots$ of $h_1, h_2, \cdots$ and a positive sequence $x_1, x_2, \cdots$ such that

$$\| y(x_i + k_i) - y(x_i + l_i) \| \geqq \epsilon \qquad (i = 1, 2, \cdots). \tag{5}$$

Choose a sub-sequence $x_1', k_1', l_1'; x_2', k_2', l_2', \cdots$ of the sequence of triples x_1, k_1, l_1; $x_2, k_2, l_2; \cdots$ such that $\lim_{i\to\infty} y(x_i' + k_i')$ and $\lim_{i\to\infty} y(x_i' + l_i')$ exist and such that $x_1' + k_1', x_2' + k_2', \cdots$ and $x_1' + l_1', x_2' + l_2', \cdots$ converge with respect to M. Then by property C, $\lim_{i\to\infty} y(x + x_i' + k_i')$ and $\lim_{i\to\infty} y(x + x_i' + l_i')$ exist for all x. Choose a sequence $\lambda_1, \lambda_2, \cdots$ of positive numbers which converges to $-(x_1 + k_1'), -(x_2 + k_2'), \cdots$ with respect to M and a sub-sequence $\lambda_1', \lambda_2', \cdots$ of $\lambda_1, \lambda_2, \cdots$ such that $\lim_{j\to\infty} \lim_{i\to\infty} y(x_i' + k_i' + \lambda_j')$ and $\lim_{j\to\infty} \lim_{i\to\infty} y(x_i' + k_i' + \lambda_j')$ exist. By property D, $\lim_{j\to\infty} \lim_{i\to\infty} y(x + x_i' + k_i' + \lambda_j')$ and $\lim_{j\to\infty} \lim_{i\to\infty} y(x + x_i' + l_i' + \lambda_j')$ exist for all x, and by B they are both equal to $y(x)$. Hence by E

$$\lim_{i\to\infty} y(x_i' + k_i') = \lim_{i\to\infty} y(x_i' + l_i'),$$

contrary to (5). Thus the assumption was incorrect and $\lim_{i\to\infty} y(x + h_i)$ exists uniformly in x for all positive x. By an argument used by the author in a paper "Almost Periodic Transformations" in the April number of the Transactions of the American Mathematical Society (1934) it can be shown that this implies that the limit exists uniformly for all x. Hence by Lemma 4, $y(x)$ is almost periodic with a module contained in M.

LEMMA 7. *If $f(x)$ is an almost periodic numerical function and $\int_0^x f(\xi)d\xi$ is bounded for positive x, then it is bounded for all x.*

To show this, let M be an upper bound for $\int_0^x f(\xi)d\xi$ for positive x, and let x_1 be any negative number. Let $\tau > -x_1$ be a 1-translation number for the almost periodic function $\int_{x_1}^x f(\xi)d\xi$. Then $\left|\int_0^{x_1} f(\xi)d\xi\right| \leqq 1 + \left|\int_{x_1}^{\tau} f(\xi)d\xi\right| \leqq 1 + \left|\int_0^{x_1+\tau} f(\xi)d\xi\right| + \left|\int_0^{\tau} f(\xi)d\xi\right| \leqq 1 + 2M$; and the lemma is proved.

Before stating Lemma 8, we will introduce the following modifications of some of the conditions mentioned above.

I$^+$. $\int_0^x \Re\,[a_{11}(\xi) + \cdots + a_{nn}(\xi)]\,d\xi$ *is bounded for positive* x.

II$^+$. *All of the solutions of* (1) *are bounded for positive* x.

V$^+$. *There are no positive symmetric solutions of* (1), *(neither real nor complex ones) except almost periodic ones.*

Obviously I and I$^+$ are equivalent, from Lemma 7.

LEMMA 8. *Let $a_{\mu\nu}(x)$ be almost periodic ($\mu, \nu = 1, \cdots, n$) and have their module contained in the module M, let the system* (1) *satisfy conditions* I$^+$ *and* II$^+$, *and let $h_1, h_2, \cdots$ and $k_1, k_2, \cdots$ be two sequences of positive numbers such that $h_1, h_2, \cdots$ converges to $-k_1, -k_2, \cdots$ with respect to M. Then if for each x and each solution $y(x)$ of* (1) $\lim_{j\to\infty} \lim_{i\to\infty} y(x + h_i + k_j)$ *exists, the transformation*

$$T\,[y(x)] = \lim_{j\to\infty} \lim_{i\to\infty} y(x + h_i + k_j)$$

is one-to-one and linear and does not alter the positive range or the range of any solution. Moreover there exists a linearly independent set of solutions $y^{(1)}(x), \cdots, y^{(n)}(x)$ (which may be complex even if the $a_{\mu\nu}(x)$ are real) such that

$$T\,[y^{(\nu)}(x)] = \lambda_\nu y^{(\nu)}(x) \qquad (\nu = 1, \cdots, n); \tag{6}$$

where the λ_ν are constants having the absolute value unity.

(*Note.* The symbol $T[y(x_1)]$ will be used to denote $T[y(x)]_{x=x_1}$).

PROOF. The fact that T is linear is obvious, as is also the fact that

$$R^{(+)}[y(x)] \supset R^{(+)}\{T[y(x)]\} \tag{7}$$

for each solution $y(x)$. Since $h_1, h_2, \cdots$ converges with respect to M $\lim_{i\to\infty} a_{\mu\nu}(x + h_i)$ exists uniformly for each μ and ν, and if the $a_{\mu\nu}(x)$ in (1) are replaced by these limits the resulting system has $\lim_{i\to\infty} y(x + h_i)$ as a solution. In the same way, since $\lim_{j\to\infty} \lim_{i\to\infty} a_{\mu\nu}(x + h_i + k_j) = a_{\mu\nu}(k)$ uniformly in x, it follows that for every solution $y(x)$ of (1), $T[y(x)]$ is also a solution. Thus the linear manifold of all solutions is transformed by T into a sub-set of itself, and its matrix can be reduced to the classic canonical form. Hence there exists a set of indices $1 = p_1 < p_2 < \cdots < p_q < p_{q+1} = n + 1$ and

a set of constant multipliers $\lambda_1, \cdots, \lambda_q$ and a set of linearly independent solutions $y^{(1)}(x), \cdots, y^{(n)}(x)$ of (1) such that

$$T[y^{(p_\rho)}(x)] = \lambda_\rho y^{(p_\rho)}(x) \qquad (\rho = 1, \cdots, q) \tag{8a}$$

$$T[y^{(\sigma)}(x)] = \lambda_\rho y^{(\sigma)}(x) + y^{(\sigma-1)}(x) \qquad (\sigma = p_\rho + 1, p_\rho + 2, \cdots, p_{\rho+1} - 1; \rho = 1, \cdots, q). \tag{8b}$$

Let $\mathfrak{T}$ be the matrix of the transformation (8), and let $\mathfrak{A}(x)$ be the matrix

$$\begin{Vmatrix} y_1^{(1)}(x) & \cdots & y_n^{(1)}(x) \\ \cdots & \cdots & \cdots \\ y_1^{(n)}(x) & \cdots & y_n^{(n)}(x) \end{Vmatrix}$$

and $\widetilde{\mathfrak{A}}(x)$ the corresponding matrix of the $T[y_\mu^{(\nu)}(x)]$. Then for all x,

$$\widetilde{\mathfrak{A}}(x) = \mathfrak{T}\mathfrak{A}(x) . \tag{9}$$

Let c be a constant such that

$$w(x) = c\, e^{\int_0^x \left[\sum_{\mu=1}^n a_{\mu\mu}(\xi)\right] d\xi}$$

is the determinant of $\mathfrak{A}(x)$. Then

$$\log \left| \frac{w(x)}{c} \right| = \int_0^x \Re \left[\sum_{\mu=1}^n a_{\mu\mu}(\xi) \right] d\xi$$

is bounded and hence almost periodic with a module contained in M'. Hence

$$\lim_{j\to\infty} \lim_{i\to\infty} | w(x + h_i' + k_j') | = | w(x) |$$

is the absolute value of the determinant of $\widetilde{\mathfrak{A}}(x)$, and from (9) the determinant of $\mathfrak{T}$ has the absolute value unity. Thus $\mathfrak{T}$ is a one-to-one transformation and

$$| \lambda_1^{p_2 - p_1} \lambda_2^{p_3 - p_2} \cdots \lambda_q^{p_{q+1} - p_q} | = 1 . \tag{10}$$

Now from (7) and (8a) we have that for all ρ, $| \lambda_\rho | \leqq 1$; and from (10) it follows that for all ρ, $| \lambda_\rho | = 1$. Moreover for each ρ, $p_{\rho+1} = p_\rho + 1$. For if $p_{\rho+1} > p_\rho + 1$, we have by iteration of (8) that

$$\left.\begin{aligned} T^m[y^{(p_\rho)}(x)] &= \lambda_\rho^m\, y^{(p_\rho)}(x) \\ \text{and} \\ T^m[y^{(p_\rho+1)}(x)] &= \lambda_\rho^m\, y^{(p_\rho+1)}(x) + m\lambda_\rho^{m-1}\, y^{(p_\rho)}(x) \end{aligned}\right\} \quad (\rho = 1, \cdots, q) .$$

But this is impossible since for all m the set $R^+\{T^m[y^{(p_\rho+1)}(x)]\}$ is contained in the bounded set $R^+[y^{(p_\rho+1)}(x)]$. Hence $\mathfrak{T}$ is in the diagonal form and (6) holds.

It remains to show that for each solution $y(x)$, $R^+[y(x)] \subset R^+\{T[y(x)]\}$. To do this, choose a sequence, $m_1, m_2, \cdots$ of positive integers such that for all ν, $\lim_{j\to\infty} \lambda_\nu^{m_j} = 1$. Then for each ν, $\lim_{j\to\infty} T^{m_j}[y^{(\nu)}(x)] = y^{(\nu)}(x)$; and since T^{m_j} is linear, for each solution $y(x)$,

$$\lim_{j\to\infty} T^{m_j}[y(x)] = y(x) .$$

But from (7) it follows that for any value x_1 of x, the points $T^{m_1}[y(x_1)]$, $T^{m_2}[y(x_1)]$, $\cdots$ all belong to the closed set $R^+\{T[y(x)]\}$. Hence their limit point $y(x_1)$ also belongs to it, and

$$R^+[y(x)] \subset R^+\{T[y(x)]\} .$$

6. We can now prove the theorems mentioned in the introduction, as well as Theorems IV and IV$^+$ which will be stated in terms of the following conditions:

VI. *There are no solutions of* (1) (*neither real nor complex ones*) *which are twistable with respect to* M *except almost periodic ones.*

VI$^+$. *There are no solutions of* (1) (*neither real nor complex ones*) *which are positive twistable with respect to* M *except almost periodic ones.*

THEOREM IV$^+$. *Let* $a_{\mu\nu}(x)$ *be almost periodic* ($\mu, \nu = 1, \cdots, n$), *and let* M *be their module. Then all the solutions of* (1) *are almost periodic if and only if conditions* I$^+$, II$^+$, *and* VI$^+$ *are satisfied. Moreover if the conditions are satisfied, the module of all the solutions of* (1) *is the same as the module* M' *of the set consisting of all the* $a_{\mu\nu}(x)$ *together with all the solutions of* (1) *which are positive twistable with respect to* M.

PROOF. The necessity of II$^+$ and VI$^+$ is obvious while for I$^+$ it follows from the fact that the Wronskian of any linearly independent set of solutions is

$$c\, e^{\int_0^x \left[\sum_{\mu=1}^{n} a_{\mu\mu}(\xi)\right] d\xi} ,$$

and from a theorem of Bochner's.[10]

To prove sufficiency, assume that the conditions hold, and apply Lemma 6 to any solution $y(x)$ of (1), using the module M'. Property A is satisfied since condition II holds. If $h_1, h_2, \cdots$ converges with respect to M', it follows that uniformly in x $\lim_{i\to\infty} a_{\mu\nu}(x + h_i)$ exists for all x. Moreover if $\lim_{i\to\infty} y(h_i)$ exists,

$$\lim_{i\to\infty} y(x + h_i) , \tag{11}$$

[10] "Remark on the Integration of Almost Periodic Functions," *Journal of the London Mathematical Society*, Vol. 8 (1933), pp. 250–254. This paper will be referred to as Bochner II.

exists for all x and satisfies the transformed differential equation. Hence property C of the Lemma is satisfied; and a second application of the argument gives property D. If we choose a linearly independent set $\mathfrak{L}$ of solutions of (1), it follows from I^+ that their Wronskian is almost periodic and does not take on values arbitrarily near to 0. If as before $h_1, h_2, \cdots$ converges with respect to M, and we choose a sub-sequence of $h_1, h_2, \cdots$ such that each element of $\mathfrak{L}$ approaches a limit when transformed by it, the transformation obtained is one-to-one, since the new Wronskian does not vanish. Hence if two distinct elements of $\mathfrak{L}$ converge under the original sequence $h_1, h_2, \cdots$, their transforms are distinct. If we apply this fact to a system which has been transformed once already, we obtain property E.

To show that property B also holds let $h_1, h_2, \cdots$ and $k_1, k_2, \cdots$ be sequences of positive numbers such that $\lim_{j\to\infty} \lim_{i\to\infty} y(x + h_i + k_j)$ exists for all x and such that $h_1, h_2, \cdots$ converges to $-k_1, -k_2, \cdots$ with respect to M'. Let $h_1', h_2', \cdots$ and $k_1', k_2', \cdots$ be sub-sequences of $h_1, h_2, \cdots$ and $k_1, k_2, \cdots$ such that for every solution $z(x)$ of (1), $\lim_{j\to\infty} \lim_{i\to\infty} z(x + h_i + h_j)$ exists for all x. (That such subsequences can be found can be seen by applying property C twice to a linearly independent set of solutions.) From Lemma 8 it follows that there exist linearly independent solutions $y^{(1)}(x), \cdots, y^{(n)}(x)$ such that

$$\lim_{j\to\infty} \lim_{i\to\infty} y^{(\nu)}(x + h_i' + k_j') = c_\nu y^{(\nu)}(x) \qquad (\nu = 1, \cdots, n),$$

where $|c_1| = |c_2| = \cdots = |c_n| = 1$. Suppose that there exists a ν such that $c_\nu \neq 1$. It follows that $y^{(\nu)}(x)$ is positive twistable with respect to M' and hence with respect to M. Thus by hypothesis it follows that $y^{(\nu)}(x)$ is almost periodic with a module contained in M', and hence $c_\nu = 1$, contrary to the assumption. Therefore $c_1 = c_2 = \cdots = c_n = 1$, and the transformation is the identity, so that $\lim_{j\to\infty} \lim_{i\to\infty} y(x + h_i + k_j) = y(x)$.

Since $y(x)$ has all the required properties, it follows from Lemma 6 that it is almost periodic and has a module contained in M'. Thus M' contains the module M'' of all the solutions of (1). On the other hand, Bochner's argument[11] shows that $M'' \supset M$ and hence $M'' = M'$. Q.E.D.

It is clear that Theorem IV, which is obtained from Theorem IV^+ in the obvious manner (by replacing condition I^+ by I, II^+ by II, etc.), is a consequence of Theorem IV^+. These two theorems and Lemma 5 imply Theorem II (stated above) and the corresponding Theorem II^+.

THEOREM III^+. *Let $a_{\mu\nu}(x)$ be almost periodic ($\mu, \nu = 1, \cdots, n$). If in addition conditions I^+ and II^+ are satisfied, and if $y(x)$ is a solution of* (1) *such that no solution $z(x)$ can have the same positive range as $y(x)$ and be distinct from $y(x)$ unless it is almost periodic, then it follows that $y(x)$ is almost periodic. Moreover the module of $y(x)$ is contained in the module M' of the set consisting of all*

[11] Bochner I, page 288.

the $a_{\mu\nu}(x)$ together with all the solutions $z(x)$ of (1) *other than $y(x)$ which satisfy* $R^+[z(x)] = R^+[y(x)]$.

PROOF. Apply Lemma 6 to $y(x)$, using the module M'. It follows as in Theorem IV$^+$ that properties A, C, D, E hold. Let $h_1, h_2, \cdots$ and $k_1, k_2, \cdots$ be sequences of positive numbers such that $\lim_{j\to\infty} \lim_{i\to\infty} y(x + h_i + k_j)$ exists for all x and such that $h_1, h_2, \cdots$ converges to $-k_1, -k_2, \cdots$ with respect to M'. As before, let $h_1', h_2', \cdots$ and $k_1', k_2', \cdots$ be sub-sequences such that for every solution $z(x)$ of (1),

$$T[z(x)] = \lim_{j\to\infty} \lim_{i\to\infty} z(x + h_i' + k_j')$$

exists for all x. By Lemma 8, the transformation T is one-to-one, and

$$R^+\{T[y(x)]\} = R^+[y(x)] .$$

Thus by hypothesis, $T[y(x)]$ is almost periodic, and its module is contained in M', or else $T[y(x)] = y(x)$. Hence from the definition of T, we have in either case

$$T\{T[y(x)]\} = T[y(x)] ;$$

and since T is one-to-one,

$$T[y(x)] = y(x) .$$

It follows that B also holds, and $y(x)$ is almost periodic with a module contained in M'. Q.E.D.

Clearly Theorem III can be proved in the same way, and I and I$^+$ follow from III and III$^+$.

It was asserted in the introduction that Theorem III can be used to prove a particular solution almost periodic even when not all of the solutions are almost periodic. This can be proved by the following

Example. Let $f(x)$ be a real almost periodic function such that $e^{i\int_0^x f(\xi)\,d\xi}$ is not almost periodic. Consider the system

$$\frac{dy_1(x)}{dx} = i f(x)\, y_1(x)$$

$$\frac{dy_2(x)}{dx} = (\cos x)\, y_2(x) .$$

It has the 2-parameter family of solutions

$$\begin{cases} y_1(x) = c_1 e^{i\int_0^x f(\xi)\,d\xi} \\ y_2(x) = c_2 e^{\sin x} \end{cases} .$$

It obviously satisfies the conditions I and II. Moreover the solution corresponding to $c_1 = 0, c_2 = 1$ has a range consisting of the points (y_1, y_2) satisfying $y_1 = 0, e^{-1} \leqq y_2 \leqq e$. It is the only solution having this range, and thus according to Theorem III it is almost periodic. Thus Theorem III applies to the solution $c_1 = 0, c_2 = 1$ even though the solution $c_1 = 1, c_2 = 0$ is not almost periodic.

7. The above theorems can all be restated so as to apply to the non-homogeneous equation (2). In fact, some of them can be restated in two different ways.

In the first place, consider the system

$$\begin{aligned} \frac{dz_\mu(x)}{dx} &= a_{\mu 1}(x)z_1(x) + \cdots + a_{\mu n}(x)z_n(x) + b_\mu(x)z_{n+1}(x) \quad (\mu = 1, \cdots, n) \\ \frac{dz_{n+1}(x)}{dx} &= 0\,. \end{aligned} \tag{12}$$

If $y(x)$ is a solution of (2), then $z_1(x) = y_1(x), \cdots, z_n(x) = y_n(x), z_{n+1}(x) = 1$ is a solution of (12). Hence theorems concerning the non-homogeneous system (2) can be obtained from theorems concerning the homogeneous system (12). For example, Theorem III can be generalized thus:

THEOREM IIIg. *Let $a_{\mu\nu}(x)$ and $b_\mu(x)$ be almost periodic $(\mu, \nu = 1, \cdots, n)$. If condition* I *is satisfied, if all the solutions of* (2) *are bounded, and if $y(x)$ is a solution of* (2) *such that no solution $z(x)$ can have the same range as $y(x)$ and be distinct from $y(x)$ unless it is almost periodic, then it follows that $y(x)$ is almost periodic. Moreover the module of $y(x)$ is contained in the module of the set consisting of all the $a_{\mu\nu}(x)$ and $b_\mu(x)$ together with all the solutions $z(x)$ of* (1) *other than $y(x)$ which satisfy*

$$R[z(x)] = R[y(x)].$$

The corresponding generalization of Bochner's theorem is

THEOREM. *Let $a_{\mu\nu}(x)$ and $b_\mu(x)$ be almost periodic $(\mu, \nu = 1, \cdots, n)$. Then all of the solutions of* (2) *are almost periodic if condition* I *as well as the following conditions are satisfied:*

IIg. *All the solutions of* (2) *are bounded.*

IIIg. *There exists a set of $n + 1$ linearly independent solutions $y^{(1)}(x), \cdots, y^{(n+1)}(x)$ of* (2) *having the property that if $y(x)$ is any solution of* (2)*, then it is not true that $R[y(x)] \subset R[y^{(\nu)}(x)]$ unless $y(x) \equiv y^{(\nu)}(x)$.*

Moreover if these conditions are satisfied, the module of the set of solutions of (2) *is the same as the module of the set consisting of the $a_{\mu\nu}(x)$ and the $b_\mu(x)$.*

It is clear that this method of generalization applies to all our theorems on homogeneous equations. However, there is another method based upon the following theorem, which gives generalizations for all but Theorems III and III$^+$.

THEOREM V. *Let $a_{\mu\nu}(x)$ and $b_\mu(x)$ be almost periodic $(\mu, \nu = 1, \cdots, n)$. Then*

a necessary and sufficient condition that all the solutions of (2) *be almost periodic is that at least one of them be bounded for all positive* x *and all the solutions of the associated homogeneous equation* (1) *be almost periodic. Moreover if the condition is satisfied, the module of all the solutions of* (2) *is the same as the module* M *of the set consisting of all the solutions of* (1) *plus the* n *functions* $b_\mu(x)$.

PROOF. The necessity of the condition is obvious. The proof of sufficiency is similar to that of IV$^+$ or III$^+$. In the first place, conditions I and II are satisfied, since the solutions of (1) are almost periodic. Moreover by Bochner's[12] argument, the modules of the $a_{\mu\nu}(x)$ are contained in M. As before, apply Lemma 6 to any solution $y(x)$ of (2). Properties A, C, D, E can be seen to hold by an argument similar to that in Theorem IV$^+$.

Let $h_1, h_2, \cdots$ and $k_1, k_2, \cdots$ be sequences of positive numbers such that

$$T[y(x)] = \lim_{j\to\infty} \lim_{i\to\infty} y(x + h_i + k_j)$$

exists for all x and such that $h_1, h_2, \cdots$ converges to $k_1, k_2, \cdots$ with respect to M. Then $T[y(x)]$ is a solution of (2), and $z(x) = T[y(x)] - y(x)$ is a solution of (1). But the transformation T obviously applies to any solution of (1) and leaves it invariant. Hence for any positive integer ρ, $z(x) = T^\rho[y(x)] - T^{\rho-1}[y(x)]$. Taking $\rho = 1, \cdots, m$ and adding, we obtain

$$T^m[y(x)] = mz(x) + y(x).$$

Hence for all positive m, $R^+[mz(x) + y(x)] \subset R^+[y(x)]$. It follows that $z(x) \equiv 0$, and $y(x)$ is positive stationary. Thus $y(x)$ is almost periodic with a module contained in M. But the module of the solutions of (2) contains the module of the solutions of (1); hence the module of the $a_{\mu\nu}(x)$ and finally the module of the $b_\mu(x)$.

If we combine Theorem V with Bochner's theorem we obtain the

THEOREM. *Let* $a_{\mu\nu}(x)$ *and* $b_\mu(x)$ *be almost periodic* ($\mu, \nu = 1, \cdots, x$). *Then all the solutions of* (2) *are almost periodic if conditions* I, IIg, III *are satisfied. Moreover if these conditions are satisfied the module of the set of solutions of* (2) *is the same as the module of the set consisting of the* $a_{\mu\nu}(x)$ *and the* $b_\mu(x)$.

As in the case of Bochner's theorem, the two generalizations of I are two distinct theorems. However, since the solution of (12) can only be symmetric or twistable if its last component is zero, the two methods of generalization lead to the same theorem in the case of Theorems II and IV. Thus we have

THEOREM Ig. *Let* $a_{\mu\nu}(x)$ *and* $b_\mu(x)$ *be almost periodic* ($\mu, \nu = 1, \cdots, n$) *and let at least one of the* $b_\mu(x)$ *be different from zero for at least one value of* x. *Then all of the solutions of* (2) *are almost periodic if and only if conditions* I *and* IIg *as well as the following condition are satisfied*:

IVg. *There exists a set of* $n + 1$ *linearly independent solutions* $y^{(1)}(x), \cdots,$

[12] Bochner I, page 288.

$y^{(n+1)}(x)$ of (2) *having the property that if $y(x)$ is any solution of* (2) *then it is not true that $R[y(x)] = R[y^{(\nu)}(x)]$ unless $y(x) \equiv y^{(\nu)}(x)$ or $y(x)$ is almost periodic.*

THEOREMS Ig′, IIg *and* IVg. *Let $a_{\mu\nu}(x)$ and $b_\mu(x)$ be almost periodic $(\mu, \nu = 1, \cdots, n)$ with the module M. Then each of the following three sets of conditions is necessary and sufficient that all the solutions of* (2) *be almost periodic*: I, IIg *and* IV; I, IIg *and* V; *or* I, IIg *and* VI.

Each of the above theorems can be completed by making a statement concerning modules. It is easy to see what this statement should be in each case.

PRINCETON, N. J.

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SUR LA STRUCTURE D'ENSEMBLE DES MOUVEMENTS STABLES AU SENS DE POISSON

PAR HEINRICH HILMY

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Considérons un système dynamique, dont les mouvements sont donnés par l'équation différentielles:

$$\frac{dx_i}{dt} = X_i(x_1, \cdots, x_n) \qquad i = 1, 2, \cdots, n, \tag{1}$$

où $x_1, \cdots, x_n$—désignent les paramètres qui déterminent la position du système dynamique et où t désigne le temps.

Nous considérons l'ensembles des valeurs $x_1, \cdots, x_n$ comme un point p avec les coordonnées $(x_1, \cdots, x_n)$ dans l'espace Euclidien E^n à n-dimensions.

Les mouvements du système dynamique seront caractérisés par les trajectoires décrites dans E^n par les points $p \subset E^n$; nous désignerons ces trajectoires par $f(p, t)$ en supposant p fixe.

Soit A quelque ensemble dans E^n. Le symbol $f(A, t)$ désigne l'ensemble des points $\{f(p, t)\}$ avec $p \subset A$.

Nous admettons que les fonctions X_i dans les seconds membres des équations différentielles sont continues et qu'elles remplissent la condition de Lipschitz dans un certain domaine fermé D. Dans ce cas a lieu le théorème de l'existence et de l'unicité des solutions ainsi que le théorème de la dépendance continue des trajectoires des conditions initiales.

Ce dernier théorème peut-être énoncé de la manière suivante.

Désignons par $\rho(x, y)$ la distance dans E^n entre les points x et y et par $S(x, \epsilon)$ le voisinage sphérique du point x du rayon ϵ.

Soit p un point arbitraire de l'espace E^n. Alors pour tout $\epsilon > 0$ et pour tout nombre positif T aussi grand que l'on veut, on peut indique un nombre $\delta > 0$, assez petit, que

$$f(S(p, \delta), t) \subset S(f(p, t), \epsilon)$$

pour toute valeur de t appartenant à l'intervalle $0 \leqq t \leqq T$.

Nous dirons que le sous-ensemble M de l'espace E^n est invariant si $f(p, t) \subset M$ à la condition que $p \subset M$.

Une trajectoire $f(p, t)$ (p étant fixe) est dite stable au sens de Poisson si pour tout nombre positif ϵ et pour tout T positif aussi grand que l'on veut, il existe des valeurs de t tant positives que négatives et surpassant T, en valeur absolue, pour lesquelles $\rho(p, f(p, t)) < \epsilon$. Il est évident que le point initial de la trajectoire $f(p, t)$ peut-être pris arbitrairement.

Désignons par $\mathfrak{P}$ l'ensemble de tous les points, situés sur les trajectoires des mouvements stables au sens de Poisson, contenues dans un ensemble invariant et fermé donné M ou dans l'espace E^n. Nous appellerons quelquefois l'ensemble $\mathfrak{P}$ ensemble maximal de mouvements stables au sens de Poisson.

Soit $f(p, t)$ un mouvement que nous supposerons instable au sens de Poisson. De la continuité des fonctions dans les seconds membres des équations différentielles (1) il s'ensuit que pour chaque point de la trajectoire du mouvement $f(p, t)$ il existe un voisinage sphérique si petit qu'après en être sorti la première fois le point quittera ce voisinage pour toujours, le temps variant dans le sens négatif ou positif.

Nous dirons que le nombre non négatif ϵ' égal a la borne supérieure des rayons de tous les voisinages sphériques du point dans lesquels il ne revient pas après en être sorti la première fois, le temps variant dans un sens positif, nous dirons que ce nombre est *l'indice de la stabilité du point p pour le temps croissant.*

Par analogie nous dirons que le nombre non négatif ϵ'' égal à la limite supérieure des rayons des voisinages sphériques du point dans lesquels il ne revient pas, après en être sorti, avec le temps variant dans un sens négatif est *l'indice de la stabilité pour le temps décroissant.*

Nous nommerons le nombre ϵ égal au plus grand entre les nombres ϵ' et ϵ'' *indice général de la stabilité* ou simplement *indice de la stabilité du point p.*

Dans le cas général les différents points d'une même trajectoire ont différents indices de stabilité. L'indice général de la stabilité du mouvement stable d'après Poisson est nul.

LEMME. *Soit $\mathfrak{P}$ un ensemble maximal de mouvements stables au sens de Poisson. Assignons-nous un nombre positif arbitraire et fixe ϵ et désignons par Q_ϵ l'ensemble de points q situés sur les trajectoires instables au sens de Poisson, et dont les indices de stabilité surpassent ou égalent ϵ. Alors autour de chaque point $p \subset \mathfrak{P}$ différent du point de repos on peut construire un voisinage suffisament petit, ne contenant de points de l'ensamble Q_ϵ.*

Soit p un point arbitraire de l'ensemble $\mathfrak{P}$ différent du point d'équilibre. Décrivons autour du point p deux sphères concentriques de rayons η_1 et η_2 en choisissant $\eta_1 < \epsilon/2$ et suffisamment petit pour que le point p quitte le η_1-voisinage de sa position initiale avec le temps variant dans les deux sens, et η_2 suffisamment petit pour que tout point, appartenant au η_2-voisinage du point p entre dans son η_1-voisinage simultanement avec le point p pendant son premier retour dans le η_1-voisinage de sa position initiale, le temps variant dans le sens positif ou négatif. Alors avec le retour du point p dans le n_1-voisinage de sa position initiale la distance entre la position initiale et la position finale de tout point appartenant au η_2-voisinage du point p ne pourra surpasser ϵ. Nous en concluons directement qu'aucun point du η_2-voisinage du point p ne peut appartenir à l'ensemble $Q\epsilon$; le lemme est demontré.

THÉORÈME. *L'ensemble maximal $\mathfrak{P}$ des mouvements stables au sens de Poisson appartient à la classe G_δ d'apres la classification de Hausdorff, c'est à dire qu'il représente la partie commune à une infinite dénombrable d'ensembles ouverts.*

Désignons par $\mathfrak{P}'$ l'ensemble de tous les points d'équilibre contenus dans $\mathfrak{P}$ et par $\mathfrak{P}''$ l'ensemble de tous les autres points: $\mathfrak{P} = \mathfrak{P}' + \mathfrak{P}''$.

L'ensemble $\mathfrak{P}'$ est un ensemble fermé, donc un ensemble appartenant à la classe G_δ.

La structure d'ensemble $\mathfrak{P}''$ peut-être élucidée de la manière suivante.

Soit ϵ un nombre positif arbitraire fixe. Construisons autour de chaque point $p \subset \mathfrak{P}''$ un voisinage ne contenant aucun point de l'ensemble Q_ϵ; en vertu du théorème précédent ceci est toujours possible. Désignons par G_ϵ la somme de tous ces voisinages. Il est clair que G_ϵ comme somme d'ensembles ouverts, sera également un ensemble ouvert et que $G_\epsilon \supset \mathfrak{P}''$

Soit ensuite

$$\epsilon_1, \epsilon_2, \cdots, \epsilon_n, \cdots$$

une suite infinie de nombres positifs tendant vers zéro.

Nous pouvons construire une suite d'ensembles du type considéré

$$G_{\epsilon_1}, G_{\epsilon_2}, \cdots, G_{\epsilon_n}, \cdots$$

correspondant aux nombres de la suite.

Examinons la partie commune de tous ces ensembles

$$G = \prod_{i=1}^{\infty} G_{\epsilon_i}$$

et démontrons que $G = \mathfrak{P}''$.

Il est clair que tout point $p \subset \mathfrak{P}''$ appartiendra à G, car il appartient à tous G_{ϵ_i}, et par conséquent à leur partie commune.

D'un autre coté aucun point situé sur la trajectoire d'un mouvement instable ne peut appartenir à G, car parmi les ensembles G_{ϵ_i} il n'appartient pas à ceux pour lesquels ϵ_i est inférieur à l'indice de la stabilité du point q.

Il en découle immédiatement que

$$\mathfrak{P}'' = G = \prod_{i=1}^{\infty} G_{\epsilon_i}$$

c'est à dire que l'ensemble $\mathfrak{P}''$ appartient à la classe G_δ d'après la classification de Hausdorff.

Mais alors l'ensemble $\mathfrak{P} = \mathfrak{P}' + \mathfrak{P}''$ comme la somme de deux ensembles du type G_δ sera également un ensemble appartenant à la classe G_δ.

Le théorème est démontré.

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EINIGE SÄTZE ÜBER TOPOLOGISCHE GRUPPEN

VON HANS FREUDENTHAL

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Im Folgenden werde ich einige einfache Sätze über topologische Gruppen beweisen, die zwar an sich nicht viel bedeuten, die aber dauernd bei allen möglichen Untersuchungen über topologische Gruppen gebraucht werden.[1] Wenn man die elementaren Sätze der reinen Gruppentheorie auf topologische Gruppen übertragen will, stösst man nämlich immer wieder auf gewisse Schwierigkeiten. Der innere Grund dieser Schwierigkeiten ist die Tatsache, dass sich schon in recht einfachen Fällen ein stetiger Homomorphismus nicht mehr durch Faktorgruppenbildung nach einem abgeschlossenen Normalteiler erzeugen zu lassen braucht. Bildet man z.B. von der Gruppe G der Translationen der (x, y)-Ebene die Faktorgruppe G' nach dem Normalteiler aller "ganzzahligen" Translationen, so geht die abgeschlossene Untergruppe H ($y = ax$, a irrational) von G in eine nicht abgeschlossene, überall dichte Untergruppe H' von G' über. Die Abbildung von H auf H' ist ein Isomorphismus, der den topologischen Charakter von H ganz ändert; sie ist nur in *einer* Richtung stetig. Wenn solche Komplikationen eintreten, kann man die Sätze der elementaren Gruppentheorie nicht ohne Weiteres übertragen. Wir wollen nun einen möglichst weiten Bereich für die Gültigkeit dieser Sätze abgrenzen.

1. Alle betrachteten topologischen Gebilde sollen separable topologische Räume sein.[2] Unter einer Gruppe verstehen wir immer eine topologische Gruppe, d.h. eine Gruppe, deren Elemente die Punkte eines topologischen Raumes (kleine lateinische Buchstaben) und deren Operationen (Bildung des Produktes und der Inversen) stetig im Sinne der Topologie des Raumes sind. Wir erinnern an die drei äquivalenten Definitionen der Stetigkeit: konvergente Folgen gehen in konvergente Folgen über; die Urbilder abgeschlossener Mengen sind abgeschlossen; die Urbilder offener Mengen sind offen.

"Homomorphes bzw. isomorphes (= eineindeutiges) Bild" bedeutet immer "*stetig* homomorphes bzw. isomorphes Bild." Eine in beiden Richtungen stetige Isomorphie heisst topologische Isomorphie (Zeichen $\cong$).

Der Durchschnitt zweier Teilmengen M und N einer Gruppe wird mit $M \vee N$ bezeichnet; MN bezeichnet die Menge aller Punkte mn mit $m \subset M$, $n \subset N$.

Eine Untergruppe ist als Teilmenge zu topologisieren, ein direktes Produkt

[1] Siehe nachfolgende Arbeit des Verf.: *Topologische Gruppen mit genügend vielen fastperiodischen Funktionen.* Annals of Math., dieses Heft.

[2] d.h. reguläre Hausdorffsche Räume mit zweitem Abzählbarkeitsaxiom (F. Hausdorff, *Grundzüge der Mengenlehre*, Leipzig 1914). Die Voraussetzung der Regularität ist jedoch überflüssig (wenn man es mit Gruppen zu tun hat), wie D. van Dantzig gezeigt hat. Siehe: *Zur topologischen Algebra* I, Math. Ann. 107 (1932), 587–626, Satz TG 14.

als topologisches Produkt der Faktoren, d.h. ist U_1 eine Umgebung in G_1, U_2 eine in G_2, so ist $U_1 \times U_2$ eine in $G_1 \times G_2$; die Faktorgruppe[3] G/H von G nach dem abgeschlossenen Normalteiler H ist zu topologisieren als Raum der Nebengruppen, d.h. ist U eine Umgebung in G, so ist die Gesamtheit der in UH enthaltenen Nebengruppen aH eine Umgebung in G/H (dass die topologischen Axiome erfüllt sind, ist klar).

2. Alle folgenden Untersuchungen gelten, ohne dass wir es bei den einzelnen Sätzen ausdrücklich erwähnen, auch für Gruppenkeime.[4] Ein Gruppenkeim ist ein Raum, in dem die Gruppenoperationen zwischen den Punkten in einer *offenen* Teilmenge erklärt sind; soweit die Gruppenoperationen erklärt sind, sollen sie den gruppentheoretischen und topologischen Voraussetzungen genügen; die offene Teilmenge soll die Identität enthalten.

3. Ein Homomorphismus heisst *gebietstreu*, wenn das Bild jeder offenen Menge eine offene Menge ist.

4. *Ein Homomorphismus ist dann und nur dann gebietstreu, wenn das Bild jeder offenen Menge einen inneren Punkt enthält.*

Beweis: Die Notwendigkeit ist evident.—Sei U eine Umgebung in G, U' ihr Bild in G'. U' enthält nach Voraussetzung eine offene Menge W', der Durchschnitt des Urbildes von W' mit U heisse W. Dann ist W' das Bild von W. Die Umgebungen von e[5] mögen $U_1, U_2, \cdots$ heissen. In U_n gibt es, wie wir sahen, eine offene Menge W_n, deren Bild offen ist. Ist O irgendeine offene Menge von G, so lässt sich zu jedem Punkt von O ein ihn enthaltendes $a_n W_n$ angeben, das in O enthalten ist. O lässt sich also als Vereinigung offener Mengen mit offenen Bildern darstellen; das Bild O' von O ist als Vereinigung offener Mengen selbst offen.

5. *Das homomorphe Bild einer im Kleinen kompakten*[6] *Gruppe ist dann und nur dann gebietstreu, wenn es von zweiter Kategorie*[7] *ist.*[8]

[3] Zuerst zu finden bei D. van Dantzig, *Studiën over topologische Algebra*, Diss. Amsterdam 1931, S. 6. Siehe auch D. van Dantzig, *Zur topologischen Algebra* I, Math. Ann. 107 (1932), 587–626, insbesondere S. 609–610. Man beachte jedoch, dass der Satz TG 20 (ähnlich wie T33) dieser Arbeit zur Hälfte unrichtig ist, wie gerade das Beispiel unserer Einleitung zeigt.

[4] Der Begriff des Gruppenkeimes wird in der Lieschen Theorie immerzu verwendet; zum ersten Mal hat ihn O. Schreier (Jahresberichte DMV 37 (1928), 113–132) vollständig formuliert.

[5] Die Identität wird—in welcher Gruppe auch—immer mit e bezeichnet.

[6] Eine echte oder unechte Teilmenge eines Raumes heisst kompakt, wenn jede ihrer unendlichen Teilmengen im Raume einen Häufungspunkt besitzt. Wichtig ist: Der Durchschnitt einer absteigenden Folge nichtleerer abgeschlossener kompakter Mengen ist nichtleer. Im Kleinen kompakt heisst: die Umgebungen sind kompakt.

[7] Ein Raum heisst von zweiter Kategorie, wenn er sich nicht als Summe abzählbar vieler nirgendsdichter Teilmengen darstellen lässt. Siehe F. Hausdorff, *Mengenlehre*, Berlin 1927, S. 142. Für die Anwendungen wichtig ist, dass jeder im Kleinen kompakte Raum von zweiter Kategorie ist.

[8] Im Wesentlichen *dieser* Satz (in der allgemeineren Form von Satz 26) findet sich auch

BEWEIS: U sei eine kompakte Umgebung in G; dann ist G die Vereinigung abzählbar vieler aU. G' ist die Vereinigung der abzählbar vielen Bilder $a'U'$ der aU. Wenn G' von zweiter Kategorie ist, muss eines der $a'U'$ irgendwo dicht sein, also muss auch U', das Bild von U, irgendwo dicht sein. Die abgeschlossene Hülle von U', die das Bild der abgeschlossenen Hülle $\overline{U}$ von U ist,[9] enthält also einen inneren Punkt, d.h. das Bild von $\overline{U}$ enthält einen inneren Punkt. Ist O nun irgendeine offene Menge, so gibt es in O eine Umgebung U, deren abgeschlossene Hülle $\overline{U}$ auch noch in O ist;[10] das Bild von O enthält das von $\overline{U}$, also einen inneren Punkt. Nach 4 ist die Abbildung also gebietstreu.

Die Umkehrung folgt aus dem nächsten Satz.

6. *Das gebietstreue homomorphe Bild einer im Kleinen kompakten Gruppe ist im Kleinen kompakt.* (Beweis klar, denn das Bild einer kompakten offenen Menge wird wieder eine kompakte offene Menge.)

7. *Ist das homomorphe Bild einer Gruppe im Kleinen kompakt, so ist es gebietstreu.*

BEWEIS: O sei eine offene Menge in G. Dann ist G die Vereinigung abzählbar vieler aO. Das Bild von O in G' heisse O'. Dann ist G' die Vereinigung abzählbar vieler $a'O'$ (Bilder der aO). W' sei eine kompakte abgeschlossene Umgebung in G'. W' ist also in der Vereinigung endlich vieler $a'O'$ enthalten; eines von ihnen enthält einen inneren Punkt,[11] also enthält O' einen inneren Punkt, also ist nach 4 die Abbildung gebietstreu.

8. *Notwendig und hinreichend für Gebietstreue eines Homomorphismus ist, dass man zu jeder konvergenten Folge eine konvergente Folge von Urbildern finden kann.*

BEWEIS: Notwendig: G ist gebietstreu auf G' abgebildet. a'_n sei eine konvergente Folge in G', a' ihr Limes. a sei ein Urbild von a'; U_m seien die Umgebungen von a, ihre Bilder sind offen, sie heissen bzw. U'_m. Fast alle a'_n sind in U'_m, d.h. alle von a'_{n_m} an. Von $a'_{n_m}, \cdots, a'_{n_{m+1}-1}$ suchen wir Urbilder in U_m: bzw. $a_{n_m}, \cdots, a_{n_{m+1}-1}$. Die Folge dieser a_n konvergiert gegen a.

Hinreichend: O sei eine offene Menge in G, O' ihr Bild in G', a' ein Punkt von O', a'_n eine gegen a' konvergente Folge und a^*_n eine (nach Voraussetzung existierende) Folge von Urbildern in G, die gegen a^* konvergiere. a sei ein Urbild von a' in O; dann gibt es auch eine gegen a konvergente Folge a_n von Urbildern der a'_n (nämlich etwa $a_n = a^*_n a^{*-1} a$). O enthält fast alle a_n, also O'

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in meiner Diss.: *Über die Enden topologischer Räume und Gruppen*, Math. Zeitschr. 33 (1931), 692–713; siehe Satz 19 auf S. 710. Allerdings finden sich dort einige Versehen.

[9] Bei der Abbildung eines Raumes auf einen andern geht eine kompakte Teilmenge in eine kompakte und ihre abgeschlossene Hülle in die abgeschlossene Hülle über.

[10] Die Existenz ergibt sich aus der Regularität des Raumes (siehe[2]).

[11] Siehe F. Hausdorff, *Mengenlehre*, Berlin 1927, S. 141, Satz VI.

fast alle a'_n. Das gilt für jede gegen a' konvergente Folge, also ist a' innerer Punkt von O'. Das gilt für jeden Punkt a' von O', also ist O' offen.

9. *Ein Isomorphismus ist dann und nur dann topologisch, wenn er gebietstreu ist.* (Beweis klar nach 8.)

10. *Ein Homomorphismus lässt sich dann und nur dann durch Faktorgruppenbildung nach einem abgeschlossenen Normalteiler erzeugen, wenn er gebietstreu ist. Genauer: Ist G homomorph abgebildet auf G' und ist N der (abgeschlossene) Normalteiler aller in (e) übergehenden Elemente, so ist dann und nur dann*

$$G' \cong G/N,$$

wenn der Homomorphismus gebietstreu ist.

BEWEIS:[12] "Nur dann": Sei O offen in G, O' sein Bild in G', a' ein Punkt aus O', a eines seiner Urbilder in O, U eine in O enthaltene Umgebung von a. UN ist nach Definition eine Umgebung in G/N und enthalten in ON, das sich als O' deuten lässt. O' enthält also zu jedem seiner Punkte a' eine Umgebung.

"Dann": G/N ist isomorph auf G' abgebildet, wenn jedem aN das Bild a' in G' (von a in G) zugeordnet ist. Jede offene Menge in G/N hat zum Urbild eine offene Menge in G (siehe ersten Teil des Beweises), und deren Bild in G' ist nach Voraussetzung offen. Also ist tatsächlich der Isomorphismus von G/N auf G' gebietstreu, also nach 9 topologisch.

11. *Ist $N \subset H \subset G$, H abgeschlossene Untergruppe und N abgeschlossener Normalteiler von G, so ist H/N abgeschlossene Untergruppe von G/N.*

BEWEIS: H/N ist als Menge die Gesamtheit der in $HN=H$ enthaltenen Nebengruppen und wegen der Abgeschlossenheit von H abgeschlossen.

12. *Ist G im Kleinen kompakt und gebietstreu auf G' abgebildet, ist ferner H abgeschlossene Untergruppe von G, so sind die folgenden zwei Aussagen äquivalent:*

a.) *H ist gebietstreu auf H' abgebildet.*

b.) *H' ist abgeschlossen.*

BEWEIS: a $\rightarrow$ b: H ist als abgeschlossene Teilmenge von G im Kleinen kompakt, also ist nach 6 auch H' im Kleinen in sich kompakt. a'_n sei eine in G' konvergente Folge aus H', $a' \subset G'$ ihr Limes. U' sei eine so kleine Umgebung von e in G', dass $H' \vee U'$ kompakt ist (in H'); dann sind auch die $H' \vee a'_n U'$ kompakt. $a'U'$ enthält fast alle a'_n, also ist a' in fast allen $a'_n U'$, z.B. in $a'_m U'$. Somit enthält $a'_m U'$, also auch $H' \vee a'_m U'$ fast alle a'_n. Da aber $H' \vee a'_m U'$ kompakt (in H') ist, muss es zu der Folge a'_n in H' einen Häufungspunkt geben: a' liegt in H', H' ist abgeschlossen.

b $\rightarrow$ a: Nach 6 ist G' im Kleinen kompakt, H' als abgeschlossene Teilmenge

[12] Den rein gruppentheoretischen Teil des Beweises lassen wir bei all solchen Sätzen weg.

von G' ist ebenfalls im Kleinen kompakt, also ist es nach 7 gebietstreues Bild von H.[13]

13. In der Gruppentheorie spielen zwei Isomorphiesätze[14] eine wichtige Rolle, die wir hier als A und B topologisch formulieren werden. In 14 und 15 werden wir den Gültigkeitsbereich dieser Sätze abgrenzen.

$$\text{A: } \boxed{HN/N \cong H/H \vee N}$$

(H abgeschlossene Untergruppe, N abgeschlossener Normalteiler.)

$$\text{B: } \boxed{G/H \cong G'/H'.}$$

(G' homomorphes Bild von G, H' abgeschlossener Normalteiler von G' und H sein Urbild.)

14. A *ist für im Kleinen kompakte G dann und nur dann gültig, wenn HN abgeschlossen in G ist.*

Beweis: "Dann": HN ist abgeschlossen und enthält N, also ist nach 11 HN/N abgeschlossene Untergruppe von G/N. Bei der homomorphen Abbildung von G auf G/N geht H in HN/N über, und wegen der Abgeschlossenheit von HN/N ist nach 12 diese Abbildung gebietstreu, also nach 10 durch den Normalteiler der in e abgebildeten Elemente von H—das ist $H \vee N$—erzeugbar, d.h. $HN/H = H/H \vee N$.

"Nur dann": Durch den Homomorphismus wird H auf HN/N abgebildet; wegen der Gültigkeit von A ist diese Abbildung gebietstreu, also ist nach 12 HN/N abgeschlossen. HN ist aber das Urbild von HN/N, also auch abgeschlossen.

15. B *ist dann und nur dann für alle H' gültig, wenn G gebietstreu auf G' abgebildet ist.*

Beweis: "Dann": Durch die Zuordnung "$aH \to a'H'$" entsteht jedenfalls ein Isomorphismus von G/H auf G'/H'; der ist gebietstreu, denn einer offenen Menge in G/H entspricht eine offene Menge in G, dieser nach Voraussetzung eine offene Menge in G' und dieser endlich nach 10 eine offene Menge in G'/H'. Nach 9 folgt daraus unser Satz.

"Nur dann": Man setze nur $H' = (e)$.

16. Eine andere Form von B ist

$$\text{C: } \boxed{G/H \cong (G/N)/(H/N).}$$

(N ist der den Homomorphismus $G \to G'$ erzeugende Normalteiler.)

[13] Die Voraussetzung der Abgeschlossenheit von H ist bei diesem Teil des Beweises also entbehrlich.

[14] Wir folgen hier B. L. van der Waerden, *Moderne Algebra I*, Berlin 1930, S. 136.

17. Die nach 10 notwendige und hinreichende Bedingung "HN abgeschlossen" für die Gültigkeit von A ist praktisch gut brauchbar wegen der beiden folgenden Sätze:

a.) *Sind M_1 und M_2 abgeschlossene Teilmengen von G und ist eine von ihnen kompakt, so ist M_1M_2 abgeschlossen.*

b.) *Sind M_1 bzw. M_2 abgeschlossene Teilmengen von G_1 bzw. G_2 und ist $G = G_1 \times G_2$, so ist M_1M_2 abgeschlossen.*

BEWEIS von a: Sei a_n eine in G konvergente Teilfolge von M_1M_2; $a_n = a_n^{(1)} \cdot a_n^{(2)}$. Ist etwa M_1 kompakt, so enthält die Folge $a_n^{(1)}$ eine konvergente Teilfolge $a_{n'}$ mit dem Limes $a^{(1)}$; dann konvergiert aber auch die Folge $a_{n'}^{(2)} = a_{n'}^{(1)-1} \cdot a_{n'}^{(1)} a_{n'}^{(2)}$, ihr Limes sei $a^{(2)}$. Wegen der Abgeschlossenheit von M_1 und M_2 sind $a^{(1)}$ und $a^{(2)}$ in ihnen bzw. enthalten, also ist $a^{(1)}a^{(2)}$ in M_1M_2. Das ist aber der Limes der gegebenen Folge.

BEWEIS von b: Nach der Definition des direkten Produktes ist eine Folge $a_n^{(1)}a_n^{(2)}$ ($a_n^{(1)} \subset G_1$, $a_n^{(2)} \subset G_2$) dann und nur dann konvergent, wenn $a_n^{(1)}$ und $a_n^{(2)}$ einzeln konvergieren. Daraus folgt der Satz.

18. Ist N abgeschlossener Normalteiler von G_2, so ist

$$\text{D: } \boxed{(G_1 \times G_2)/N \cong G_1 \times (G_2/N).}$$

Beweis klar.

19. *Die Komponente der Identität*[15] *eines Normalteilers ist selbst Normalteiler.* (Beweis wie der des—spezielleren—Satzes von O. Schreier[16].)

20. *Das homomorphe Bild einer im Kleinen kompakten nulldimensionalen*[17] *Gruppe ist nulldimensional.*[18]

BEWEIS: G besitzt beliebig kleine kompakte abgeschlossene offene Untergruppen H[19], G ist die Vereinigung abzählbar vieler Nebengruppen von H, G' die Vereinigung ihrer Bilder; diese sind kompakt und haben alle dieselbe Dimension wie H', das Bild von H. Nach dem Summensatz der Dimensionstheorie ist dann dim G' = dim H'. Das in sich kompakte H' ist von zweiter Kategorie (siehe Fussnote[7]), die Abbildung von H auf H' ist also nach 5 gebietstreu. H besitzt wieder beliebig kleine offene Untergruppen H_1, ihre Bilder H_1' sind auch

[15] Komponente = maximale zusammenhängende Teilmenge. Komponente der Identität = *die* Komponente, die e enthält.

[16] Abh. Math. Sem. Hamburg 4 (1926), 15–32, Theorem 1.

[17] Wir verwenden hier—jedoch in recht unwesentlicher Weise—einige Begriffe und Sätze der Dimensionstheorie.

[18] Wahrscheinlich gibt es überhaupt keine dimensionserhöhenden Homomorphismen.

[19] Nach D. van Dantzig, Diss. (siehe[2]), 17–18, besitzt eine im Kleinen kompakte, nulldimensionale Gruppe beliebig kleine offene Untergruppen, während—wovon wir auch gleich Gebrauch machen werden—jede zusammenhängende, im Kleinen kompakte Gruppe, die beliebig kleine offene Untergruppen besitzt, allein aus der Identität besteht.

offen. Ist C' die Komponente der Identität von H', so ist also $H_1' \vee C'$ eine offene Untergruppe von C', und zwar ist $H_1' \vee C'$ beliebig klein (wenn nur H_1 genügend klein ist). Da C' zusammenhängend ist, ist also $C' = (e)$, also dim $H' = 0$, also dim $G' = 0$.

21. *Ist G im Kleinen kompakt und gebietstreu homomorph auf G' abgebildet, ist ferner G' zusammenhängend und im Kleinen zusammonhängend und G_0 die Komponente der Identität von G, so ist bereits das Bild G_0' von G_0 gleich G'.*

Beweis: U sei eine abgeschlossene kompakte Umgebung von e in G, U' ihr Bild in G' (wegen der Gebietstreue auch eine abgeschlossene Umgebung); U bzw. U' lassen sich als Keime auffassen; U' erzeugt das zusammenhängende G', und $U \vee G_0 = U_0$ erzeugt G_0. U' darf als zusammenhängend vorausgesetzt werden (wegen des Zusammenhangs im Kleinen von G'). Das Bild U_0' von U_0 ist abgeschlossen im U. Bei dem durch den gegebenen induzierten Homomorphismus von U/U_0 auf U'/U_0' muss nach 20 das Bild des kompakten nulldimensionalen U/U_0 nulldimensional sein; da es zusammenhängend ist, ist es (e). Also $U' = U_0'$. Da U' bzw. $U_0'G'$ bzw. G_0' erzeugen, ist also tatsächlich $G' = G_0'$.

22. *Sind der Normalteiler H und die Faktorgruppe G/H von G in sich kompakt, so ist es auch G.*

Beweis: a_n sei eine Folge in G. Wegen der Kompaktheit von G/H lässt sich aus der Folge a_nH eine konvergente Teilfolge aussuchen; für geeignete $h_{n'}$ aus H muss also $a_{n'}h_{n'}$ konvergieren. Wegen der Kompaktheit von H lässt sich aus den $h_{n'}$ eine konvergente Teilfolge $h_{n''}$ aussuchen; da $a_{n''}h_{n''}$ konvergierte, konvergiert dann auch $a_{n''}$.

23. Wir sprechen von einer Darstellung der Gruppe G im "Wirkungsraum" W,[20] wenn jedem a von G eine eineindeutige Abbildung $f_a(x)$ von W in sich zugeordnet ist mit den Eigenschaften:

$$f_{ab}(x) = f_a(f_b(x));$$

aus $\lim a_n = a, \quad \lim x_n = x$

folgt $\lim f_{a_n}(x_n) = f_a(x).$

Die Darstellung heisst *transitiv*, wenn zu je zwei Punkten x und y ein a existiert mit $f_a(x) = y$.

Die Darstellung heisst *treu*, wenn aus $a \neq b$ folgt $f_a \neq f_b$.

Unter $f_A(X)$ verstehen wir die Menge aller $f_a(x)$ mit $a \subset A$ und $x \subset X$.

Die Darstellung heisst *asystatisch*,[21] wenn jeder Punkt x charakterisiert ist

[20] Wir folgen hier der Definition a.a.O.[8]

[21] Der Begriff stammt von Lie. Siehe *Theorie der Transformationsgruppen*, Leipzig 1888, Bd. I, Kap. 24.

durch seine "festgelegte" Untergruppe H_x—das ist die grösste Untergruppe von G mit der Eigenschaft $f_{H_x}(x) = x$.

Die Darstellung heisst *gebietstreu*, wenn, falls O offen ist, auch $f_o(x)$ offen ist (für jedes feste x).

Zwei Darstellungen f and f' derselben Gruppe in den Wirkungsräumen W und W' heissen topologisch ähnlich,[22] wenn zwischen W und W' eine topologische Beziehung $(x \leftrightarrow x')$ existiert, bei der für jedes a $f_a(x)$ und $f'_a(x')$ einander entsprechen.

24. *Ist bei einer transitiven Darstellung $f_a(x) = y$, so ist $H_y = aH_xa^{-1}$. Dann und nur dann existiert eine transitive (stetige) treue bzw. asystatische Darstellung von G mit (abgeschlossenem) H als einer festgelegten Untergruppe, wenn H ausser (e) keinen (abgeschlossenen) Normalteiler von G enthält, bzw. wenn H nicht echter Normalteiler einer (abgeschlossenen) Untergruppe von H ist.* (Beweis für "nur dann" klar; für "dann" Folgerung von 25.)

25. *Ist H eine abgeschlossene Untergruppe von G, und nennt man W den Raum der Nebengruppen uH (topologisiert wie der Faktorgruppenraum in* 1), *so erhält man in $f_a(uH) = auH$ eine transitive stetige gebietstreue Darstellung von G in W mit H als einer festgelegten Untergruppe. Alle stetigen transitiven gebietstreuen Darstellungen sind derartigen Darstellungen topologisch ähnlich.* (Beweis wie der von 10.)

26. *Eine stetige transitive Darstellung eines im Kleinen kompakten G ist dann und nur dann gebietstreu, wenn W von zweiter Kategorie ist.* (Beweis wie der von 5.)

Ebenso lassen sich die übrigen Sätze auf Darstellungen übertragen.

Anhang[23]

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Verschiedene der vorstehend behandelten Sätze lassen sich noch ausdehnen und übersichtlicher darstellen, wenn man sich mit *vollständigen*[24] statt mit im Kleinen kompakten Gruppen beschäftigt.

27. Die Folge a_n von Elementen der Gruppe G heisse Fundamentalfolge, wenn

$$\lim_{m,n\to\infty} a_m^{-1} a_n = e$$

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[22] Natürlich kann man auch homomorphe Abbildungen zwischen Darstellungen einführen und entsprechende Sätze beweisen.

[23] Die Anregung zu diesem Anhang verdanke ich Herrn D. van Dantzig.

[24] Die hier folgende metrikfreie Definition der Vollständigkeit einer Gruppe rührt von D. van Dantzig her (a.a.O.[3]), 6–7, bzw. 612).

ist. Die Gruppe G heisst vollständig, wenn in ihr jede Fundamentalfolge konvergiert.

Die Vollständigkeit von G fällt, wie man ohne Weiteres bemerkt, mit der folgenden Eigenschaft zusammen: Ist U_n eine Folge von Umgebungen der Identität mit der Identität als gemeinsamem Durchschnitt, und haben die Mengen $c_m U_m$ zu je endlich vielen einen nichtleeren Durchschnitt, so haben sie alle zusammen einen nichtleeren Durchschnitt.

28. *Jede vollständige Gruppe ist von zweiter Kategorie.*

Beweis: $U_1, U_2, \cdots$ sei eine Folge von Umgebungen von e mit (e) als gemeinsamem Durchschnitt. Sei im Widerspruch zur Behauptung G die Vereinigung abzählbar vieler nirgends dichter Mengen M_n. Es gibt dann in G ein zu M_1 fremdes $c_1 U_k$. In der offenen Menge $c_1 U_k$ gibt es wieder ein zu M_2 fremdes $c_2 U_{k_2}$. Fährt man so fort, so erhält man eine Folge ineinandergeschachtelter $c_n U_{k_n}$, deren Durchschnitt einerseits (siehe 27) nicht leer, andererseits zu allen M_n fremd ist; da G die Vereinigung der M_n ist, ergibt sich also ein Widerspruch.

29. *Jede im Kleinen kompakte Gruppe ist vollständig.*

Beweis: Sei $a_1, a_2, \cdots$ eine Fundamentalfolge und U eine kompakte Umgebung von e. Dann ist für alle $m, n \geqq k$: $a_m^{-1} a_n \subset U$. Also sind fast alle a_n in dem ebenfalls kompakten $a_k U$. Die Fundamentalfolge a_n hat demnach einen Häufungspunkt, ist also konvergent.

Bemerkung: Auf Grund dieses Satzes folgen verschiedene der vorstehenden Sätze aus Sätzen des Anhangs.

30. *Eine homomorphe Abbildung der vollständigen Gruppe G auf die Gruppe G' ist dann und nur dann gebietstreu, wenn G' ebenfalls vollständig ist.*

Beweis: „Dann": Nach 28 ist G' von zweiter Kategorie. U sei eine Umgebung in G, U' ihr Bild in G'. G lässt sich als Vereinigung abzählbar vieler $a_n U$ schreiben, also G' als Vereinigung der Bilder $a_n' U'$. Eines der $a_n' U'$, also U' selbst muss irgendwo dicht sein; in der abgeschlossenen Hülle von U' gibt es also eine Umgebung V', der Durchschnitt ihres Urbildes mit U heisse V. Dann ist das Bild von V überall dicht in V'. Nennt man eine Menge in G', die in einer offenen Menge (enthalten und) überall dicht ist, eine Grundmenge, so hat man: In jeder Umgebung U aus G gibt es eine offene Menge V, deren Bild eine Grundmenge ist. Da aber jede offene Menge Vereinigung abzählbar vieler Umgebungen ist, gilt sogar: Das Bild jeder offenen Menge aus G ist eine Grundmenge.

$U_0, U_1, \cdots$ sei eine Folge von Umgebungen in G mit dem Durchschnitt (e) und mit der Eigenschaft

$$\overline{U}_{n+1}^2 \subset U_n.^{25}$$

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[25] Wegen der Existenz siehe a.a.O.[3], Math. Ann., 611.

Dann ist

$$U_{n+1}U_{n+2}\cdots U_{n+k-1}U_{n+k} \subset U_{n+1}U_{n+2}\cdots U_{n+k-1}^2 \subset U_{n+1}U_{n+2}\cdots U_{n+k-2}^2 \subset \cdots \subset U_{n+1}^2 \subset U_n,$$

also

$$(*) \qquad U_{n+1}U_{n+2}\cdots U_{n+k} \subset U_n.$$

Die Bilder der U_n sind Grundmengen, die zugehörigen offenen Mengen mögen U_n' heissen; wir werden zeigen, dass das Bild von U_n sogar gleich U_n' ist.

a' sei ein Punkt aus U_1'. Im Durchschnitt von U_1' und $a'U_2'^{-1}$, der ja offen ist, liegen die Bilder der Punkte von U_1 überall dicht; eines möge b_1' heissen, das Bild von b_1. Dann ist also

$$(\S_1) \qquad b_1 \subset U_1,$$

$$(\S\S_1) \qquad a_1' = b_1'^{-1}a' \subset U_2'.$$

$$(**_1) \qquad b_1' = a'a_1'^{-1}$$

Sei im Durchschnitt von U_n' und $a_{n-1}'U_{n+1}'$ bereits ein Element b_n' als Bild eines Elements b_n von U_n bestimmt, so dass also

$$(\S_n) \qquad b_n \subset U_n,$$

$$(\S\S_n) \qquad a_n' = b_n'^{-1}a_{n-1}' \subset U_{n+1}'$$

gilt; sei ferner

$$(**_n) \qquad b_1'b_2'\cdots b_n' = a'a_n'^{-1},$$

Dann bestimmen wir im Durchschnitt von U_{n+1}' und $a_n'U_{n+2}'$, der wegen $(\S\S_n)$ nichtleer ist und in dem die Bilder der Punkte von U_{n+1} überall dicht liegen, ein Element b_{n+1}, Bild eines Elementes $b_{n+1} \subset U_{n+1}$. Dann ist also

$$b_{n+1} \subset U_{n+1}, \qquad (\S_{n+1})$$

$$a_{n+1}' = b_{n+1}'^{-1}a_n' \subset U_{n+2}'. \qquad (\S\S_{n+1})$$

Ferner ist nach $(**_n)$

$$b_1'b_2'\cdots b_{n+1}' = a'a_n'^{-1}\cdot a_n'a_{n+1}'^{-1} = a'\cdot a_{n+1}'^{-1}. \qquad (**_{n+1})$$

Wir können daher die Formeln $(\S_n)$, $(\S\S_n)$, $(**_n)$ als für alle n gültig annehmen.

Nun ist wegen $(\S_n)$ und (*)

$$b_{n+1}b_{n+2}\cdots b_{n+k} \subset U_{n+1}U_{n+2}\cdots U_{n+k} \subset U_n.$$

Die Folge der Produkte $b_1b_2\cdots b_n$ ist demnach eine Fundamentalfolge, ihr Limes heisse a. Die zugehörige Bildfolge ist wegen $(**_n)$ identisch mit der Folge $a'a_n'^{-1}$. Wegen $(\S_n)$ ist $\lim a_n' = e$, also auch $\lim a_n'^{-1} = e$, also $\lim a'a_n'^{-1} = a'$. Demnach ist a' Bildpunkt von $a = \lim b_1b_2\cdots b_n = b_1 \lim b_2b_3\cdots b_n$. Da die $b_2\cdots b_n$ in U_1 liegen, ist $\lim b_2\cdots b_n$ in $\overline{U}_1$, also a in

U_0. Zu jedem Punkt a' aus U_1' gibt es also ein Urbild in U_0. Das Bild jeder offenen Menge U_0 enthält also eine volle Umgebung U_1', und nach 4 ist der Homomorphismus gebietstreu.

„Nur dann": Sei $a_1', a_2', \cdots$ eine Fundamentalfolge in G', lim $a_k'^{-1}a_l' = e$; es genügt, in ihr eine konvergente Teilfolge zu finden. Wir wählen k_n so, dass für $k, l \geqq k_n$

$$a_k'^{-1}a_l' \subset U_n'$$

ist; die U_n' seien die (nach Voraussetzung offenen) Bilder von Umgebungen U_n; die U_n seien wieder so gewählt, dass der Bedingung (*) Genüge geschieht. Dann ist, wenn wir $b_n' = a_{k_n}'$ setzen,

$$b_n'^{-1}b_{n+1}' \subset U_n'.$$

Jedes $b_n'^{-1}b_{n+1}'$ hat ein Urbild c_n in U_n, c_0 sei ein Urbild von b_1'. Wie oben sieht man, dass die Folge der Produkte $c_0c_1c_2 \cdots c_n$ eine Fundamentalfolge ist, also konvergiert; demnach muss auch die Bildfolge, b_{n+1}', eine Teilfolge von a_n', konvergieren, wzbw.

31. Wir können nun 12 aussprechen in der Form:

Ist G vollständig und G' gebietstreues homomorphes Bild von G, ist ferner H abgeschlossene Untergruppe von G, so ist die Abbildung von H auf sein Bild H' dann und nur dann ebenfalls gebietstreu, wenn H' abgeschlossen ist. (Beweis klar, da eine Untergruppe einer vollständigen Gruppe dann und nur dann vollständig ist, wenn sie abgeschlossen ist.)

32. Wir können nun 14 aussprechen in der Form:

A *ist für vollständige G dann und nur dann gültig, wenn HN abgeschlossen in G ist.*

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TOPOLOGISCHE GRUPPEN MIT GENÜGEND VIELEN FASTPERIODISCHEN FUNKTIONEN

VON HANS FREUDENTHAL

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S. Bochner[1] und J. v. Neumann[2] haben den Bohrschen[3] Begriff der Fastperiodizität auf Funktionen in beliebigen Gruppen übertragen (zu den Bohrschen fastperiodischen Funktionen gehört in diesem Zusammenhang als Gruppe die der linearen Translationen). Eine auf einer Gruppe G definierte Funktion $f(y)$ wird fastperiodisch genannt, wenn der Raum der zu f "rechts-kongruenten" Funktionen $g_a(y) = f(ya)$ und ebenso der Raum der zu f "links-kongruenten" Funktionen $h_a(y) = f(ay)$ F in seiner vollständigen Hülle kompakt ist.

Von einem wirklichen System fastperiodischer Funktionen auf einer Gruppe G kann man allerdings erst dann sprechen, wenn zu irgend zwei Elementen a und b der Gruppe ($a \neq b$) eine fastperiodische Funktion existiert, die an diesen beiden Stellen verschiedene Werte annimmt: $f(a) \neq f(b)$. Ist G eine topologische Gruppe, so wird man von f ausser der Fastperiodizität noch Stetigkeit im Sinne der Topologie von G verlangen. Existiert in der topologischen Gruppe G also zu a, b eine stetige fastperiodische Funktion f mit $f(a) \neq f(b)$, so wollen wir sagen, dass G genügend viel fastperiodische Funktionen besitzt.[3a]

Dass auf *kompakten* Gruppen mit zweitem Abzählbarkeitsaxiom genügend viel fastperiodische Funktionen existieren, ist klar: auf ihnen ist jede stetige Funktion fastperiodisch.[4] Für die im *Kleinen* kompakten *abelschen* Gruppen wird die Existenz genügend vieler fastperiodischer Funktionen von v. Neumann bewiesen.[4] Wir werden nun zeigen, dass sich (was die zusammenhängenden, im Kleinen kompakten Gruppen betrifft) in diesen beiden Typen die Gruppen mit genügend vielen fastperiodischen Funktionen im Wesentlichen erschöpfen. Genauer:

Jede zusammenhängende, im Kleinen kompakte Gruppe mit zweitem Abzählbarkeitsaxiom ist direktes Produkt der (abelschen) Gruppe der Translationen des n-dimensionalen euklidischen Raumes ($n = 0, 1, \cdots$) und einer (zusammenhängenden) kompakten Gruppe (mit zweitem Abzählbarkeitsaxiom).[5]

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[1] *Beiträge zur Theorie der fastperiodischen Funktionen.* Math. Ann. *96* (1927), 119–147.

[2] *Almost periodic functions in a group.* I. Trans. Amer. Math. Soc. *36* (1934), 445–492.

[3] *Zur Theorie der fast periodischen Funktionen.* Acta Math. *45* (1925), 29–127; *46* (1925), 101–214.

[3a] Das ist also der v. Neumannsche Begriff der Maximal-Fastperiodizität einer Gruppe, a.a.O.[2] , S. 481.

[4] a.a.O. [2], Theorem 36.

[5] Man kommt also über die klassischen fastperiodischen Funktionen nur unwesentlich hinaus.—Zusatz bei der Korrektur: Wie mir inzwischen Herr André Weil mündlich mitteilte, ist es auch ihm gelungen, diesen Satz zu beweisen.

Über den ersten Faktor ist weiter nichts zu sagen. Aber auch der zweite Faktor ist einfach zu übersehen. Wir zeigen nämlich:

Jede zusammenhängende kompakte Gruppe mit zweitem Abzählbarkeitsaxiom lässt sich G_n-adisch[6] *als Limesgruppe kompakter Liescher Gruppen darstellen.*[7]

Dies Ergebnis ist völlig befriedigend, denn man kann alle kompakten Lieschen Gruppen aufzählen: ihre universellen Überlagerungsgruppen sind direkte Produkte kompakter halbeinfacher Liescher Gruppen (die man sämtlich kennt[8]) und einparametriger abelscher Gruppen, und sie selber ergeben sich aus ihren Überlagerungsgruppen durch Faktorbildung nach diskreten Normalteilern, die auch alle bekannt sind.[9]

Es lassen sich schliesslich Kriterien dafür aufstellen, wann eine Gruppe G mit genügend viel fastperiodischen Funktionen endlich-dimensional und wann sie im Kleinen zusammenhängend ist. Wir erwähnen:

Dann und nur dann ist eine im Kleinen kompakte, zusammenhängende Gruppe mit zweitem Abzählbarkeitsaxiom, die genügend viel fastperiodische Funktionen besitzt, im Kleinen zusammenhängend und endlich-dimensional, wenn sie Liesch ist.[7] *Dann und nur dann ist sie im Kleinen zusammenziehbar, wenn sie Liesch ist.*

Lässt man nun die Bedingung der Kompaktheit im Kleinen fallen, so kann man noch beweisen:

Jede zusammenhängende Gruppe mit zweitem Abzählbarkeitsaxiom, die genügend viel stetige fastperiodische Funktionen besitzt, lässt sich stetig isomorph einbetten in eine kompakte Gruppe mit zweitem Abzählbarkeitsaxiom.

Auch über nicht zusammenhängende Gruppen liessen sich einige einschneidende Sätze aussprechen; wir gehen auf sie nicht ein, da sie die Frage doch nicht erschöpfend beantworten.

In I bzw. II haben wir einige einfache Hilfssätze über topologische bzw. Liesche Gruppen bewiesen; wo es uns interessant schien, haben wir mehr bewiesen als für die Anwendung in III unmittelbar nötig war.

I

1. Wir übernehmen alle Bezeichnungen und Definitionen vorstehender Arbeit *Einige Sätze über topologische Gruppen* (zitiert als T); siehe insbesondere T1 – 3.

Wir verabreden ferner, dass ein Gruppenkeim stets als so klein angenommen werden soll, wie es im betreffenden Fall nötig ist.

Ein Gruppenkeim heisst $\times$-*Gruppenkeim*, wenn in ihm jeder abgeschlossene Normalteilerkeim direkter Faktor ist. Genauer: Wenn G ein $\times$-Gruppenkeim

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[6] Das wird in 21 definiert. Historische Daten über diesen Begriff finden sich in [30].

[7] Dieser Satz steht bereits bei L. Pontrjagin, *Sur les groupes topologiques compacts et le cinquième problème de Hilbert*, C.R. *198* (1934), 170–190. Verf. hat ihn unabhängig gefunden und Mitte 1933 Herrn J. v. Neumann brieflich mitgeteilt im Anschluß an dessen Arbeit in Annals of Math. (2) *34* (1933), 170–190.

[8] E. Cartan, *Les tenseurs irréductibles et les groupes linéaires simples et semi-simples.* Bulletin Sc. Math. (2) *49* (1925), 130–152 (150 ff.)

[9] E. Cartan, *La géométrie des groupes simples.* Annali di Mat. (4) *4* (1927), 209–256 (227–230).

ist, soll es zu jedem abgeschlossenen Normalteilerkeim H einen abgeschlossenen Normalteilerkeim L geben, derart dass sich ein geeigneter offener Teilkeim von G als direktes Produkt $H \times L$ darstellen lässt.—Eine Gruppe heisst $\times$-*Gruppe*, wenn eine gewisse Umgebung der Identität $\times$-Gruppenkeim ist.

K bezeichne stets eine kompakte Gruppe, V eine Vektorengruppe, d.h. eine Gruppe, die topologisch isomorph der Additionsgruppe der Vektoren eines n-dimensionalen Vektorraums ist.

2. *Eine zusammenhängende Untergruppe eines V ist abgeschlossen, direkter Faktor und wieder ein V. Ein abgeschlossener Untergruppenkeim eines V erzeugt eine abgeschlossene Untergruppe. Die einzige kompakte Untergruppe eines V ist (e) (die Identität). Ist A abgeschlossene Untergruppe eines V, so ist V/A dann und nur dann ein V, wenn A zusammenhängend ist, und dann und nur dann ein K, wenn keine zusammenhängende, A enthaltende, echte Untergruppe von V existiert.* (Beweise klar.)

3. *Ein abgeschlossener Normalteilerkeim eines kompakten $\times$-Gruppenkeimes ist ein $\times$-Gruppenkeim.*

Beweis: H sei ein abgeschlossener Normalteilerkeim von G, dann ist also $G = H \times H_1$. Ist L wieder ein Normalteilerkeim von H, so ist L auch Normalteilerkeim von G. $L \times H_1$ ist erklärt, Normalteilerkeim von G, abgeschlossen wegen T 17a, also direkter Faktor: $G = L \times H_1 \times L_1$. Dann ist nach T 18: $H \cong G/H_1 \cong L \times L_1$. Also ist L in H direkter Faktor.

4. *Ist $G = K \times V$,[10] so ist (α) K die (eindeutig bestimmte) maximal kompakte Untergruppe von G, (β) V eine (nicht notwendig eindeutig bestimmte) maximale abgeschlossene V-Untergruppe von G, und so ist (γ) für jede abgeschlossene V-Untergruppe V_1 von G der Durchschnitt mit K gleich (e).*

Beweis: (α) Sei K^* eine kompakte Untergruppe von G. Dann ist KK^* kompakt, wie man analog zu T 17a einsieht. KK^*/K ist eine kompakte Untergruppe von G/K, das nach T 18 eine Vektorengruppe ist. Nach 2 ist also $KK^*/K = (e)$, also $K^* = K$.

(β) Sei $V_1 \supset V$. Dann ist nach 2 V_1/V eine Vektorengruppe; andererseits ist V_1/V nach T 11 abgeschlossene Untergruppe von G/V, das (wegen T 18 $\cong K$, also) kompakt ist, also selber in sich kompakt. Nach 2 ist dann $V_1/V = (e)$, $V_1 = V$.

(γ) $K \vee V_1$ ist als Untergruppe von K (abgeschlossen und) kompakt; andererseits ist es Untergruppe von V_1. Also ist $K \vee V_1 = (e)$.

5. *Sei $G = K \times V$, B ein abgeschlossener zusammenhängender Untergruppenkeim von G mit $K \vee B = (e)$; sei H die von B erzeugte Untergruppe von G. Dann ist $K \vee H = (e)$, H ist eine V-Gruppe und abgeschlossen in G.*

[10] Man beachte im Folgenden, dass die $K \times V$ im Kleinen kompakt, also sicherlich auch von zweiter Kategorie sind.

BEWEIS: Bei der homomorphen Abbildung von G auf G/K geht das kompakte abgeschlossene B in einen abgeschlossenen zusammenhängenden Untergruppenkeim B' von G/K über. Wegen $G/K \cong V$ (siehe T 18) ist die von B' erzeugte Untergruppe H' von G/K abgeschlossen und eine V-Gruppe (siehe 2). H' ist ferner bei dem betrachteten Homomorphismus das Bild von H.

$h' \subset H'$ sei das Bild von $h \subset H$. Von dieser Abbildung lässt sich eine im Kleinen eindeutige stetige Umkehrung erklären: Wegen $K \vee B = (e)$ ist e das einzige Element von B, das in die Identität von B' übergeht; verschiedene Elemente von B gehen also in verschiedene Elemente von B' über. Ist h' irgendein Element von H', so enthält $h'B'$ eine Umgebung von h'. In dieser Umgebung erklären wir die Umkehrung der Abbildung so, dass wir einem $h'b'$(mit $b' \subset B'$)hb zuordnen; dabei ist h ein festes Urbild von h' und b das (eindeutig bestimmte) Urbild von b' innerhalb B.

Die so definierte Umkehrung von $h \to h'$ ist im Kleinen eindeutig und stetig; nach dem Monodromieprinzip[11] ist sie es wegen des einfachen Zusammenhanges von H' auch im Grossen. Die Abbildung $H \to H'$ (die durch die Faktorbildung nach K erklärt war) ist also topologisch, H ist ebenso wie H' eine V-Gruppe und $K \vee H = (e)$.

HK ist als Urbild von H' abgeschlossen in G. Ist $h_1, h_2, \cdots$ eine Folge aus H, die in G konvergiert, so lässt sich ihr Limes schreiben: hk $(h \subset H, k \subset K)$. Die Bildfolge von $h_1h^{-1}, h_2h^{-1}, \cdots$ konvergiert also gegen e, und da die Beziehung zwischen H und H' topologisch ist, konvergiert auch $h_1h^{-1}, h_2h^{-1}, \cdots$ gegen e, also die gegebene Folge tatsächlich gegen ein Element von H. Demnach ist H abgeschlossen.

6. *Sei G eine $\times$-Gruppe der Form $K \times V$ und H ein abgeschlossener, zusammenhängender, im Kleinen zusammenhängender*[12] *Normalteiler von G. Dann lässt sich G derart als $K \times V_1 \times V_2$ schreiben, dass $H = (K \vee H) \times V_1$ ist.*

BEWEIS: $H \vee K$ ist abgeschlossener Normalteiler von G, also nach 3 eine $\times$-Gruppe. Daher gilt die Darstellung:

$$(*) \qquad \begin{aligned} &\text{Keim von } H = H_0 \times H_1 \\ &(H_0 = \text{Keim von } K \vee H). \end{aligned}$$

Dabei darf man den Keim von H, also auch den von H_1 als zusammenhängend voraussetzen. Wegen $K \vee H_1 = (e)$ erzeugt H_1 nach 5 eine abgeschlossene Vektorengruppe V_1, und es ist $K \vee V_1 = (e)$. Da H_1 direkter Faktorkeim in H, also Normalteilerkeim in G war, ist auch das von H_1 erzeugte V_1 Normalteiler in G. Also ist $K \times V_1$ definiert.

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[11] Das Monodromieprinzip besagt: Ist in einer einfach zusammenhängenden Mannigfaltigkeit eine Abbildung im Kleinen eindeutig (und stetig) definiert, so ist sie es auch im Grossen.

[12] Zusammenhang im Kleinen heisst: Alle Komponenten aller offenen Mengen sind offen.—Nach F. Hausdorff, Mengenlehre (Berlin 1927), 156, Satz X, stimmt diese Definition mit der üblichen überein; für unsere Zwecke ist sie bequemer als die übliche.

Weiter ist wegen (*) H_1, also auch V_1, also auch $(K \vee H) \times V_1$ in H enthalten. Da aber wegen (*) $(K \vee H) \times V_1$ den Keim $H_0 \times H_1$ des zusammenhängenden H enthält, ist andererseits H in $(K \vee H) \times V_1$ enthalten. Daraus folgt ein Teil der Behauptung: $(K \vee H) \times V_1 = H$.

Bei dem Homomorphismus von G auf G/K gehe V in V' und V_1 in V_1' über. Da KV_1 nach T 17a abgeschlossen ist, ist nach T11 V_1' abgeschlossener Normalteiler von V' und obendrein wegen des Zusammenhangs von V_1 zusammenhängend. V_1' ist also tatsächlich Vektorengruppe, und V' ist darstellbar als $V_1' \times V_2'$. Wegen $G = K \times V$ lässt sich das Urbild von V_2' schreiben als $K \times V_2$ mit $V_2 \subset V$. Das Bild von V_2 ist V_2'. $(K \times V_1) \vee V_2$ ist erstens in K enthalten (denn sein Bild ist $V_1' \vee V_2' = (e)$), also kompakt, zweitens (wegen T 17a) abgeschlossen, drittens Untergruppe von V_2. Also ist es $= (e)$: $(K \times V_1) \vee V_2 = (e)$. Ferner ist V_2 Normalteiler. $K \times V_1 \times V_2$ ist demnach definiert, das ist aber $= G$.

7. *Sei G eine $\times$-Gruppe der Form $K \times V$, und sei jeder abgeschlossene zusammenhängende Normalteiler von G im Kleinen zusammenhängend. Sei G homomorph auf $G' = K' \times V'$ abgebildet. Dann lässt sich G derart als $K \times V_1 \times V_2$ schreiben, dass V_1 auf V' abgebildet wird. Ist der Komponentenraum des den Homomorphismus definierenden Normalteilers*[13] *H von G kompakt, so lässt sich obendrein erreichen, dass $V_2 = (e)$ ist und K auf K' abgebildet wird.*

Beweis: W sei das Urbild von V', W_0 die Komponente der Identität von W. Nach 6 hat man

$$(*) \qquad \begin{aligned} G &= K \times V_1 \times V_2, \\ W_0 &= (K \vee W_0) \times V_1, \end{aligned}$$

V_1 und V_2 sind abgeschlossene Vektorengruppen. Nach T 21 erschöpft das Bild von W_0 bereits V'; da der Faktor $K \vee W_0$ in eine kompakte Untergruppe von V', also in (e) übergeht, ist sogar

$$(**) \qquad \text{Bild von } V_1 = V'.$$

H_0 sei die Komponente der Identität von H; $H \subset W$, $H_0 \subset W_0$. H/H_0 ist nach Voraussetzung (in sich) kompakt. Also ist auch

$$(\S) \qquad (KH/K)/(KH_0/K)$$

kompakt, denn es lässt sich als Bild von H/H_0 (bei der Faktorgruppenbildung nach K) auffassen. (§) ist aber Untergruppe von

$$(\S\S) \qquad (G/K)/(KH_0/K)$$

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[13] Nach T 5 und T 10 wird der Homomorphismus tatsächlich durch einen Normalteiler definiert.

und zwar nach T 11 abgeschlossene Untergruppe, weil nach T 17a KH abgeschlossen in G, also nach T 11 auch KH/K abgeschlossen in G/K ist. Nach T 18 ist $G/K \cong V$ und, da KH_0/K als Bild des zusammenhängenden H_0 zusammenhängend ist, auch (§§) eine Vektorengruppe. Das kompakte (§) ist als Untergruppe von (§§) gleich (e), also

$$(^{*}_{**}) \qquad \begin{gathered} KH = KH_0 \\ H \subset KH_0 \subset KW_0 \subset K \times V_1 \end{gathered}$$

(die letzte Inklusion wegen (*)).

Nach T 17b ist V_2H abgeschlossen in G; ferner ist $V_2 \vee H = (e)$ wegen $(^{*}_{**})$; also ist nach T 14 V_2 topologisch isomorph seinem Bild V_2' (bei dem gegebenen Homomorphismus), ausserdem ist V_2' nach T 11 abgeschlossen. Also ist nach 4γ: $V_2' \vee K' = (e)$.

Aus T 15, T 18 und $(^{*}_{*})$ folgt $G/V_1H \cong G'/V' \cong K'$, also kompakt. Nach T 17b ist V_2V_1H abgeschlossen in G, also V_2V_1H/V_1H abgeschlossen in dem kompakten G/V_1H, also in sich kompakt; andererseits ist V_2V_1H/V_1H Vektorengruppe (da $V_1H \subset K \times V_1$ und daher $V_1H \vee V_2 = (e)$ ist). Nach 2 ist dann $V_2 = (e)$.

Das Bild K^* von K ist kompakt, also nach 4α in K'. Andererseits ist G' (als Bild von $G = K \times V_1$) gleich $K^* \times V'$, also nach 4α auch $K' \subset K^*$. Also $K' = K^* =$ Bild von K.

II

8. *Eine abgeschlossene zusammenhängende Untergruppe einer Lieschen Gruppe*[14] *ist eine Liesche Untergruppe, und entsprechend für Gruppenkeime.* Beides lässt sich im Wesentlichen genauso beweisen wie bei J. v. Neumann[15] die Analytizität der linearen Gruppen (Untergruppen der allgemeinen linearen Gruppe); siehe auch E. Cartan,[16] S. 24.

Die Komponenten einer abgeschlossenen Lieschen Untergruppe häufen sich

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[14] Eine Liesche Gruppe ist eine Gruppe, deren Gruppenraum eine analytische Mannigfaltigkeit ist, und deren Operationen analytisch sind (wir nehmen also in den Begriff der Lieschen Gruppe den Zusammenhang des Gruppenraumes auf).

Soweit es sich mit der Kürze der Darstellung irgendwie vereinbaren liess, haben wir die Verwendung von Sätzen der Lieschen Theorie vermieden. Wir erwähnen nur noch für das Folgende die Tatsache: Ist H eine Liesche Untergruppe der Lieschen Gruppe G, so lässt sich in einer Umgebung von e, ein analytisches Koordinatensystem einführen, in dem sich H als eine lineare Mannigfaltigkeit durch e und die Nebengruppen zu H als die zu ihr parallelen linearen Mannigfaltigkeiten darstellen. Daraus kann man leicht ableiten, dass die Faktorgruppe G/H wieder Liesch ist.—Entsprechendes gilt auch für Gruppenkeime.

Man beachte im Folgenden noch, dass die Lieschen Gruppen im Kleinen kompakt, von zweiter Kategorie, im Kleinen zusammenhängend, endlichdimensional und im Kleinen zusammenziehbar sind.

[15] Sitzungsberichte Akad. Berlin *1927*, 76–90. Math. Zeitschr. *30* (1929), 3–42.

nirgends, der Komponentenraum besitzt keinen Häufungspunkt; siehe E. Cartan[16]), S. 24. *Insbesondere enthält also jede abgeschlossene Untergruppe, die Häufungspunkte besitzt, eine Liesche Untergruppe* $\neq e$.

9. *Eine abelsche Liesche Gruppe ist direktes Produkt von Kreisdrehungsgruppen und eindimensionalen Vektorengruppen, also auch direktes Produkt eines Lieschen* K *und eines* V. (Beweis klar.)

10. *Ist* G *direktes Produkt eines halbeinfachen*[17] *Lieschen Keimes* H *und eines abelschen Lieschen Keimes* A, *so ist* G *eine* $\times$*-Gruppe.*

Beweis: Der Satz ist richtig für $H = (e)$ (siehe 2) und für $A = (e)$ (siehe z.B. E. Cartan, Thèse,[18] S. 53).

Sei N Normalteilerkeim von $A \times H$; jedes seiner Elemente lässt sich in der Form $a \cdot h$ schreiben ($a \subset A$, $h \subset H$). Die bei dieser Darstellung auftretenden Elemente von H bilden einen in H abgeschlossenen Gruppenkeim H_1, einen Normalteilerkeim von H, der nach 3 wieder $\times$-Gruppenkeim ist. Der Kommutatorkeim H_1' von H_1 ist also direkter Faktor: $H_1 = H_1' \times H_1''$. Der Kommutatorkeim von H_1'' ist in dem von H_1, also in H_1' enthalten, also in $H_1' \vee H_1'' = (e)$. H_1'' ist also abelscher Normalteilerkeim von H_1, also auch von H, also $H_1'' = (e)$ und $H_1 = H_1'$. Nun ist aber der Kommutatorkeim von N gleich dem von H_1, also gleich H_1, also $H_1 \subset N$ und $N = A_1 \times H_1$ mit $A_1 \subset A$. Ferner ist $A = A_1 \times A_2$ und $H = H_1 \times H_2$, also tatsächlich $A \times H = N \times (A_1 \times H_2)$.

11. *Eine kompakte Liesche halbeinfache Gruppe ist endlichfach zusammenhängend, ihre universelle Überlagerungsgruppe*[19] *also kompakt.*

Beweis: Die Faktorgruppe nach dem (diskontinuierlichen) Zentrum, die sogen. adjungierte Gruppe, lässt sich nämlich nach der Lieschen Theorie als lineare Gruppe auffassen; wegen der Kompaktheit darf man nach F. Peter und

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[16] *La théorie des groupes finis et continus et l'analysis situs*, Mémorial sc. math. XLII.— Cartan setzt allerdings voraus, dass die Untergruppe "continu" (d.h. eine Mannigfaltigkeitsgruppe) ist, macht davon jedoch nirgends Gebrauch. Deutet man "continu" im Sinne von "Kontinuum," so bleibt die Beweisführung wörtlich richtig; man erhält dann den oben von uns angeführten Satz.

[17] Halbeinfach heisst eine Gruppe (ein Gruppenkeim), die (der) keinen zusammenhängenden abelschen Normalteiler ausser (e) besitzt. Man sieht ohne Weiteres, dass der Keim einer halbeinfachen Gruppe halbeinfach ist, und umgekehrt. Wir erwähnen noch aus der Theorie der halbeinfachen Gruppen, dass die halbeinfachen Gruppen im Kleinen direkte Produkte (nichtabelscher) einfacher Liescher Gruppen sind, und umgekehrt.

[18] *Sur la structure des groupes de transformations finis et continus* (Paris 1894).

[19] Wir verwenden von jetzt an häufig Begriffe und Sätze von O. Schreier (*Abstrakte kontinuierliche Gruppen*, Abh. Math. Sem. Hamburg *4* (1926), 15–32). Wir erwähnen insbesondere: Ist G^* eine Überlagerungsgruppe von G, so ist $G = G^*/M$, wo M ein abgeschlossener Normalteiler ohne Häufungspunkt ist; und umgekehrt: ist M ein abgeschlossener Normalteiler ohne Häufungspunkt von G^*, so lässt sich G^* als Überlagerungsgruppe von G^*/M deuten.

H. Weyl[20] annehmen, dass sie unitär ist. Daraus folgt nach H. Weyl,[21] S. 380, (siehe auch E. Cartan, a.a.O.[16], S. 38–41) der endlichfache Zusammenhang der adjungierten Gruppe, also erst recht der der gegebenen, die sich als Überlagerungsgruppe der adjungierten auffassen lässt.

12. *Ein vollständig reduzibler linearer Keim ist direktes Produkt einfacher abgeschlossener Untergruppenkeime.*

BEWEIS: Wir folgen E. Cartan.[22] Sei zunächst der gegebene lineare Keim G *irreduzibel.* Sei A ein maximal abelscher (abgeschlossener) Normalteilerkeim von G. A besitzt einen Eigenvektor u: $A(u) = \lambda u$ (λ abhängig vom betreffenden Element von A). Da A Normalteilerkeim ist, ist für jedes $g \subset G$: $Ag(u) = \lambda g(u)$. Da G irreduzibel ist, also die $g(u)$ den ganzen Vektorraum aufspannen, folgt daraus: $A(u) = \lambda u$ für jeden Vektor u. Die Elemente von A sind also Multiplikationen,[23] A ist im Zentrum. Der lineare Gruppenkeim G' entstehe aus G, indem man die Elemente von G so mit Skalaren multipliziert, dass die Determinanten eins werden. In G' gibt es keine Multiplikationen mehr, G' ist aber noch irreduzibel; nach dem soeben für G bewiesenen hat G' keinen abelschen Normalteilerkeim, ist also halbeinfach. Der Kommutatorkeim von G' ist also (siehe Beweis von 10) gleich G'; andererseits haben G und G' denselben Kommutatorkeim. Somit ist G' der Kommutatorkeim von G, und G ist darstellbar als $A \times G'$ (direktes Produkt eines abelschen und eines halbeinfachen Lieschen Gruppenkeimes).

Sei nun G beliebig, aber vollständig reduzibel. Seien $L_1, \cdots, L_n$ invariante lineare Mannigfaltigkeiten, die zusammen den ganzen Vektorraum aufspannen, und in denen G irreduzibel ist. Wir betrachten den linearen (irreduziblen) Gruppenkeim G_k, gemäss dem E_k durch G transformiert wird; G_k ist, wie wir im vorigen Absatz zeigten, direktes Produkt eines Lieschen abelschen und eines Lieschen halbeinfachen Gruppenkeimes. Wir bilden abstrakt das direkte Produkt der G_k; es hat dieselbe Eigenschaft.[24] G lässt sich als homomorphes Bild von $G_1 \times \cdots \times G_n$ auffassen, ist also nach 10 und 3 direkter Faktorkeim

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[20] *Die Vollständigkeit der primitiven Darstellungen einer geschlossenen kontinuierlichen Gruppe.* Math. Ann. *97* (1927), 737–755.

[21] *Theorie der Darstellung kontinuierlicher halbeinfacher Gruppen durch lineare Transformationen.* III. Math. Zeitschr. *24* (1926), 377–395.

[22] *Les groupes de transformations continus, infinis, simples.* Annales École Norm. (3) *26* (1909), 93–161 (p. 148, théorème XVIII).

[23] Zunächst hat man nämlich für jedes a aus A: $a(u) = \lambda u$ für alle u einer Menge M von Vektoren, die den ganzen l-dimensionalen Vektorraum aufspannt; dabei kann der Wert von λ mit u variieren. Die Transformation a induziert eine projektive Transformation im zugehörigen $(l-1)$-dimensionalen projektiven Raum, bei der l Punkte festbleiben, die keinem niederdimensionalen Teilraum angehören; a induziert also die identische projektive Transformation und muss selber alle Vektoren mit einem Skalar multiplizieren, der, wie man leicht sieht, für alle Vektoren derselbe sein muss. Die Elemente von A sind also tatsächlich Multiplikationen.

[24] Siehe [17].

in $G_1 \times \cdots \times G_n$ und daher auch direktes Produkt eines halbeinfachen Lieschen und eines abelschen Lieschen Gruppenkeimes, also direktes Produkt einfacher Liescher Gruppenkeime.

13. *Ist G ein linearer unitärer halbeinfacher Gruppenkeim, so ist die von G erzeugte lineare Gruppe G^* abgeschlossen im Gebiet aller linearen Transformationen.*[25]

Beweis: Die abgeschlossene Hülle $\overline{G^*}$ von G^* ist eine unitäre Gruppe, also vollständig reduzibel und nach 12 eine $\times$-Gruppe. G ist daher direkter Faktorkeim in $\overline{G^*}$: $\overline{G^*} = G \times H$ (im Kleinen). Ist H^* die von H erzeugte lineare Gruppe und $\overline{H^*}$ ihre abgeschlossene Hülle, so kann $G \vee \overline{H^*}$ keinen Häufungspunkt besitzen, weil es sonst nach 8 (zweiter Absatz) einen Lieschen Keim enthielte, der wegen der Vertauschbarkeit von G und H zum Zentrum des halbeinfachen G gehörte und nicht allein aus e bestünde, was nicht möglich ist. $\overline{G^*}/\overline{H^*}$ ist also im Kleinen isomorph mit G. Andererseits ist $\overline{G^*}/\overline{H^*}$ als Bild von $\overline{G^*}$ kompakt und besitzt nach 11 eine kompakte universelle Überlagerung; G als Bild dieser universellen Überlagerung ist auch kompakt (in sich), also abgeschlossen im Gebiete aller linearen Transformationen.

14. *Eine im Kleinen kompakte Gruppe G mit treuer stetiger unitärer Darstellung $D(G)$ ist im Kleinen direktes Produkt einfacher Liescher Normalteilerkeime, also $\times$-Gruppe. Ihre universelle Überlagerungsgruppe ist direktes Produkt einer kompakten halbeinfachen Gruppe K und eines V.*

Beweis: U sei eine genügend kleine kompakte abgeschlossene Umgebung von e in G, $D(U)$ ist also abgeschlossen im Gebiet aller linearen Transformationen, also Liesch. Die abgeschlossene Hülle $\overline{D(G)}$ von $D(G)$ ist auch noch unitär, also vollständig reduzibel, also ist auch $D(G)$, also auch $D(U)$ vollständig reduzibel. Daraus folgt nach 12 der erste Teil unseres Satzes.

G lässt sich also im Kleinen als Produkt eines Lieschen linearen abelschen Keimes A und eines Lieschen halbeinfachen unitären Keimes H darstellen. A lässt sich als Keim eines V auffassen, während H nach 13 eine *kompakte* halbeinfache Liesche Gruppe H^* erzeugt. $H^* \times V$ (abstrakt gebildet) ist im Kleinen isomorph G. Die universelle Überlagerung von $H^* \times V$ ist das direkte Produkt von V und der universellen Überlagerung von H^*, also nach 11 auch wieder direktes Produkt einer kompakten halbeinfachen Gruppe K und eines V.

15. *Ist die universelle Überlagerungsgruppe G^* von G von der Form $K^* \times V^*$, so ist auch G selbst von der Form $K \times V$.*

Beweis: $G = G^*/M$, M abgeschlossener Normalteiler ohne Häufungspunkt. M_{K^*} bzw. M_{V^*} seien die Mengen von Elementen von K^* bzw. V^*, die bei der Produktdarstellung ($K^* \times V^*$) der Elemente von M auftreten. V_1^* sei die minimale zusammenhängende, M_{V^*} enthaltende Untergruppe von V^*. Dann

[25] Siehe auch E. Cartan, a.a.O.[16], S. 37–41.

ist $V^* = V_1^* \times V_2^*$. Wir setzen $K = (K^* \times V_1^*)/M, V = V_2^* M/M$. Dann ist in der Tat $G \cong K \times V$ und V eine Vektorengruppe. Dass K tatsächlich kompakt ist, sieht man so ein: $V_1^* K^*$ ist nach T 17a abgeschlossen in G^*, also $V_1^* K^*/K^*M \cong V_1^*/(V_1^* \vee K^*M)$ nach T 14. Nach der Definition von V_1^* und 2 ist diese letzte Gruppe aber kompakt (in sich), also ist auch $V_1^* K^*/K^*M$ kompakt. Nun ist aber nach T 15

$$K^* V_1^*/K^*M \cong (K^* V_1^*/M)/(K^*M/M);$$

die Gruppe links haben wir gerade als kompakt nachgewiesen, der Divisor rechts ist K^* isomorph, also kompakt, also ist nach T22 $K = K^* V_1^*/M$ kompakt.

16. *Eine im Kleinen kompakte Gruppe mit im Kleinen treuer unitärer Darstellung ist eine* $\times$*-Gruppe der Form* $K \times V$. (Beweis folgt unmittelbar aus 14 und 15.)

III

17. Wir betrachten nun die im Kleinen kompakten zusammenhängenden Gruppen G mit genügend vielen stetigen fastperiodischen Funktionen (siehe Einleitung). Sie sind nach J. v. Neumann, a.a.O.[2], S. 480, Theorem 33, dadurch charakterisiert, dass es zu je zwei verschiedenen Elementen a und b eine stetige unitäre Darstellung $D(G)$ gibt mit $D(a) \neq D(b)$. Insbesondere gibt es dann zu jedem $a \neq e$ aus G eine Darstellung mit $D(a) \neq 1$. Bei einer solchen Darstellung gibt es eine ganze Umgebung U_a von a, so dass $D(U_a)$ die identische lineare Transformation nicht enthält. Mit abzählbar vielen solcher U_a lässt sich $G - e$ überdecken. Es gibt also eine Folge stetiger unitärer Darstellungen $D_n(G)$ mit der Eigenschaft, dass zu jedem a ein n existiert mit $D_n(a) \neq 1$.

G erscheint also (nach 14) homomorph abgebildet auf gewisse Liesche Gruppen G_n, die alle im Kleinen direkte Produkte einfacher unitärer Liescher Gruppen sind. Die die Homomorphismen definierenden Normalteiler H_n von G haben (wegen der letzten Bemerkung des vorigen Absatzes) den Durchschnitt (e).

18. Gestatten G/H_m und G/H_n treue stetige unitäre Darstellungen, so gilt dasselbe für $G/(H_m \vee H_n)$; man braucht nur die Darstellungen von G/H_m und G/H_n längs der Hauptdiagonale aneinanderzureihen und sonst Nullen zu setzen, um die gewünschte Darstellung zu erhalten. Daher dürfen wir von nun an voraussetzen, dass die *H_n eine absteigende Folge abgeschlossener Normalteiler von G* sind (andernfalls bilde man einfach die Durchschnitte).

H_n^0 sei die Komponente der Identität von H_n, C_n die Komponentengruppe von H_n, d.h. $C_n = H_n/H_n^0$ (siehe auch T 19). Die Komponenten von C_n sind Punkte, während G/H_n^0 zusammenhängend ist. Also gehört C_n zum Zentrum[26]

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[26] Wir verwenden hier den Satz von O. Schreier (a.a.O.[19]): Jeder Normalteiler einer zusammenhängenden Gruppe, dessen Komponenten Punkte sind, liegt im Zentrum.

von G/H_n^0, ist also sicherlich abelsch. Dann[27] besitzt C_n eine abgeschlossene offene *kompakte* Untergruppe C_n', die nun auch *Normalteiler* von G/H_n^0 ist. H_n' sei der Normalteiler von G, der bei dem Homomorphismus $H_n \to C_n$ auf C_n' abgebildet wird. $H_n' \subset H_n$; H_n' ist offen in H_n als Urbild von C_n', das offen in C_n ist.

H_n'' sei der Durchschnitt der H_k' $(k = 1, \cdots, n)$. $H_k' \vee H_n$ ist offen in $H_k \vee H_n = H_n$. Dasselbe gilt für den Durchschnitt von 1 bis n: $H_n'' \vee H_n = H_n''$ *ist offen in* H_n. *Die* H_n'' *bilden eine absteigende Folge abgeschlossener Normalteiler von* G *mit* (e) *als Durchschnitt. Ihre Komponentengruppen* C_n'' *sind (als abgeschlossene Untergruppen der* C_n'*) in sich kompakt.*

Wir setzen nun $G_n'' = G/H_n''$. Da H_n'' offen in H_n war, sind G_n'' und G_n im Kleinen isomorph, also gilt nach 16:

Die $G_n'' = G/H_n''$ sind Liesche $\times$-Gruppen der Form $K_n \times V_n$.

19. Sei U eine kompakte Umgebung von (e) in G und $R(U)$ der Rand von U. Der Durchschnitt aller $H_n'' \vee R(U)$ ist leer, und da sie alle in sich kompakt sind, sind fast alle $H_n'' \vee R(U)$ leer. Die Komponente der Identität fast aller H_n'' ist (wegen $e \subset U$) in U enthalten, also kompakt. Da der Komponentenraum der H_n'' kompakt ist (siehe 18), folgt daraus nach T 22, dass *fast alle* H_n'' *kompakt sind.* Wenn wir auf endlich viel H_n'' verzichten und alle Striche weglassen, können wir sagen:

In G *gibt es eine Folge absteigender kompakter abgeschlossener Normalteiler* H_n *mit dem Durchschnitte* (e)*, so dass die* $G_n = G/H_n$ *Liesche* $\times$*-Gruppen der Form* $K_n \times V_n$ *sind.*

20. Durch G wird ein Homomorphismus von G_{n+1} auf G_n erzeugt: $G_n \cong G_{n+1}/(H_n/H_{n+1})$. Nach 7 (die Voraussetzungen von 7 sind erfüllt, denn jeder abgeschlossene zusammenhängende Normalteiler von G_{n+1} ist Liesch, also sicher im Kleinen zusammenhängend, und der Normalteiler H_n/H_{n+1} ist kompakt) können wir annehmen, dass die V_n in den G_n so gewählt sind, dass durch *die Homomorphismen stets* K_{n+1} *auf* K_n *und* V_{n+1} *auf* V_n *abgebildet wird.*

Sei a_n *eine Folge von Elementen bzw. aus* G_n *mit der Eigenschaft, dass bei den vorliegenden Homomorphismen stets* a_n *das Bild von* a_{n+1} *ist. Dann gibt es genau ein* a *in* G*, von dem all diese* a_n *die Bilder sind.* Sei nämlich A_n die Urbildmenge von a_n in G. Es ist $A_n = H_n A_{n+1}$. Die A_n bilden also eine absteigende Folge kompakter abgeschlossener Mengen, ihr Durchschnitt ist nicht leer, kann aber auch keine zwei verschiedenen Elemente enthalten, denn deren Quotient wäre ein Element $\neq e$ von G, das allen H_n angehörte.

K_n^* bzw. V_n^* seien die Urbilder von K_n bzw. V_n in G; der Durchschnitt der K_n^* heisse K, der der V_n^* heisse V. Der Durchschnitt von K und V ist gleich dem der H_n, also (e). Sei g ein Element von G; seien g_n seine Bilder bzw. in G_n.

[27] Siehe D. van Dantzig, *Studiën over topologische algebra*, Diss. Amsterdam 1931, S. 18, TG 37.—Derselbe, *Zur topologischen Algebra* III, erscheint in Compositio mathematica.

g_n lässt sich schreiben als $b_n c_n$ mit $b_n \subset K_n$, $c_n \subset V_n$. Der erste Absatz dieser Nummer lehrt uns, dass wir annehmen dürfen, dass bei den vorliegenden Homomorphismen stets b_{n+1} auf b_n und c_{n+1} auf c_n abgebildet ist. Der zweite Absatz dieser Nummer zeigt dann, dass zu den b_n *ein* b in K und zu den c_n *ein* c in V gehört, von denen die b_n bzw. c_n die Bilder sind. bc hat dann als Bilder in den G_n die $b_n c_n = g_n$. Also ist $g = bc$, $b \subset K$, $c \subset V$. Schliesslich sind K und V abgeschlossene Normalteiler von G. Rein gruppentheoretisch ist also G *direktes Produkt* von K und V. Nun ist aber $K \times V$ (stetig) isomorph abgebildet auf G, da bc stetig von b und c abhängt. Nach T 5 und T 9 wissen wir, dass diese Beziehung eine *topologische Isomorphie* ist, sobald wir wissen, dass $K \times V$ im Kleinen kompakt ist. Das wird sich bei den folgenden Überlegungen mit ergeben.

Wir haben nämlich noch die Bezeichnungen K und V für die erklärten Normalteiler von G zu rechtfertigen, d.h. zu zeigen, dass K kompakt und V eine Vektorengruppe ist.

Zunächst V: Da die Normalteiler H_n/H_{n+1} der G_{n+1}, die die Homomorphismen auf G_n erzeugen, kompakt sind, sind alle V_n *isomorph* aufeinander abgebildet. Zu jedem c_n gibt es also *ein* c_{n+1}, von dem es Bild ist, also auch *ein* $c \subset V$, von dem es Bild ist; umgekehrt gibt es auch zu jedem $c \subset V$ *ein* Bild c_n in V_n, und diese letzte Beziehung ist stetig. Nach T 5 und T 9 sind V und V_n also topologisch isomorph. *V ist Vektorengruppe derselben Dimension wie die V_n.*

Nun die Kompaktheit von K: Ist O_n offen in G_n und O_n^* das Urbild von O_n in G, so ist O_n^* offen in G und beliebig klein (da der Durchschnitt der H_n gleich (e) war), falls nur O_n genügend klein und n genügend gross gewählt war. Die O_n^* sind also als Umgebungssystem in G brauchbar. Daraus folgt, dass die *Konvergenz einer Folge in G äquivalent ist mit der Konvergenz der Folge ihrer n-ten Bilder (für alle n) in G_n.* Nach dem Diagonalverfahren schliesst man daraus aus der Kompaktheit der K_n auf die *Kompaktheit von K.*

Wir haben also den

HAUPTSATZ I: *Jede zusammenhängende, im Kleinen kompakte Gruppe mit zweitem Abzählbarkeitsaxiom, die genügend viel stetige fastperiodische Funktionen besitzt, ist direktes Produkt der (abelschen) Gruppe der Translationen des n-dimensionalen euklidischen Raumes ($n = 0, 1, \cdots$) und einer (zusammenhängenden) kompakten Gruppe (mit zweitem Abzählbarkeitsaxiom).*

Dass diese Gruppen auch wirklich genügend viel stetige fastperiodische Funktionen besitzen, ist evident; man nehme etwa Funktionen, die für jedes feste Element von K periodisch auf V sind.[28]

21. Sei A_n irgendeine Folge irgendwelcher Gruppen (nicht einmal topologisch brauchen sie zu sein), von denen jede homomorphes Bild der folgenden ist. Die A_n erzeugen eine "Limesgruppe" A. Die Elemente von A sind die Folgen

[28] Ferner ist auch klar, dass in diesen Gruppen die durch die fastperiodischen Funktionen induzierte Topologie mit der gegebenen übereinstimmt (siehe J. v. Neumann, a.a.O. [2], Theorem 36).

von Elementen $a_n \subset A_n$, bei denen jedes Element Bild des folgenden ist; a_n heisst die n-te Koordinate des Elementes $(a_1, a_2, \cdots)$ von A. Unter dem Produkt der Elemente a und b von A versteht man das Element von A, dessen n-te Koordinate das Produkt der n-ten Koordinaten von a und b ist. Es ist klar, dass A eine Gruppe ist. Ordnet man jedem Element von A seine n-te Koordinate zu, so erhält man einen Homomorphismus.

Sind die A_n topologische Gruppen und ist die zwischen ihnen bestehende Beziehung gebietstreu, so topologisiere man A wie folgt: Ist U_n eine Umgebung in A_n, so nenne man die Menge aller Elemente von A, deren n-te Koordinaten in U_n liegen, eine Umgebung in A.[29] Dass die topologischen Axiome usw. in A erfüllt sind, ist klar, ebenso, dass A auf die A_n gebietstreu abgebildet ist, falls die gegebenen Homomorphismen der A_n gebietstreu waren. Erfüllen die A_n das zweite Abzählbarkeitsaxiom, bzw. sind sie kompakt, so gilt dasselbe bzw. für A.

Ist eine Gruppe A topologisch isomorph der von der Folge A_n erzeugten Limesgruppe, so sprechen wir von einer *G_n-adischen Erzeugung*[30] *der Gruppe A mit Hilfe der A_n*.

Wenden wir das auf die zwischen den K_n und K bestehende Beziehung an (siehe 20), so können wir sagen:

HAUPTSATZ II[7]: *Jede zusammenhängende kompakte Gruppe mit zweitem Abzählbarkeitsaxiom lässt sich G_n-adisch mit Hilfe kompakter Liescher Gruppen erzeugen.*[31]

Die kompakten Lieschen Gruppen sind alle bekannt (siehe E. Cartan, a.a.O.[16]), S. 37–42). Ihre universellen Überlagerungen sind (siehe 12) direkte Produkte

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[29] Eine Punktfolge in A ist also dann und nur dann konvergent, wenn ihre n-ten Koordinaten (für jedes n) konvergieren.

[30] Das erste Beispiel G_n-adisch erzeugter topologischer Gruppen findet sich in einer unbeachtet gebliebenen Arbeit von L. E. J. Brouwer: *On the structure of perfect sets of points*, Proceedings Amsterdam *12* (1910), 784–795, §4–5.

Das Prinzip tritt dort schon trotz des speziellen Charakters des Beispiels in seiner vollen Allgemeinheit hervor. L. E. J. Brouwer erhält in seinem Beispiel die Cantorschen Gruppen.

Ohne Kenntnis der Brouwerschen Note hat D. van Dantzig (*Über topologisch homogene Kontinua*, Fundamenta *15* (1930), 102–125) mit der G_n-adischen Methode die Solenoide definiert (siehe Nr. 25 unserer Arbeit). Derselbe hat ein der G_n-adischen Erzeugung nahe verwandtes Prinzip a.a.O.[27], S. 20, entwickelt. Hier findet sich auch die Bezeichnung "G_n-adisch", die an die ganz ähnliche Erzeugung der p-adischen Zahlen erinnern soll.

Für endliche abelsche G_n finden sich ferner G_n-adische Folgen bei L. Pontrjagin (*Über den algebraischen Inhalt topologischer Dualitätssätze*, Math. Ann. *105* (1931), 165–205 (195) jedoch mit der irreführenden Bemerkung, dass diese Folgen keine Limesgruppe definierten.

Die G_n-adische Erzeugung findet sich auch definiert und angewandt bei H. Freudenthal: *Die Hopfsche Gruppe, eine topologische Begründung kombinatorischer Begriffe*, Compositio mathematica *2* (1935), 134–162.

Auf demselben Prinzip wie die G_n-adische Erzeugung topologischer Gruppen beruht die Alexandroffsche Spektraldarstellung kompakter Räume (a.a.O. [33]), S. 107–108).

Zusatz bei der Korrektur: Eine systematische Entwicklung dieser Begriffe findet sich bei Hans Freudenthal, *Die R_n-adische Entwicklung von Räumen und Gruppen*, Proceedings Amsterdam *38* (1935), 414–418.

[31] Siehe [7].

von Vektorengruppen und kompakten halbeinfachen Lieschen Gruppen. Die kompakten, einfach zusammenhängenden, halbeinfachen Lieschen Gruppen hat E. Cartan[8]) aufgestellt. Man braucht sie nur noch mit Vektorengruppen zu multiplizieren und nach einem (im Zentrum enthaltenen) diskreten Normalteiler zu reduzieren, um alle kompakten Lieschen Gruppen zu erhalten; die diskreten Normalteiler ergeben sich alle aus den Untersuchungen von E. Cartan[9]).

Nach dem Hauptsatz II ist damit das Problem der Aufstellung aller kompakten Gruppen mit zweitem Abzählbarkeitsaxiom vollständig gelöst.

22. Ist irgend eine topologische Gruppe G homomorph auf $G' = G/H$ abgebildet, so können wir die Abbildung in zwei Schritten vollziehen, indem wir eine Gruppe G^* einschalten: $G^* = G/H_0$ (H_0 = Komponente der Identität von H; siehe T 19), $G' = G^*/(H/H_0)$. Beim ersten Schritt wird die Faktorgruppe nach einer Lieschen Untergruppe gebildet; die Dimension sinkt (wenn der Schritt überhaupt wesentlich ist). Beim zweiten wird nach einem diskreten Normalteiler reduziert; die Gruppe bleibt sich im Kleinen isomorph. Wenn wir von G' zu G übergehen, wollen wir den Schritt von G' nach G^* den "überlagernden," den von G^* nach G den "dimensionserhöhenden" nennen.

Ist die kompakte zusammenhängende Gruppe K von einer Folge Liescher kompakter Gruppen K_n G_n-adisch erzeugt, so dürfen wir also voraussetzen, dass jeder Schritt von K_n zu K_{n+1} rein überlagernd oder rein dimensionserhöhend ist (die unwesentlichen Schritte lassen wir weg).

U_1 sei eine genügend kleine abgeschlossene Umgebung von e und zwar eine Vollkugel[32] in K_1. Wenn U_n schon erklärt ist als abgeschlossene Umgebung von e in K_n und zwar als Vollkugel, erklären wir U_{n+1} wie folgt: Ist der Schritt von K_n nach K_{n+1} überlagernd, so setzen wir U_{n+1} gleich der Komponente der Identität des Urbildes von U_n in K_{n+1}; U_{n+1} ist dann mit U_n homöomorph. Ist dagegen der Schritt von K_n nach K_{n+1} dimensionserhöhend, so nehmen wir U_{n+1} (nachdem wir K_{n+1} im Kleinen als direktes Produkt von H_n/H_{n+1} und $K_{n+1}/(H_n/H_{n+1})$ dargestellt haben) als Produkt einer genügend kleinen Vollkugel aus H_n/H_{n+1}, die e als inneren Punkt enthält, und einer Vollkugel aus $K_{n+1}/(H_n/H_{n+1})$ an; die zweite Vollkugel muss dabei (bei dem Isomorphismus $K_n \cong K_{n+1}/(H_n/H_{n+1})$) Bild von U_n sein.

Die Folge U_n definiert dann eine Teilmenge M in G. M ist entweder einem endlichdimensionalen Quader homöomorph oder dem (unendlichdimensionalen) Fundamentalquader des Hilbertschen Raumes. $\dim M = \lim \dim U_n$ (dabei ist unter dem Limes einer über alle Grenzen wachsenden Folge ∞ zu verstehen). Da U_n eine Umgebung in K_n war, ist $\dim U_n = \dim K_n$; da M Teilmenge von K ist, ist $\dim M \leqq \dim K$. Also

$$\dim K \geqq \lim \dim K_n.$$

[32] *Vollkugel* ist stets topologisch zu verstehen; wir haben das Wort *Element* vermieden, weil auch von Gruppenelementen die Rede ist.

Nun lässt sich das kompakte K durch beliebig kleine Transformationen in fast alle K_n überführen (d.h. die Urbildmenge jedes Punktes, eine Nebenklasse von H_n, ist für genügend grosses n beliebig klein). Daraus folgt nach dem P. Alexandroffschen Überführungssatz[33]

$$\dim K \leqq \lim \dim K_n.$$

HAUPTSATZ III: *Ist die kompakte Gruppe K mit zweitem Abzählbarkeitsaxiom G_n-adisch erzeugt mit Hilfe der kompakten Lieschen Gruppen K_n, so ist* $\dim K = \lim \dim K_n$, *und K ist dann und nur dann endlichdimensional, wenn bei der G_n-adischen Erzeugung von K nur endlich viel dimensionserhöhende Schritte vorkommen.*

23. Wir nehmen nun dim K als endlich an und fragen, was geschieht, wenn K obendrein im Kleinen zusammenhängend ist. Indem wir endlich viel K_n aus der G_n-adischen Erzeugung von K weglassen, erreichen wir, dass in der Folge K_n keine dimensionserhöhenden Schritte vorkommen; alle Schritte sind überlagernd. U_1 sei eine offene einfach zusammenhängende Menge in K_1, U_n ihr Urbild in K_n. Die Komponenten von U_n bezeichnen wir mit $U_n^{k_1 \cdots k_n}$ und zwar so, dass das Urbild von $U_n^{k_1 \cdots k_n}$ in G_{n+1} aus den $U_n^{k_1 \cdots k_n k_{n+1}}$ besteht. (k_1 ist stets = 1.)

Ich behaupte, dass der Index k_{n+1} sovieler Werte fähig ist (unabhängig von der Wahl der vorangehenden Indizes), wie H_n/H_{n+1} Komponenten, d.h. Elemente) besitzt, und dass die Komponenten sämtlicher U_n (also auch für verschiedene n) einander homöomorph sind. Der Beweis[34] wird induktiv geführt:

Wir schreiben W statt $U_n^{k_1 \cdots k_n}$ und W^k statt $U_n^{k_1 \cdots k_n k_{n+1}}$. Statt H_n/H_{n+1} schreiben wir H, das lässt sich dann auffassen als der Normalteiler, der den Homomorphismus von K_{n+1} auf K_n erzeugt. H ist diskret. Man beachte, dass die Abbildung von K_{n+1} auf K_n unverzweigt, d.h. im Kleinen umkehrbar, ist.

Ein W^k kann keine zwei Punkte enthalten, die zur selben Nebengruppe von H in K_{n+1} gehören; denn zwei derartige Punkte liessen sich in W^k durch einen Weg verbinden, dessen Bild in W geschlossen, also zusammenziehbar wäre; eine solche Zusammenziehung liesse sich aber auf das Urbild in K_{n+1} übertragen, was nicht möglich ist.

Andererseits enthält jedes W^k von einem Punkt von W mindestens ein Urbild, wie man aus der Gebietstreue der Abbildung leicht entnimmt.

W^k und W sind also wirklich topologisch aufeinander abgebildet, und es gibt soviel W^k, wie eine Nebengruppe von H Elemente hat, also soviel, wie H selbst Elemente hat, w. z. b. w.

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[33] *Untersuchungen über Gestalt und Lage abgeschlossener Mengen beliebiger Dimension*, Annals of Math. *30* (1929), 101–187 (120).—Erst hier beginnen wir, wesentlich dimensionstheoretische Begriffe zu verwenden, während wir bis jetzt mit dem elementaren Dimensionsbegriff auskamen.

[34] Wir skizzieren den Beweis nur; er beruht auf geläufigen Tatsachen über die Beziehung zwischen einer Mannigfaltigkeit und ihren Überlagerungen.

Die Komponenten des Urbildes von U^k in K sind die von den Folgen $U_1^{k_1}$, $U_2^{k_1 k_2}$, $U_3^{k_1 k_2 k_3}$, $\cdots$ erzeugten Mengen von K. Der so entstehende Komponentenraum ist entweder eine endliche oder eine Cantorsche Menge; im zweiten Fall können die Komponenten keine offenen Mengen sein. Ist K im Kleinen zusammenhängend, so sind aber die Komponenten jeder offenen Menge offen; es kann also nur endlich viel verschiedene Folgen $U_1^{k_1}$, $U_2^{k_1 k_2}$, $\cdots$ geben, von einem gewissen n an findet keine Aufspaltung mehr statt, fast alle H_n/H_{n+1} bestehen allein aus e, also auch fast alle H_n. Fast alle K_n sind dann topologisch isomorph untereinander und mit K. Daraus folgt:

HAUPTSATZ IVa[7]: *Dann und nur dann ist eine im Kleinen kompakte Gruppe mit zweitem Abzählbarkeitsaxiom Liesch, wenn sie endlichdimensional und im Kleinen zusammenhängend ist.*

HAUPTSATZ IVb: *Dann und nur dann ist eine im Kleinen kompakte, zusammenhängende Gruppe mit zweitem Abzählbarkeitsaxiom, die genügend viel stetige fastperiodische Funktionen besitzt, endlichdimensional und im Kleinen zusammenhängend, wenn sie Liesch ist.*

IVb folgt aus IVa unmittelbar, wenn man berücksichtigt, dass nach Hauptsatz I nur eine Vektorengruppe als Faktor hinzukommen kann, die ja selber endlichdimensional und im Kleinen zusammenhängend ist.

24. Wir untersuchen weiter, wie die im Kleinen zusammenziehbaren[35] K aussehen.[36]

Entsprechend den Überlegungen von 23 darf man annehmen, daß alle H_n zusammenhängend sind. U sei eine Umgebung von e in K in sich zusammenziehbar. Man darf annehmen, daß alle H_n in U liegen. U_n sei das Bild von K in $K_n = K/H_n$; $U_{n+1} \supset H_n/H_{n+1}$. H_n/H_{n+1} ist als kompakte Liesche Gruppe, wie man leicht sieht, Träger eines höchstdimensionalen, in K_{n+1} nicht berandenden Zyklus, der also in U_{n+1} liegt. Zu ihm gibt es als Urbild in U einen in K nicht berandenden ϵ_n-Zyklus ($\epsilon_n \to 0$) derselben Dimension. Da U zusammenziehbar war, geht das nur, wenn dim $H_n/H_{n+1} = 0$, also $H_n = H_{n+1}$, also $K = K_n$ ist für fast alle n. Dann ist aber K Liesch. Daraus folgt:

HAUPTSATZ Va: *Dann und nur dann ist eine kompakte zusammenhängende Gruppe mit zweitem Abzählbarkeitsaxiom Liesch, wenn sie im Kleinen zusammenziehbar ist.*

HAUPTSATZ Vb: analog zu IVb.

25. Ohne Beweis erwähnen wir noch.

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[35] *Im Kleinen zusammenziehbar* soll heissen: es gibt zu jedem Punkt beliebig kleine, ihn enthaltende, offene Mengen, die sich in sich auf diesen Punkt stetig zusammenziehen lassen.

[36] Die hier folgende Vorbereitung von Hauptsatz V hat erst in der Korrektur ihre gegenwärtige vereinfachte Gestalt erhalten. Auch die Beweise der Hauptsätze III und IV ließen sich durch solche Homologiebetrachtungen vereinfachen.

HAUPTSATZ VIa: *Zu jeder kompakten zusammenhängenden Gruppe mit zweitem Abzählbarkeitsaxiom gibt es eine nulldimensionalblättrige kompakte Überlagerungsgruppe, die direktes Produkt eines abelschen Solenoids und endlich oder unendlich vieler einfacher nichtabelscher Liescher Gruppen ist.*

Dabei nennen wir K^* nulldimensionalblättrige Überlagerung von K, wenn K Faktorgruppe von K^* nach einem nulldimensionalen Normalteiler ist.

Ein abelsches Solenoid[37] wird so definiert: Sei V^∞ die ∞-dimensionale Vektorengruppe, d.h. die Gruppe, deren Elemente die unendlichen Folgen reeller Zahlen $(r_1, r_2, \cdots)$ sind mit der Addition der r_n als Gruppenoperation und der folgenden Topologie: n-te Umgebung von $(r_1, r_2, \cdots)$ ist die Menge aller $(s_1, s_2, \cdots)$ mit $|r_k - s_k| < \frac{1}{n}$ für $k \leqq n$. Sei A_n eine absteigende Folge abgeschlossener Untergruppen; A_n sei enthalten in der Untergruppe von V^∞ die entsteht, wenn man alle Koordinaten von der $(n+1)$-ten an null setzt. Die G_n-adische Folge V^∞/A_n definiert dann ein *abelsches Solenoid.*

Schliesslich ist in Hauptsatz VI das direkte Produkt unendlich vieler Gruppen F_n als Limesgruppe der G_n-adischen Folge F_1, $F_1 \times F_2$, $F_1 \times F_2 \times F_3$, $\cdots$ zu verstehen.

HAUPTSATZ VIb: analog zu IVb.

26. Wir lassen nun die Voraussetzung der Kompaktheit im Kleinen fallen und beschäftigen uns mit zusammenhängenden Gruppen mit zweitem Abzählbarkeitsaxiom, die genügend viel stetige fastperiodische Funktionen besitzen. Wir gehen dazu von den Ergebnissen von 18 aus. Dann gibt es also in G eine absteigende Folge von abgeschlossenen Normalteilern H_n mit e als Durchschnitt und kompakten Komponentenräumen, derart dass $G_n = G/H_n = K_n \times V_n$ ist (K_n Liesch; die Dimensionen der V_n dürfen jetzt über alle Grenzen wachsen). Ebenso wie im ersten Absatz von 20 haben wir wieder einen Homomorphismus von G_{n+1} auf G_n: $G_n = G_{n+1}/(H_n/H_{n+1})$. Ebenso wie in 20 dürfen wir wegen 7 annehmen, dass dabei K_{n+1} auf K_n und V_{n+1} auf V_n abgebildet wird. (Die Voraussetzungen sind erfüllt, da H_n/H_{n+1} als Bild von H_n einen kompakten Komponentenraum hat.)

Die G_n-adische Folge $K_n \times V_n$ besitzt eine Limesgruppe $G^* = K \times V$ (hier darf V auch ein V^∞ sein; in dem Falle ist das Produkt nicht mehr im Kleinen kompakt, doch gilt noch das zweite Abzählbarkeitsaxiom). Einem Element a von G, dessen Bild in G_n a_n ist, ordnen wir das Element $(a_1, a_2, \cdots)$ von G^* zu. Die Abbildung ist (stetig) isomorph; es liesse sich sogar zeigen, dass das Bild von G in G^* überall dicht ist, man beachte jedoch, dass die Abbildung nicht notwendig topologisch isomorph ist.

V^∞ (also auch jedes endlichdimensionale V) lässt sich stetig isomorph in eine kompakte zusammenhängende Gruppe mit zweitem Abzählbarkeitsaxiom einbetten, nämlich in das direkte Produkt unendlich vieler Kreisdrehungsgruppen.

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[37] Siehe auch [30].

Man ordne nämlich einem Element $(r_1, r_2, \cdots)$ von V^∞ die Folge $(\eta_1, \eta_2, \cdots)$ zu, wobei gesetzt ist

$$\text{für gerades } n \quad : \eta_n = e^{i(r_{n-1}+r_n)},$$

$$\text{für ungerades } n\text{: } \eta_n = e^{i(r_n+r_{n+1})\rho}, \ (\rho \text{ reell irrational}).$$

Daraus ergibt sich für das in $K \times V$ eingebettete G:

HAUPTSATZ VII: *Jede zusammenhängende Gruppe mit zweitem Abzählbarkeitsaxiom, die genügend viel fastperiodische stetige Funktionen besitzt, lässt sich stetig isomorph einbetten in eine zusammenhängende kompakte Gruppe mit zweitem Abzählbarkeitsaxiom.*

Anhang

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27. In diesem Anhang zeigen wir erstens, dass *jede einseitig fastperiodische Funktion (beiderseits) fastperiodisch* ist (auch in nichtkommutativen Gruppen und ohne alle topologischen Voraussetzungen). Zweitens beweisen wir:

HAUPTSATZ VIII: *Bei den zusammenhängenden, im Kleinen kompakten Gruppen mit zweitem Abzählbarkeitsaxiom sind folgende vier Eigenschaften miteinander äquivalent:*

1) *G besitzt genügend viele stetige fastperiodische Funktionen,*

2) *G ist von der Form $K \times V$ (K zusammenhängend),*

3) *G transformiert e-Folgen in e-Folgen, d.h. ist* $\lim a_n = e$, *so ist auch* $\lim s_n a_n s_n^{-1} = e$.

4) *G ist invariant metrisierbar, d.h. es gibt eine Funktion $\rho(a, b)$ mit den üblichen Eigenschaften der Abstandsfunktion und der Invarianzeigenschaft $\rho(a, b) = \rho(as, bs) = \rho(sa, sb)$.*

Dabei ist die Äquivalenz von 1 und 2 aus Hauptsatz I und einem Satz von J. v. Neumann[4] bekannt; die Äquivalenz von 3 und 4 (ohne topologische Voraussetzungen ausser dem Abzählbarkeitsaxiom) ist von D. van Dantzig bewiesen[38]; dass 3 aus 1 folgt wird sich leicht ergeben, das Umgekehrte nur unter Verwendung der Äquivalenz von 1 mit 2 und von 3 mit 4.[39]

28. *G sei eine Gruppe (ohne alle topologischen Voraussetzungen). $f(x)$ sei eine auf G rechtsfastperiodische Funktion, d.h. der Raum der Funktionen $f(xa)$ sei totalbeschränkt.*[40] *Wir beweisen, dass f dann auch linksfastperiodisch ist, dass also der Raum der Funktionen $f(ax)$ totalbeschränkt ist.*

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[38] *Zur topologischen Algebra* I, Math. Ann. *107* (1932), 587–626, §5.

[39] Die Äquivalenz von 2 und 3 ist zuerst von Herrn L. Pontrjagin erkannt worden, der sie mir (ohne Angabe des Beweises) vor einiger Zeit brieflich mitteilte.

[40] Totalbeschränktheit eines metrischen Raumes nennt man die folgende Eigenschaft: zu jedem $\epsilon > 0$ lassen sich endlich viel Punkte derart finden, dass jeder Punkt von einem dieser endlich vielen einen Abstand $< \epsilon$ hat. Sie ist äquivalent mit der folgenden Eigenschaft: aus jeder unendlichen Teilmenge des Raumes lässt sich eine Fundamentalfolge auswählen. (Siehe F. Hausdorff, Mengenlehre, Berlin-Leipzig 1927, §21, 4.)

Zu jedem $\epsilon > 0$ gibt es endlich viel Elemente $y_1, \cdots, y_p$, so dass zu jedem $y \subset G$ ein y_k $(k = 1, \cdots, p)$ existiert mit

$$|f(xy) - f(xy_k)| < \frac{\epsilon}{3}. \qquad (*)$$

Insbesondere ist f also beschränkt. Sei irgendeine Teilmenge von G gegeben. Aus ihr können wir eine Folge $a_1, a_2, \cdots$ so auswählen, dass $f(a_n y)$ konvergiert in den endlich vielen Punkten $y = y_1, \cdots, y = y_p$. Es gibt also ein q, so dass

$$|f(a_m y_k) - f(a_n y_k)| < \frac{\epsilon}{3} \text{ für } m \geqq q, n \geqq q, k = 1, \cdots, p.$$

Ersetzt man in (*) x durch a_m bzw. durch a_n, so erhält man zusammen mit der letzten Ungleichung für alle y

$$|f(a_m y) - f(a_n y)| < \epsilon \text{ für } m \geqq q, n \geqq q.$$

Lässt man nun ϵ eine Nullfolge durchlaufen, und wendet man dabei in bekannter Weise das Diagonalverfahren an, so erhält man tatsächlich aus der gegebenen Menge eine Teilfolge a_n, für die $f(a_n y)$ gleichmässig konvergiert; der Raum der (ay) ist also totalbeschränkt, f ist linksfastperiodisch.

29. *Sei G eine topologische Gruppe und f eine stetige fastperiodische Funktion. Ist e Häufungspunkt der zu $a \subset G$ konjugierten Elemente, so ist notwendig $f(a) = f(e)$.*

Beweis: Sei $\lim s_n a s_n^{-1} = e$. Die Folge $f(s_n^{-1} x s_n)$ besitzt eine (gegen $g(x)$) gleichmässig konvergente Teilfolge.[41] Zu gegebenem $\epsilon > 0$ gibt es also unendlich viel n mit

$$|f(s_n^{-1} x s_n) - g(x)| < \epsilon;$$

für diese n gilt dann auch

$$|f(x) - g(s_n x s_n^{-1})| < \epsilon.$$

Setzt man hier $x = e$, so erhält man $f(e) = g(e)$; setzt man $x = a$, so erhält man (wegen $\lim s_n a s_n^{-1} = e$) $f(a) = g(e)$; zusammen also $f(a) = f(e)$, w. z. b. w.

30. Aus 29 folgt unmittelbar, dass, wenn G genügend viel fastperiodische Funktionen besitzt, e-Folgen in e-Folgen transformiert werden. Aus Eigenschaft 1 folgt also Eigenschaft 3.

31. G sei nun im Kleinen kompakt und erfülle das zweite Abzählbarkeitsaxiom G besitze ferner die Eigenschaft 3 aus 27. G lässt sich dann invariant metrisieren.[42] In dieser Metrik sei U eine genügend kleine, d.h. kompakte Vollkugel

[41] Siehe J. v. Neumann, a.a.O. [4], Theorem 9.

[42] D. van Dantzig, *Studiën over topologische algebra*, Diss. Amsterdam 1931, TG 32. *Zur topologischen Algebra* III, erscheint in Compositio Math.

um e. Die Transformationen, die x auf sxs^{-1} abbilden, sind dann isometrische Abbildungen von U auf sich; ihre Gesamtheit ist gleichartig gleichmässig stetig. Fügt man zu ihr alle Limesabbildungen hinzu, so entsteht also eine kompakte Gruppe, in die G/Z isomorph abgebildet ist; dabei ist Z die Gesamtheit aller s mit $sx = xs$ für alle x aus U. Ist G obendrein zusammenhängend, so erzeugt U ganz G[42], dann ist Z also das Zentrum.

Von jetzt an setzen wir G stets als *im Kleinen kompakt, zusammenhängend und dem zweiten Abzählbarkeitsaxiom genügend* voraus; ferner nehmen wir an, dass G *die Bedingung* 3 erfüllt. Dann hat sich also gezeigt, dass G/Z in eine kompakte Gruppe (Z = Zentrum von G) isomorph einbettbar ist; G/Z besitzt also genügend viel stetige fastperiodische Funktionen und ist nach Hauptsatz I von der Form $K \times V$. Wir müssen zeigen, dass G selbst von der Form $K \times V$ ist.

32. Im Folgenden beachte man:

a) *Eine Untergruppe oder Faktorgruppe einer Gruppe mit der Eigenschaft* 3 *hat auch die Eigenschaft* 3 (siehe 27).

b) *Das direkte Produkt zweier Gruppen der Form* $K \times V$ *hat dieselbe Form.*

c) *Eine Faktorgruppe einer Gruppe von der Form* $K \times V$ *hat dieselbe Form.* (Beweis analog dem von 6 und 7.)

d) *Besitzt G eine Folge von Normalteilern H mit (e) als Durchschnitt, für die die G/H genügend viel stetige fastperiodische Funktionen besitzen, so besitzt G selbst genügend viel fastperiodische Funktionen* (fastperiodische Funktionen auf G/H lassen sich nämlich als fastperiodische Funktionen auf G auffassen). Oder: *Sind die G/H von der Form $K \times V$, so ist es auch G selbst.*

e) *Ist G/N von der Form $K \times V$ und N nulldimensional, so ist G selbst von der Form $K \times V$.* (Folgt aus Hauptsatz II, wenn man berücksichtigt, dass der Übergang von G/N zu G für die erzeugenden Lieschen Gruppen den Übergang zu Überlagerungsgruppen nach sich zieht.)

33. Die Komponente von e in Z heisse Z_0. Dann ist Z/Z_0 nulldimensional. $G/Z \cong (G/Z_0)/(Z/Z_0)$ (siehe T 16 und 19). Da G/Z von der Form $K \times V$ ist, hat G/Z_0 nach 32e dieselbe Eigenschaft. Nach Hauptsatz I und II hat Z_0 beliebig kleine Normalteiler N, für die Z_0/N Liesch ist. $G/Z_0 \cong (G/N)/(Z_0/N)$. Nach 32d genügt es hier, zu beweisen, dass all diese G/N von der Form $K \times V$ sind. Wenn wir wieder G statt G/N und Z statt Z_0/N schreiben, brauchen wir also wegen 32a nur zu zeigen: Besitzt G die Eigenschaft 3, ist Z eine Untergruppe des Zentrums von G, ist Z Liesch und G/Z von der Form $K \times V$, so ist auch G von der Form $K \times V$. Wenn wir das bewiesen haben, sind wir fertig.

Es ist $G/Z = K \times V$. Bei der Faktorbildung nach Z sei L das Urbild von K und W das Urbild von V. Da sich G als eine Faktorgruppe von $L \times W$ auffassen lässt, genügt es nach 32c, zu beweisen, dass L und W von der Form $K \times V$ sind. Das soll in 34 bzw. 35 geschehen.

34. Z ist abelsch und Liesch, besitzt also einen diskreten Normalteiler N, so dass Z/N kompakt ist. $L/Z = (L/N)/(Z/N) \cong K$; nach T 22 ist also L/N kompakt und nach 32e ist L selbst von der Form $K \times V$.

35. $W/Z = V$; die Kommutatorgruppe von W ist also, da V abelsch ist, in Z enthalten. a und x seien Elemente von W; wir setzen $\varphi(x) = axa^{-1}x^{-1}$ (für ein festes a). Dann ist also $\varphi(x) \subset Z$ und $axa^{-1} = \varphi(x) \cdot x$, also $a^n x a^{-n} = \varphi(x)^n \cdot x$, da Z zum Zentrum gehört. Gäbe es eine Folge x_n, mit $\lim x_n = e$ und $\varphi(x_n) \neq e$ so könnten wir, da Z Liesch ist, also nicht beliebig kleine Untergruppen (ausser (e)) besitzt, Exponenten k_n so bestimmen, dass $\varphi(x_n)^{k_n}$ nicht nach e konvergiert. Dann würde auch $\varphi(x_n)^n x_n$ nicht nach e konvergieren, während andererseits die Eigenschaft 3 besagt, dass $a^n x_n a^{-n} = \varphi(x_n)^n x_n$ wohl nach e konvergiert. Es kann also keine Folge x_n geben mit $\lim x_n = e$ und $\varphi(x_n) \neq e$, d.h. es gibt eine Umgebung von e, in der überall $\varphi(x) = e$ ist. Für alle x aus dieser Umgebung ist dann $ax = xa$; W war aber zusammenhängend (denn V und Z sind es), also ist für alle x aus W überhaupt $ax = xa$. Diese Überlegung lässt sich für jedes a aus W durchführen, also ist W abelsch und nach Hauptsatz I von der Form $K \times V$.

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ALMOST PERIODIC FUNCTIONS AND COMPACT GROUPS

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I. Introduction

The theory of any given almost periodic function on a group G can be reduced to the study of a continuous function on a uniquely determined compact group F. Then there is a one-to-one correspondence between continuous functions on F and a certain class of a.p. functions on G, such that convolutions of corresponding functions are again corresponding functions. This class of functions on G is the smallest *modul* containing the given function. In II we repeat the definition of modul and give, with some indications for the proofs, certain of their properties that are not affected by the knowledge of the compact group F. In III we give the definition of the compactified group F and in IV we prove our statement concerning the class of functions on G related to the continuous functions on F.

The approximation theorem for moduls of a.p. functions proved by Bochner and von Neumann[1] is put in an essentially different light if considered on the compact group. We give a proof of this theorem in V, together with a generalization of the theorem concerning Bohr a.p. functions with positive Fourier coefficients. In VI we prove the product theorem for expansions of a.p. functions. This theorem might just as well have been proved on the basis of the approximation theorem given in (1) p. 475.

In VII we give some definitions and results on the relation between different moduls on one group, applying this in VIII to moduls on compact groups. As a consequence we find in IX a theorem (th. 6) implicitly contained in Pontrjagin's paper (3), but giving a somewhat better insight into the character of a compact group. The proof of theorem 7 is nothing but an adaption of Pontrjagin's proof of the same theorem (cf. (3)), using theorem 6 as a starting point. Theorem 7 and its proof are a refinement of von Neumann's work in (6). Theorem 6 can be used as the foundation for an analysis of the structure of a compact connected group (7).

In X and XI we give two convergence theorems for a.p. functions and an analysis of infinite direct products of compact groups. Theorem 8 will be found in (2) as theorem 34, while theorem 10 is related to theorem 35 of (2). Theorems 4 and 10 are generalizations of results of H. Bohr: Math. Zeitschr., 23 (1925), pp. 38–44.

Though abelian compact groups have not been specially mentioned, a com-

[1] See (2), th. 32. The numbers in brackets refer to the list at the end of this paper.

parison of the present treatment of compact groups with the treatment of abelian groups in (4) and (5) will be found interesting. The same is true for von Neumann's closed families of functions ((1), Ch. IV). This paper has been written in such a way that explicit knowledge is only supposed of the general theory of a.p. functions in the first three chapters of (1) and some topological properties of compact groups, to be found in the first chapter of (5) and in many other places. Except in a few remarks all the topologies occurring in this paper are separable metric.

Some of the results common to this paper and (2) were found independently. Later the authors of (2) had the kindness to put a manuscript of (2) at my disposal, for which I wish to extend my thanks. Also Professor von Neumann communicated to me an essentially different proof of the product theorem (IV).

II. Moduls of a.p. functions and representations

Definition: A class of a.p. functions on a group G is called a modul $\mathfrak{M}$ of functions if it satisfies the following conditions:

1. *If $f(x)$ and $g(x)$ are in $\mathfrak{M}$, then $f(x) + g(x)$, $\overline{f(x)}$ and $\alpha f(x)$ are in $\mathfrak{M}$.*
2. *If $f(x)$ is in $\mathfrak{M}$, then all $f(ax)$ and $f(xa)$ are in $\mathfrak{M}$.*
3. *If $f(x)$ and $g(x)$ are in $\mathfrak{M}$, then $f(x)g(x)$ is in $\mathfrak{M}$.*
4. *If the functions $f_n(x)$, $n = 1, 2, \cdots$ are in $\mathfrak{M}$ and the sequence $f_n(x)$ converges uniformly to $f(x)$, then $f(x)$ is in $\mathfrak{M}$.*

The following statements are rapid consequences of this definition, and the general theory of a.p. functions:

If $f(x)$ belongs to a modul $\mathfrak{M}$ then the convolution of $f(x)$ and any other a.p. function belongs to $\mathfrak{M}$. Compare (1), p. 457, remark.

If $f(x)$ belongs to a modul $\mathfrak{M}$ and $u(y)$ is a uniformly continuous complex valued function defined for all values taken by $f(x)$ then $u[f(x)]$ belongs to $\mathfrak{M}$. For on that range $u(y)$ can be approximated uniformly by polynomials in y and $\bar{y}$.

As a consequence of the expansion theory for a.p. functions ((1), th. 24 to 30) any element of a representation actually occurring in the expansion of an a.p. function $f(x)$ can be obtained from $f(x)$ by means of repeated convolutions (compare (1), th. 31). Applying also our other statements on moduls we find:

The class of irreducible representations of G of which the elements occur in a modul $\mathfrak{M}$ is equal to the class of representations actually occurring in the expansion of at least one function of $\mathfrak{M}$, and is closed under the operations of taking conjugate complex representations and decomposing direct products of representations (see Wedderburn, Lectures on Matrices, p. 74).

A class of representations closed under these two operations is called a *modul* of representations. The correspondence established above between moduls of functions and moduls of representations is obviously one-to-one, so that we can consider a *modul* as an abstract entity identified either by a class of functions or a class of irreducible bounded representations.

Finally we note:

If G is a topological group, then the class of all continuous a.p. functions on G is a modul.

III. Compactification of a group by means of an a.p. function[1a]

a. If $f(x)$ is a complex valued function of one variable x on a group G, then the distinct ones among the functions $f(xay)$ of two variables x and y are in one-to-one correspondence with the elements of G/H. Here H is the maximal invariant subgroup of G such that $f(x)$ is constant on any co-set of H.

To prove this we assign elements a and b of G to the same class if and only if $f(xay) = f(xby)$ for all values of x and y. Now if p and q are arbitrary elements of G and a and b are in the same class, then paq and pbq also are in the same class. It follows immediately that the classes are co-sets of an invariant subgroup H of G and that $f(x)$ is constant on each co-set of H. On the other hand if a and b are in the same co-set of an invariant subgroup with that property, then $f(xay) = f(xby)$, so H is the maximal invariant subgroup with that property.

We now identify the group G/H with the class of functions $f(xay)$ and define $f(a)$ on G/H by its values on the co-sets of H in G.

b. In the space $\mathfrak{G}_2$ of complex valued functions of two variables, each ranging over a given group G, we define a metric by the definition:

$$\delta[f(x, y), g(x, y)] = \text{Min}\,[\text{l.u.b.}_{x,y} \mid f(x, y) - g(x, y) \mid ; 1]\,.$$

The metric of $\mathfrak{G}_2$ defines an invariant metric on G/H. Under that metric $f(a)$ is uniformly continuous.

This follows from:

$$\text{l.u.b.}_{x,y} \mid f(xay) - f(xby) \mid = \text{l.u.b.}_{x,y} \mid f(xpaqy) - f(xpbqy) \mid ;$$

$$\mid f(a) - f(b) \mid \leqq \text{l.u.b.}_{x,y} \mid f(xay) - f(xby) \mid .$$

As a consequence the closure F of the group G/H of $\mathfrak{G}_2$ is a complete group, uniquely determined by G and $f(x)$, and such that $f(a)$ can be extended to a uniformly continuous function on F.

c. One of von Neumann's results in (2) is that $f(x)$ is almost periodic on G if and only if the class of functions $f(xay)$ is totally bounded in $\mathfrak{G}_2$, that means if and only if the group F is compact.

So we have succeeded in constructing *a compact group F uniquely determined by any given group G and an almost periodic function $f(x)$ on that group.*

d. To any continuous function on this group F there corresponds uniquely a function on G. It is defined on each co-set of H as constant and equal to the value of the given function in the element of F corresponding to that co-set. As the function on F is a.p. ((1), p. 483, (5), p. 451), the same is true for the

[1a] After completion of this paper we found the contents of this section also in a paper by A. Weil: Comptes Rendus, Paris, 200, p. 38.

corresponding function on G. This follows immediately from the definition of almost periodicity.

The continuous functions on F form a modul and the same is true for the corresponding functions on G. Moreover, to every continuous representation of F there corresponds a bounded representation of G, and the expansion matrix belonging to a continuous function on F and a continuous representation of F is equal to the expansion matrix of the corresponding function and the corresponding representation. We see that as far as their structure is concerned, from the point of view of convolutions and expansions, there is no difference between the continuous functions on F and the corresponding a.p. functions on G. The same is also true with respect to convergence and approximation questions.

Remark. An interesting conclusion can be drawn in case the group G is bicompact but not separable. Then G/H is bicompact and thus identical with F. But F is compact metric, thus separable. In other words:

Any continuous function on a bicompact group G is constant on all co-sets of a closed invariant subgroup H of G, such that G/H is separable.

This implies an existence theorem of very many invariant subgroups for any bicompact nonseparable group. For instance we can conclude that any such group has a well ordered decreasing composition series with compact separable factors of composition.

IV. The modul $\mathfrak{M}f$ of an a.p. function $f(x)$ on a group G

The modul $\mathfrak{M}f$ of an a.p. function $f(x)$ on a group G is defined as the common part of all moduls containing the function $f(x)$. It is obviously determined by all irreducible representations of G, together with all others obtained from these by taking conjugates and decomposing direct products.

THEOREM 1. *If to a group G and an a.p. function $f(x)$ on G there corresponds the compact group F, then the class of functions on G corresponding to the continuous functions on F is the modul $\mathfrak{M}f$.*

Considering the character of $f(x)$ as a function defined on F we see that it is sufficient to prove the following statement: If $f(x)$ is continuous on a compact group F and not constant on all co-sets of any closed invariant subgroup of F (except the identity element), then the modul of $f(x)$ is equal to the class of all continuous functions on F.

Given any element a of F distinct from the identity element 1, then we can find other elements b and c in F, such that $f(bac)$ and $f(bc)$ are distinct, otherwise $f(x)$ would be constant on all co-sets of the minimal closed invariant subgroup of F containing a. The function $g_1(x) = f(bxc)$ belongs to $\mathfrak{M}f$ and takes distinct values in 1 and $a \neq 1$. We can define a uniformly continuous real valued function $u(y)$, such that $u[g_1(1)] = 1$, $0 \leqq u(y) \leqq 1$ and $u(y) = 0$ if $2 \mid g_1(1) - y \mid \geqq \mid g_1(1) - g_1(a) \mid$. Then $g_2(x) = u[g_1(x)]$ is a real valued function belonging to $\mathfrak{M}f$, such that $g_2(x) = 0$ if x belongs to some open set $\mathfrak{U}_a$ containing a.

Now we give an arbitrary nucleus $\mathfrak{V}$ (see (5), p. 449) of F and determine for each point a of F outside of $\mathfrak{V}$ such a function $g_2(x)$ and open set $\mathfrak{U}_a$. The complement of $\mathfrak{V}$ will be covered by a finite number of these open sets. The product $h(x)$ of all corresponding functions $g_2(x)$ has the following properties:

$$h(x) \text{ belongs to } \mathfrak{M}f\,,$$

$$h(x) = 0 \text{ unless } x \text{ belongs to } \mathfrak{V},$$

$$0 \leqq h(x) \leqq 1\,, \qquad h(1) = 1\,.$$

If $g(x)$ is any continuous function on F and $\epsilon > 0$ is given we determine $\mathfrak{V}$ by the condition that: $|\, g(x) - g(y^{-1}x)\,| < \epsilon$ for all x if y belongs to $\mathfrak{V}$. Then:

$$|\, h(y)\,(g(x) - g(y^{-1}x))\,| \leqq \epsilon h(y)\,,$$

for both sides are equal to zero unless y belongs to $\mathfrak{V}$. So after taking the mean over y:

$$\left| g(x) - \frac{1}{M_y\, h(y)}\, h \times g(x) \right| \leqq \epsilon\,.$$

The function $h \times g(x)$ belongs to $\mathfrak{M}f$ (as a convolution with $h(x)$), so $g(x)$ can be approximated uniformly by functions of $\mathfrak{M}f$. In other words $g(x)$ belongs to $\mathfrak{M}f$. This concludes the proof of theorem 1.

Remark 1. An interesting corollary is obtained if we consider the modul as a modul of representation:

COROLLARY. *All continuous representations of any compact group F can be obtained, by taking conjugates and decomposing direct products, from any collection of such representations, not all equal on any two distinct points of F.* In particular all continuous representations of a given closed group of unitary transformations are obtained by the same two operations, starting only with the given group.

Remark 2. Theorem 1 shows, that the compact group F constructed in III by means of a group G and an a.p. function $f(x)$, is associated more closely with the totality of functions in $\mathfrak{M}f$ than with $f(x)$ itself. Furthermore if any modul $\mathfrak{M}$ is given a corresponding function $f(x)$ and so the corresponding compact group F can always be found, provided the number of distinct classes of inequivalent representations in $\mathfrak{M}$ is at most countable. We multiply an element of each representation in the modul with a constant different from zero but small enough to insure the uniform convergence of the sum of all terms obtained. The modul of the function so obtained is the given modul $\mathfrak{M}$.

A consequence is that whenever in this paper we discuss the continuous functions on a compact group, the results are applicable to any modul with a countable number of representations in any group.

Remark 3. For an arbitrary modul we can construct (by Pontrjagin's methods in (3)) a bicompact group F, such that the modul consists of all continuous functions on F. This allows the extension of most of the results of this paper to arbitrary moduls and bicompact groups. We mention in particular that

VIII e can be extended to transfinite sequences, that in IX theorem 6 is true for bicompact groups, and that in XI theorem 9 is extendable to the case of an arbitrary (uncountable) collection of independent moduls.

V. Approximation

In this section we will prove an approximation theorem for all continuous functions on a compact group. The method used is a refinement of the method used to prove theorem 1.

In F we take an arbitrary nucleus $\mathfrak{U}$ and in $\mathfrak{U}$ another nucleus $\mathfrak{V}$ satisfying[1b] $a\mathfrak{V}a^{-1}$, $= \mathfrak{V}^{-1} = \mathfrak{V}$ and $\mathfrak{V}^2 \subset \mathfrak{U}$. Then we determine a continuous real function $h_1(x)$ on F, equal to 1 if $x = 1$, equal to 0 unless x belongs to $\mathfrak{V}$, and satisfying $0 \leqq h_1(x) \leqq 1$.

Constructing $h_2(x) = 1/2\ (h_1(x) + h_1(x^{-1}))$ and $h_3(x) = \text{l.u.b.}_y\ h_2(yxy^{-1})$, we obtain a new continuous real function satisfying: $0 \leqq h_3(x) \leqq 1$; $h_3(x) = 0$ unless x belongs to $\mathfrak{V}$; $h_3(1) = 1$; $h_3(xy) = h_3(yx)$ and $h_3(x^{-1}) = h_3(x)$.
Finally we put

$$g(x) = \frac{h_3 \times h_3(x)}{M_x[h_3 \times h_3(x)]}.$$

Then $g(x)$ is zero except for points of $\mathfrak{U}$. For $h_3(y) \neq 0$ and $h_3(y^{-1}x) \neq 0$ implies $y \subset \mathfrak{V}$ and $yx \subset \mathfrak{V}$, so $x \subset \mathfrak{V}^2 \subset \mathfrak{U}$. The function is constant on classes of equivalent elements, so its expansion matrices are multiples of the unit matrix. The function is equal on inverse elements and real, so the diagonal elements are real and equal for conjugate complex representations. They are (except for a constant positive factor) the squares of similar elements defined for $h_3(x)$, so they are non-negative. As $g(x)$ is a convolution its expansion is absolutely and uniformly convergent. So we have proved:

THEOREM 2. *Given a compact group F and an arbitrary nucleus $\mathfrak{U}$ there is a continuous real, non-negative function $g(x)$ on F satisfying:*

$$g(x) = 0 \text{ unless } x \text{ belongs to } \mathfrak{U},$$
$$M_x\, g(x) = 1, \text{ so } g(x) \text{ is not identically zero},$$
$$g(xy) = g(yx),\ g(x) = g(x^{-1}),$$
$$g(x) = \sum_p a_p \mathrm{Tr}[D_p(x)],\ a_p \geqq 0.$$

Here the sequence is absolutely and uniformly convergent. The subscript p changes over the inequivalent classes of irreducible continuous representations $D_p(x)$ of F and $a_p = a_q$ if $D_p(x) = \overline{D_q(x)}$.

Remark 1. Without dropping essential properties of $g(x)$ we can suppose its expansion to be finite. For that purpose we break off the expansion of $g(x)$

[1b] See (5), appendix.

such that the sum of the deleted terms is in absolute value less than a given $\epsilon > 0$ and add ϵ. Then the expansion of the new function $g'(x)$ is finite and $0 \leqq g'(x) - g(x) \leqq 2\epsilon$. The sequence should be broken off without separating any representation and its conjugate representation.

THEOREM 3. *Any continuous function $f(x)$ on F is approximated uniformly by the convolutions $f \times g(x)$ or $f \times g'(x)$ if the nucleus $\mathfrak{U}$ converges to the identity element of F.*

If $\delta > 0$ is given we can choose the nucleus $\mathfrak{U}$ such that

$$| f(x) - f(xy^{-1}) | < \delta \text{ if } y \subset \mathfrak{U}.$$

For the corresponding function $g(x)$ of theorem 2 we have then

$$| g(y) \{f(x) - f(xy^{-1})\} | < \delta g(y)$$

so if we put

$$g \times f(x) = f \times g(x) = f_\delta(x), \quad \text{then} \quad | f(x) - f_\delta(x) | < \delta;$$

$$g \times f'(x) = f \times g(x) = f'_\delta(x), \quad \text{then} \quad | f(x) - f'_\delta(x) | < \delta + 2\epsilon M_x f(x).$$

Remark 2. The functions $f_\delta(x)$ are derived from f by multiplication of the expansion matrices with the positive constants a_p. For each fixed p the numbers a_p must approach the limit 1 as δ approaches zero. This follows immediately from (1), proof of theorem 26.

Remark 3. It is according to III, remark 2, possible to construct functions $g(x)$ and $g'(x)$, as mentioned in theorem 3, for any almost periodic function $f(x)$, the functions only depending on the modul $\mathfrak{M}f$.

Remark 4. One consequence, that could have been obtained in several other ways, is that the number of inequivalent irreducible continuous representations of a compact group is at most countable. In fact they must all be present in the expansions of a countable number of functions $g(x)$ belonging to a countable number of nuclei $\mathfrak{U}$.

THEOREM 4. *If the expansion matrices[2] α_p of an a.p. function $f(x)$ are positive Hermitian, its expansion is absolutely and uniformly convergent.*

The properties supposed for $f(x)$ are still true for the approximating functions $f_\delta(x)$. Accordingly $f_\delta(1) = \sum_p s_p a_p \operatorname{Tr}[\alpha_p]$ is non-negative, so $f(1)$ is non-negative.

The functions obtained from $f(x)$ by taking away any finite number of terms in its expansion still have the same properties, so they are non-negative for $x = 1$. It immediately follows that

$$\sum s_p \operatorname{Tr}[\alpha_p] \leqq f(1)$$

---

[2] We define the expansion matrices α_p as the transpose of the matrices so defined by von Neumann. A term of the expansion acquires now the form $\Sigma_p s_p \text{Trace } [\alpha_p D_p(x)]$. Compare (1), th. 26. We will write Tr for Trace.

for any finite sum. So the infinite sum is convergent. And as we can suppose that the α_p are diagonal matrices with non-negative elements it follows that the sequence

$$\sum_p s_p \operatorname{Tr}[\alpha_p D_p(x)] = f(x)$$

is absolutely and uniformly convergent.

Remark 5. As in the case of Bohr a.p. functions this theorem can be generalized to the case where the expansion matrices α_p have the following property: For any vector ξ (of the correct number of components) the number $\xi\alpha_p\xi^*$ lies in a part of the complex plane, independent of p, and bounded by two half lines through the origin making an angle less than π. Supposing the part to be symmetric with respect to the real axis, theorem 3 applies to $f(x) + f(x^{-1})$, with the expansion matrices $\alpha_p + \alpha_p^*$. Next we see that $|\operatorname{Tr}[\alpha_p - \alpha_p^*]|$ divided by $\operatorname{Tr}[\alpha_p + \alpha_p^*]$ is uniformly bounded, from which the statement follows immediately.

VI. The product theorem

The direct product of two irreducible unitary representations can be decomposed into the sum of a finite number of such representations, so any product of elements of such representations can be written as a finite linear combination with constant coefficients of elements of other representations of the same type.

It follows immediately that if $f(x)$ and $g(x)$ are a.p. functions a formal product can be written down for their developments simply by substituting the finite linear combinations mentioned above for each product of representation elements, taking all terms containing any one element of any one representation and adding all coefficients of those terms.

THEOREM 5. *The expansion of $f(x)g(x)$ is the formal product of the expansions of $f(x)$ and $g(x)$.*

The proof is divided into two steps:

a. The sequence obtained for the coefficient of any one term in the formal product is convergent.

Suppose $\sum_q s_q \operatorname{Tr}[\alpha_q D_q(x)]$ and $\sum_r s_r \operatorname{Tr}[\beta_r D_r(x)]$ are the expansions of $f(x)$ and $g(x)$. The coefficients of the elements of $D_p(x)$ in the formal product of these expansions will be obtained in the form:

$$M_x\{(\sum_q s_q \operatorname{Tr}[\alpha_q D_q(x)])(\sum_r s_r \operatorname{Tr}[\beta_r D_r(x)])D_p(x^{-1})\},$$

and it can be easily verified that the elements of this matrix are determined as formal infinite symmetric bilinear forms in the elements of all matrices α_p and all matrices β_p.

The quadratic forms corresponding to these bilinear forms are bounded (if a factor $s_p^{\frac{1}{2}}$ is absorbed into each component of α_p). For if $C(x)$ is written for any

element of $D_p(x^{-1})$ we know $|\, C(x)\,| \leqq 1$. So if all summations are taken over the same finite range:

$$
\begin{aligned}
&|\, M_x\{(\sum_q s_q \operatorname{Tr}[\alpha_q D_q(x)])^2\, C(x)\}\,| \\
&\leqq M_x\{|\sum_q s_q \operatorname{Tr}[\alpha_q D_q(x)]\,|^2\,|\, C(x)\,|\} \\
&\leqq M_x\{\sum_q \sum_r s_q s_r \operatorname{Tr}[\alpha_q D_q(x)]\, \overline{\operatorname{Tr}[\alpha_r D_r(x)]}\} \\
&= \sum_q s_q\,|\,\alpha_q\,|^2 \leqq N_f^2.
\end{aligned}
$$

The last equality follows from $D_r^*(x) = D_r(x^{-1})$ and the orthogonality relations of the unitary representations ((1) Th. 24). From well known theorems it follows now, that the coefficients in the formal product are determined by means of convergent series.

b. The series so obtained is the expansion of $f(x)g(x)$.

To prove this we apply the summation method of section V on $f(x)$ and $g(x)$. The products $f_\delta(x)g_\delta(x)$ approach uniformly the product $f(x)g(x)$. As the expansions of $f_\delta(x)$ and $g_\delta(x)$ are absolutely and uniformly convergent, the theorem for these functions is evident. But the series for the coefficients in the expansion $f_\delta(x)g_\delta(x)$ are obtained from the series for the corresponding coefficients of the formal product of $f(x)g(x)$ by multiplication with positive constants ($a_q a_r$, see Th. 2 in V), approaching 1 as δ goes to zero. As the latter series are convergent, the sums of the former series must converge to the sums of the latter series. As $f_\delta(x)g_\delta(x)$ converges uniformly to $f(x)g(x)$ the coefficients in their expansions converge uniformly to the corresponding coefficients in the expansion of $f(x)g(x)$. So b follows and theorem 5 is proved.

VII. Relations between different moduls on a group G

We will say that a modul $\mathfrak{M}_1$ is contained in another modul $\mathfrak{M}_2 (\mathfrak{M}_1 \subset \mathfrak{M}_2)$ if each function (representation) in $\mathfrak{M}_1$ is also in $\mathfrak{M}_2$. The functions (representations) common to two moduls $\mathfrak{M}_1$ and $\mathfrak{M}_2$ form a third modul $\mathfrak{M}_1 \cdot \mathfrak{M}_2$. If $\mathfrak{M}_1 \cdot \mathfrak{M}_2$ contains only constant functions (the identical representation) then $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are said to be *independent*.

The *sum* $\mathfrak{M} = \Sigma_\alpha \mathfrak{M}_\alpha$ of any collection of moduls $\mathfrak{M}_\alpha$ is defined as the common part of all moduls containing the moduls $\mathfrak{M}_\alpha$. Obviously $\mathfrak{M}$ contains exactly all irreducible unitary representations to be constructed by decomposing direct products of representations each contained in at least one of the $\mathfrak{M}_\alpha$. So each representation in $\mathfrak{M}$ is contained in the sum of a finite number of the $\mathfrak{M}_\alpha$ and each function in $\mathfrak{M}$ can be approximated uniformly by functions in finite sums of moduls $\mathfrak{M}_\alpha$. This simplifies considerably the treatment of infinite sums of moduls.

A collection of moduls is said to be *independent* if any modul in the collection and the sum of all other moduls are independent. As a result of our remark

on infinite sums of moduls we see immediately: *A collection of moduls is independent if each of its finite subcollections is independent.* If we distribute all moduls of an independent collection in subcollections (each modul occurring only in one subcollection), then the sums of the subcollections are also independent. As a result of the form of the definition of independence and the remark made about infinite sums of moduls, this statement reduces immediately to the special case of the distribution of a finite independent collection of moduls into two subcollections; and by an induction to the total number of moduls this in its turn reduces to: α: *If* $\mathfrak{M}_1$ *and* $\mathfrak{M}_2$ *are independent moduls and also* $\mathfrak{M}_3$ *and* $\mathfrak{M}_1 + \mathfrak{M}_2$ *then all three moduls are independent.*

Another result to be obtained as a consequence of α is the following: A (well ordered) sequence of moduls is independent if each modul and the sum of the preceding moduls are independent.

The proof of α goes back to the case where the moduls $\mathfrak{M}_1$, $\mathfrak{M}_2$ and $\mathfrak{M}_3$ contain only a countable number of representations. For if α is not true, so that for instance $\mathfrak{M}_1$ and $\mathfrak{M}_2 + \mathfrak{M}_3$ are not independent, then there are three irreducible unitary representations $D_i(x)$ in $\mathfrak{M}_i$, $i = 1, 2, 3$, neither of which is the identical one, such that $D_1(x)$ is an irreducible part of the direct product $D_2(x) \times D_3(x)$. Then we can replace $\mathfrak{M}_i$ by the smallest modul containing $D_i(x)$, of course satisfying the additional condition. As a consequence we can assume that the group on which the moduls are defined is compact. This case will be treated in the next section.

VIII. The moduls of a compact group

For any modul $\mathfrak{M}$ of continuous functions on a compact group F we can construct a function $f(x)$ such that $\mathfrak{M} = \mathfrak{M}f$. (Compare IV, remark 2 and V, remark 4). If H is the invariant subgroup constructed for F and $f(x)$ by the method of III, then H has the property that any function in $\mathfrak{M}$ is constant on any co-set of H and the same is not true for any invariant subgroup not contained in H.

a. *The correspondence between the moduls* $\mathfrak{M}$ *and the closed invariant subgroups* H *is one-to-one.* $\mathfrak{M}$ determines H in the way indicated above and on the other hand $\mathfrak{M}$ consists of *all* continuous functions constant on all co-sets of H (compare theorem 1).

If $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are two moduls and H_1 and H_2 the corresponding groups, then $H_1 \supset H_2$ implies $\mathfrak{M}_1 \subset \mathfrak{M}_2$ and conversely.

b. *To* $\mathfrak{M}_1 \cdot \mathfrak{M}_2$ *there corresponds the group* $[H_1, H_2]$ *generated by* H_1 *and* H_2. Obviously the modul of $[H_1, H_2]$ is contained in both $\mathfrak{M}_1$ and $\mathfrak{M}_2$ and a function belonging to both $\mathfrak{M}_1$ and $\mathfrak{M}_2$ is constant on all co-sets of $[H_1, H_2,]$ and so belongs to the modul of $[H_1, H_2]$.

c. *To* $\mathfrak{M}_1 + \mathfrak{M}_2$ *there corresponds the common part* $H_1 \cdot H_2$ *of* H_1 *and* H_2. For the modul of $H_1 \cdot H_2$ certainly contains $\mathfrak{M}_1$ and $\mathfrak{M}_2$. Suppose it contained more than the modul $\mathfrak{M}_1 + \mathfrak{M}_2$. Then the group H corresponding to $\mathfrak{M}_1 + \mathfrak{M}_2$

would have $H_1 \cdot H_2$ as proper subgroup and it would be easy to find a function in $\mathfrak{M}_1$ or in $\mathfrak{M}_2$ not constant on H.

d. *The modul $\mathfrak{M}$ of a compact group F is the sum of two independent moduls if and only if F is the direct product of two groups.* Suppose $\mathfrak{M}$ is the sum of two independent moduls $\mathfrak{M}_1$ and $\mathfrak{M}_2$. Then the corresponding invariant subgroups H_1 and H_2 generate F because $\mathfrak{M}_1 \cdot \mathfrak{M}_2$ consists of constants only and have only the identity element in common because $\mathfrak{M} = \mathfrak{M}_1 + \mathfrak{M}_2$. So F is the direct product of the two groups H_1 and H_2. Furthermore $\mathfrak{M}_1$ is the modul of continuous functions on H_2 and $\mathfrak{M}_2$ is the modul of continuous functions on H_1. A similar reasoning can be made starting with the direct product of two groups H_1 and H_2. The proof of α in VII is now evident: Under the conditions of α the compact group F with modul $\mathfrak{M}_1 + \mathfrak{M}_2 + \mathfrak{M}_3$ is the direct product of two groups F_3 and G with moduls $\mathfrak{M}_3$ and $\mathfrak{M}_1 + \mathfrak{M}_2$ and G is the direct product of groups F_1 and F_2 with moduls $\mathfrak{M}_1$ and $\mathfrak{M}_2$. So F is the direct product of F_1, F_2 and F_3 and α follows immediately.

e. *If $\mathfrak{M}_1, \mathfrak{M}_2, \cdots$ is an increasing sequence of moduls, with the sum $\mathfrak{M}$, then the corresponding decreasing sequence of closed invariant subgroups $H_1, H_2, \cdots$ has the group H corresponding to $\mathfrak{M}$ as common part.*

In fact if H' is the common part of $H_1, H_2, \cdots$, then all representations in any of the $\mathfrak{M}_i$, so also in $\mathfrak{M}$ are constant on co-sets of H', so H' is contained in H. On the other hand H is certainly contained in the common part of all H_i.

IX. The structure of a compact group

In particular we can take the increasing sequence of moduls on the compact group F in the following way: Arrange all distinct classes of irreducible continuous representations of F in a sequence. Take for $\mathfrak{M}_n$ the smallest modul containing the first n representations. Then the group F/H_n belonging to that modul can be given as a unitary group (by arranging the n representations as blocks along the diagonal of a larger matrix). As such groups are (not necessarily connected) Lie groups and the corresponding H_n have only the identity element in common we have proved:

THEOREM 6: *If U is any nucleus of a compact group F then U contains a closed invariant subgroup H such that F/H is a Lie group.* (Not necessarily connected, including the case of a finite group.)

For some classes of compact groups F certain properties can be proved for the subgroup H:

If F is locally connected, H is also locally connected. As the number of components of H is then finite we can replace H by a connected group, satisfying theorem 6. A proof of this statement together with a more detailed proof of the next statement will shortly appear in the American Journal of Mathematics.

If F is finite dimensional and the given nucleus is sufficiently small, then H must be 0-dimensional.

By means of theorem 6 we can find a decreasing sequence of invariant subgroups such that the factor groups are Lie groups. Subdividing these Lie groups in a suitable way into complexes we obtain a spectrum for F.[3] It follows now from the dimension theory and a study of relative cycles on the spectrum of F, that the dimension of F is equal to the upper limit of the dimensions of F/H. On the other hand if F is finite dimensional and H is sufficiently small, the dimension of F/H is at least the dimension of F. But as soon as the dimensions of F and F/H are equal, the subgroup H must be 0-dimensional. Otherwise we could find an invariant subgroup G of H such that H/G is a positive dimensional Lie group and then F/G would be a Lie group of dimension higher than that of F.

Combining these two results, taking into account that a 0-dimensional connected set consists of one element, we find Pontrjagin's theorem:

THEOREM 7. *If a compact group is finite dimensional and locally connected it is a Lie group* (again not necessarily connected).

X. Decomposition of continuous functions with respect to moduls in F

Suppose $f(x)$ is a continuous function in the compact group F and $\mathfrak{M}$ a modul of continuous functions in F corresponding to the closed invariant subgroup H of F.

If x is a fixed element of F and y a variable element of H then $f(yx)$ is a continuous function on H. We define $f_H(x) = M_y f(yx)$. Then $f_H(x)$ is a function on F obviously constant on all co-sets of H. It is continuous because whenever ab^{-1} is contained in a sufficiently small nucleus of F, then $f(ya) - f(yb) < \epsilon$. (Compare (5), I, 5.)

We will say that $f_H(x)$ is the result of *taking the mean of $f(x)$ over the co-sets of H*. From (2) theorem 8 it follows immediately that $M_x f(x) = M_x f_H(x) = M_x M_y f(yx)$. Here y varies again on H.

The expansion of $f_H(x)$ is obtained from the expansion of $f(x)$ by selecting the terms corresponding to representations in the modul $\mathfrak{M}$.

Of course all terms in the expansion of $f_H(x)$ belong to that modul, so it is sufficient to prove: If $g(x)$ is constant on all co-sets of H, then $f \times g(x) = f_H \times g(x)$.

But if y varies on H and z varies on F then:

$$f \times g(x) = M_z f(z) g(z^{-1}x) = M_z M_y f(zy) g(y^{-1}z^{-1}x) .$$

As $g(x)$ is constant on co-sets of H we find:

$$f \times g(x) = M_z[M_y f(zy)] g(z^{-1}x) = M_z f_H(z) g(z^{-1}x) = f_H \times g(x) .$$

We apply this on the increasing sequence of moduls $\mathfrak{M}_1, \mathfrak{M}_2, \cdots$ of which the sum is equal to the modul of all continuous functions of x. The corre-

[3] Compare: Alexandroff, *Gestalt und Lage*, Annals of Math. **30** (1928), 101–187.

sponding subgroups we call $H_1, H_2, \cdots$ and the results of taking the mean of a function $f(x)$ over these groups we call $f_1, f_2, \cdots$. As the groups $H_1, H_2, \cdots$ decrease and have only the identity element in common, the sequence of functions $f_1, f_2, \cdots$ must converge uniformly to $f(x)$. In fact we can find a number n so large, that the oscillation of $f(x)$ on any co-set of H_n is less than a given $\epsilon > 0$. But then $| f(x) - f_n(x) | < \epsilon$. This proves (compare IV, remark 2) the following theorem:

THEOREM 8: *If $f(x)$ is any a.p. function, $\mathfrak{M}_1, \mathfrak{M}_2, \cdots$ an increasing sequence of moduls whose sum contains $\mathfrak{M}_f$, and $f_k(x)$ the a.p. function whose expansion is obtained by selecting the terms in the expansion of $f(x)$ belonging to $\mathfrak{M}_k$, then $f_k(x)$ converges uniformly to $f(x)$.*

XI. Independent moduls and direct products

Suppose the compact group F is the direct product of a countable number of groups $F_1, F_2, \cdots$.[4] Then there is on F, corresponding to each of its subgroups F_i, a modul $\mathfrak{M}_i$ consisting of all continuous functions constant on the co-sets of all F_l, $l \neq i$. According to VIII, c and e, any sum of moduls $\mathfrak{M}_i$ consists of all continuous functions constant on the product of all groups F_l not corresponding to any of the $\mathfrak{M}_i$ in the sum. From VIII d we can now see that the moduls $\mathfrak{M}_i$ are independent. Furthermore the sum of the moduls $\mathfrak{M}_i$ is the modul $\mathfrak{M}$ of all continuous functions on F.

Suppose on the other hand that the modul $\mathfrak{M}$ of a compact group F is the sum of a sequence of independent moduls $\mathfrak{M}_1, \mathfrak{M}_2, \cdots$. Applying VIII d, we can prove by induction that F is the direct product of compact groups $F_1, F_2, \cdots, F_k$, of which the moduls are $\mathfrak{M}_1, \mathfrak{M}_2, \cdots, \mathfrak{M}_k$, and a group F_k' of which the modul is $\sum_{l>k} \mathfrak{M}_l$. The group F_k' is determined in F as the subgroup on the co-sets of which all functions of $\sum_{l \leq k} \mathfrak{M}_l$ are constant. According to VIII e, the common part of all groups F_k' is the identity element of F. It is now very easy to establish a (topological) simple isomorphism between F and the direct product of all groups F_i. So we have proved:

THEOREM 9. *The modul $\mathfrak{M}$ of continuous functions on a compact group F is a sum of independent moduls $\mathfrak{M}_i$ if and only if the group F is the direct product of the groups F_i determined by the moduls $\mathfrak{M}_i$* (according to IV, remark 2 and not according to VIII).

Suppose a continuous function $f(x)$ on such a group F has the property that all the representations occurring in its expansion belong to one of the moduls $\mathfrak{M}_k$. We call $f_k(x)$ the function obtained from $f(x)$ by taking the terms in the expansion of $f(x)$ belonging to $\mathfrak{M}_k$. As $\sum_k \mathfrak{M}_k = \mathfrak{M}$ is independent of the order in which the sum is taken we find, according to theorem 8, that the sum $\sum_k f_k(x)$ converges to $f(x)$ *independent of the order in which the sum* is taken.

[4] Compare (4), appendix 1; footnote 5, Annals of Math. **35** (1934), p. 905; (5), III 6m. We use the definition in (5); the equivalence with the definition at the other places cited follows directly from theorem 9.

So the series $\sum_k |f_k(x)|$ is convergent. The function $f_k(x)$ can be considered as a function on F_k. We call y_k the element of F_k where $|f_k(x)|$ obtains its maximum value and y the element of F corresponding to the sequence $y_1, y_2, \cdots$ of elements of F_k. Then $|f_k(y)| = |f_k(y_k)|$, so the sequence $|f_k(y)|$ majorizes the sequence $|f_k(x)|$. This proves:

THEOREM 10. *If all terms occurring in the expansion of an* a.p. *function $f(x)$ belong to one of a collection of independent moduls $\mathfrak{M}_1, \mathfrak{M}_2, \cdots$ and the terms in $\mathfrak{M}_k$ form the expansion of the function $f_k(x)$, then the series $\sum_k f_k(x)$ converges absolutely and uniformly, with $f(x)$ as sum.*

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CELL SPACES

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This paper[1] lays abstract foundations for a general theory of combinatorial topology; it is an outgrowth, and in part a refinement, of some features of the author's thesis.[2] The basic notion is that of cell space, i.e. a space whose fundamental elements are cells rather than points. In such a space there is perfect duality between "closed" and "open." We study two dual ways of mapping one cell space on another, in terms of which we are able to characterize formally one cell space as a subdivision of another. Through incidence numbers and correspondence numbers we introduce algebraic considerations which are a natural counterpart of our set theory and which combine with the set theory to provide a broad working basis for further operations.

1. **Definition of a cell space and its dual.** A *cell space*, or discrete space,[3] S, is a (finite or infinite) set $\{x\}$ of elements, called *cells*, together with a relation $\prec$, read *is on the boundary of* (cf. *thesis* p. 193), which is

irreflexive	$(x \not\prec x)$
asymmetric	$(y \not\prec x \text{ if } x \prec y)$
transitive	$(x \prec z \text{ if } x \prec y \text{ and } y \prec z)$.

The cell space is not ordered by the relation $\prec$ since $y \not\prec x$ does not necessarily imply $x \prec y$. A familiar example of a cell space is a simplicial complex; the (open) simplexes s_p of all dimensions p constitute the cells, and $s_p \prec s_q$ if and only if s_p is a (p-dimensional) face of s_q. However this example is a quite specialized one (see Alexandroff, loc. cit.,[3] p. 1651).

For convenience we write $x \prec y$ also as $y \succ x$ ("y has x on its boundary"). Then we can define the *dual* S^* of the cell space S as a formal copy of S in which the relations $\prec$, $\succ$ are reversed; i.e. S^* is a cell space related to S by a one-one correspondence $x^* \leftrightarrow x$ such that

$$x^* \prec y^* \text{ if and only if } x \succ y.$$

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[1] Presented to the American Mathematical Society in preliminary form, February 25, 1933. See Abstract 39-3-71, *Bull. A. M. S.*, 39 (1933) p. 196. At that time the author was a National Research Fellow.

[2] Tucker, "An abstract approach to manifolds," these *Annals*, 34 (1933) 191–243. We shall refer to this as *thesis*.

[3] Cf. Alexandroff, "Sur les espaces discrets," *Comptes Rendus Acad. Sci.*, Paris, 200 (1935) 1649–1651.

This abstraction is drawn from the familiar idea of "dual complex" in the combinatorial theory of manifolds. Although the dual of an arbitrary complex may not exist in a strict geometric sense, it does exist abstractly, and proves very useful.[4]

Of course the dual of S^* is the original cell space S, i.e. $(S^*)^* = S$.

2. **Cell space compared with point space.** Let X be a set of cells of S. We say that X is *closed* [*open*] if

$$x \underset{[>]}{\prec} y \text{ and } y \in X \text{ imply } x \in X.$$

The closed [open] sets of S possess the following properties:

a) the sum of any number of closed [open] sets is closed [open]; the null set 0 is closed [open]

b) the common part of any number of closed [open] sets is closed [open]; S itself is closed [open]

c) given any two elements x, y there is a closed [open] set which contains x or y but not both.

Conversely these properties taken as postulates[5] characterize S. According to a) the class of closed [open] sets of S forms an inverseless Abelian group[6] (i.e. satisfies all the axioms for a commutative group except the one which postulates that each element has an inverse) with respect to (finite or infinite) logical addition. According to b) the class of closed [open] sets of S forms an inverseless Abelian group with respect to (finite or infinite) logical multiplication. Taking a) and b) together we see that the class of closed [open] sets of S forms an interesting sort of ring.

It requires but slight changes of wording to make a), b), c) into postulates for a topological (*point*) space in the weakest sense of Hausdorff.[7] We must weaken a) [or b) if we are dealing with open sets] to read "any finite number," and we must strengthen c) to read "which contains x but not both." The weakening of a) [or b)] spoils the symmetry between a) and b), on which the existence of the dual space S^* depends. The strengthening of c) makes each element a closed set, thereby putting the relation $\prec$ out of use. If c) were strengthened without weakening a) [or b)] we should have the trivial case of a totally discontinuous space in which each set is both open and closed.

Let X^* denote the set of cells of S^* which correspond to the cells of a set X of S. Then X^* is open [closed] if X is closed [open]. Of course the complement $S - X$ of a closed [open] set is open [closed].

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[4] An elegant instance of this may be seen in a recent paper of Alexander, "On the chains of a complex and their duals," *Proc. Nat. Acad. Sci.*, 21 (1935) 509–511. Kolmogoroff reported on similar investigations at the Moscow Topology Conference.

[5] Alexandroff has also given these postulates, loc. cit.[3] His separation postulate, attributed to Kolmogoroff, is equivalent to 3), but is otherwise formulated.

[6] Or an Abelian groupoid—cf. Garrett Birkhoff, these *Annals*, 35 (1934), p. 351.

[7] See Hausdorff, *Mengenlehre* (1927) 226–232.

3. Cell space compared with chain structure on a complex. We use the following terminology and notation:

closure	CX	set of all cells $\leqq$ some cell of X
aperture	AX	set of all cells $\geqq$ some cell of X
boundary	BX	set of all cells $<$ some but $\neq$ any cell of X
derived	DX	set of all cells $>$ some but $\neq$ any cell of X.[8]

These definitions imply such relations as the following:

$$BX = CX - X \qquad DX = AX - X$$
$$A^*X = CX^* \qquad D^*X = BX^*$$
$$AX = C^*X^* \qquad DX = B^*X^*.$$

(Here A^*X means $(AX)^*$, etc.) The last two lines are just two ways of stating that the notions of aperture and closure are dual and also that those of derived and boundary are dual.

The boundary operator B possesses the following properties which taken as postulates characterize the cell space S:

$$\begin{cases} \sum x \text{ and } B\sum x \text{ have no cells in common} \\ B\sum x = \sum Bx \mod \sum x; \qquad B0 = 0 \\ BBx = 0. \end{cases}$$

(Here $\sum x$ denotes an arbitrary, finite or infinite, sum of cells of S.) These properties bear a striking analogy to those of the chain-boundary operator F so familiar in present-day homology theory:

$$\begin{cases} \text{a chain } c \text{ and its boundary } Fc \text{ have no cells in common} \\ F\sum \lambda c = \sum \lambda Fc \qquad (\lambda\text{'s are coefficients}) \\ FFc = 0. \end{cases}$$

The essential difference lies in the types of addition used in the two cases; the logical addition used with B is an inverseless operation, whereas the group addition used with F is an operation which has a well-defined inverse.

We have noticed above that the work of the derived operator D in S is duplicated by the work of the boundary operator B in S^*. Hence D is characterized by the same three properties as B was above. There is of course a group operator[9] related to D as F is to B.

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[8] In *thesis* "star" and "fringe" were used in much the same sense as "aperture" and "derived" are used here. The term "derived" is due to Alexander, loc. cit.[4]

[9] The δ operator of Alexander, loc. cit.[4] Previously this was used in the indirect form F^*c^*; cf. *thesis*.

4. **Closed and open mappings.** Consider two cells spaces, S and S', and a correspondence between them. If the mapping $S \to S'$ is many-one (i.e. single-valued) we say it is *continuous* provided

$$x \prec y \qquad \text{implies} \qquad x' \preceq y'.$$

An equivalent condition is that the inverse one-many mapping $S' \to S$ preserves closed [open] sets; this is the familiar criterion for continuity in point topology. Although such continuity yields some results in cell topology (cf. *thesis* p. 208) it seems that we normally need the stronger types of continuity introduced in the next paragraph.

We say that the mapping $S \to S'$ is *closed* [*open*] if for each pair of cells x, y' it is true that

$$x \underset{[\preceq]}{\overset{\succ}{=}} \text{some } y \to y' \qquad \text{if and only if} \qquad x \to \text{some } x' \underset{[\preceq]}{\overset{\succ}{=}} y'.$$

A closed [open] mapping, if many-one, is continuous in the sense stated above and maps closed [open] sets into closed [open] sets.[10] But the definition of closed [open] mapping can be applied whether the mapping is many-one or not.

The following theorems follow directly from the definitions. A closed [open] mapping preserves closures [apertures]—i.e. the image of a closure [aperture] = the closure [aperture] of the image. Closed [open] mappings are transitive—i.e. the resultant $S \to S''$ of one closed [open] mapping $S \to S'$ followed by another $S' \to S''$ is closed [open]. The inverse $S' \to S$ of a closed [open] mapping $S \to S'$ is open [closed]. The mapping $S^* \to S'^*$ of dual complexes determined by a closed [open] mapping $S \to S'$ is open [closed].

A closed [open] one-many mapping may be "broken in two" as follows. We form a cell space S° by combining a copy Y° of the image Y' of an arbitrary closed [open] set Y of S with a copy Z° of the open [closed] complement $Z = S - Y$; given any pair of cells y°, z° of Y°, Z° we require that

$$y^\circ \underset{[\succ]}{\overset{\prec}{}} z^\circ \qquad \text{if and only if} \qquad y \underset{[\succ]}{\overset{\prec}{}} z \qquad \text{where } y \to y'.$$

Let $S \to S^\circ$ map Y on Y° as $S \to S'$ maps Y on Y' and map Z on Z° identically, and let $S^\circ \to S'$ map Y° on Y' identically and Z° on Z' as $S \to S'$ maps Z on Z'. It is easy to verify that $S \to S^\circ$ and $S^\circ \to S'$ are closed [open] one-many mappings which have $S \to S'$ as their resultant. The inverse mappings $S' \to S^\circ$ and $S^\circ \to S$ are open [closed] many-one mappings which have the open [closed] many-one mapping $S' \to S$ (the inverse of $S \to S'$) as their resultant. Hence an open [closed] many-one mapping may also be "broken in two."

5. **Subdivision of a cell space.** Let S be an ordinary simplicial complex

[10] Mappings thus characterized have also been recently introduced by Alexandroff, "Sur les suites d'espaces topologiques," *Comptes Rendus Acad. Sci.*, Paris, 200 (1935) 1708–1709.

and let S' be a simplicial subdivision of S. We denote by $S \to S'$ the one-many mapping which partitions each simplex of S into a collection of simplexes of S'. $S \to S'$ is a closed mapping; its many-one inverse $S' \to S$ is open. We denote by $S' \to S'' = S$ a simplicial, or barycentric, mapping of S' on $S'' = S$ determined by "projecting" each vertex of S' into a vertex of the simplex of S on which it lies. Then each simplex of S' goes into the simplex of S of which it is a piece or else into a simplex on the boundary of that simplex. $S' \to S''$ is a closed many-one mapping. The resultant $S \to S'' = S$ of $S \to S'$ followed by $S' \to S''$ is a closed many-many mapping of S into itself which is characterized by the fact that each simplex of S goes into itself modulo its boundary.

By abstraction from the last paragraph we get the following formal characterization of one cell space as a subdivision of another. We say that S' is a *subdivision* of S if there exist mappings

$$S \to S' \qquad \text{one-many and closed}$$

$$S' \to S'' = S \qquad \text{many-one and closed}$$

whose resultant $S \to S'' = S$ is such that

$$x \to x'' = x \qquad \text{mod } Bx.$$

This notion of subdivision is necessarily quite crude since our present tools are very rudimentary; later, however, we shall be able to give it more content.

Since closed mappings are transitive, subdivision is transitive. Moreover subdivision may be "broken in two" by interpolating a cell space S° of the type described in §4—provided of course that more than one cell of S is partitioned.

The notion of subdivision has a dual which we call *expansion* for want of a better term.[11] It is defined by substituting "open" for "closed" and "Dx" for "Bx" in the definition of subdivision. From our definitions it follows that a cell space S' is an expansion of a cell space S if and only if S'^* is a subdivision of S^*. We can accordingly obtain an expansion of S by passing first to the dual S^*, then to a subdivision S'^* of this dual, and finally to $(S'^*)^* = S'$.

6. Incidence and correspondence numbers. Let us assign to each cell x a *dimension* dim x subject to the condition

$$\dim x < \dim y \qquad \text{if } x \prec y.$$

The values of the single-valued function dim x are elements of an ordered set, the "dimension set." Given an element p of this ordered set we denote by $p \mp 1$ the predecessor (successor) of p, i.e. the last element $< p$ (the first element $> p$), if it exists.

With each pair of cells x_{p-1}, x_p of successive dimensions of S we associate an

[11] For implicit use of this notion see *thesis*, p. 210 (the 0-weld) and p. 220 (the complex $M^{(1)*}$).

integer $[x_{p-1} : x_p] = [x_p : x_{p-1}]$, called their *incidence number*, subject to the condition

$$x_{p-1} < x_p \qquad \text{if} \qquad [x_{p-1} : x_p] \neq 0.$$

Intuitively the incidence number of x_{p-1} and x_p represents the algebraic number of times x_{p-1} is on the boundary of x_p. We agree that it is permissible to change the sign of an incidence number $[x_{p-1} : x_p]$ at pleasure, provided we change at the same time the signs of all the incidence numbers $[y_{p-1} : x_p]$, $[x_p : y_{p+1}]$ which involve x_p, or else the signs of all those which involve x_{p-1}. This operation is described as changing the *orientation* of x_p, or of x_{p-1}.

Given a correspondence between S and S', we associate with each pair of cells x_p, x_p' of the same dimensions, the one cell from S and the other from S', an integer $[x_p : x_p'] = [x_p' : x_p]$, called their *correspondence number*, subject to the condition that

$$x_p \text{ corresponds to } x_p' \qquad \text{if} \qquad [x_p : x_p'] \neq 0.$$

Intuitively the correspondence number of x_p and x_p' represents the algebraic number of times x_p corresponds to x_p'. We agree that the sign of this correspondence number must be changed if the orientation of either x_p or x_p' is changed.

We form various secondary numbers from our incidence and correspondence numbers as follows:

$$[x_{p-1} : x_{p+1}] = \sum [x_{p-1} : x_p]\,[x_p : x_{p+1}]$$
$$[x_{p\mp1} : x_p'] = \sum [x_{p\mp1} : x_{p\mp1}']\,[x_{p\mp1}' : x_p']$$
$$[x_p : x_p''] = \sum [x_p : x_p']\,[x_p' : x_p''].$$

In each case the summation extends over all values of the repeated cell-symbol. Of course these secondary numbers exist only if there are at most a finite number of non-zero terms in the sums.

7. **Abstract complex.** We say that a cell space S, whose cells are endowed with dimensions and incidence numbers, is an *(abstract) complex* C if for each pair of cells x_{p-1}, x_{p+1} of S the number $[x_{p-1} : x_{p+1}]$ exists and is zero. All the usual complexes of combinatorial topology fall within the scope of this definition.[12]

The dual C^* of a complex C is a complex. We assume that the dimensions of the cells x^* of C^* form a subset of our "dimension set" which is ordered by $>$ just as the subset of dimensions of C is ordered by $<$. This is indicated symbolically by writing $\dim x^* = n - p$ when $\dim x = p$. We assume that the incidence numbers of C^* are given by the rule

$$[x^*_{n-p} : x^*_{n-p+1}] = [x_{p-1} : x_p].$$

[12] Also some more general ones. For instance, the set of all singular cells on a space forms a complex in the sense of our definition. (This example was suggested by a remark of Professor Veblen).

Hence

$$[x^*_{n-p-1} : x^*_{n-p+1}] = 0 \qquad \text{if and only if} \qquad [x_{p-1} : x_{p+1}] = 0.$$

It may be easily verified that any closed [open] subset of a complex is a complex.

8. **Mappings of complexes.** We say that a mapping $C \to C'$ of one complex on another is *closed* [*open*] if it is closed [open] in the set-theoretic sense of §4 and if it satisfies the algebraic requirement that the numbers

$$[x_p : x'_{p\mp1}] = \sum [x_p : x'_p][x'_p : x'_{p\mp1}], \qquad [x'_{p\mp1} : x_p] = \sum [x'_{p\mp1} : x_{p\mp1}][x_{p\mp1} : x_p]$$

exist for each pair of cells x_p, $x'_{p\mp1}$ and be equal—i.e.

$$[x_p : x'_{p\mp1}] = [x'_{p\mp1} : x_p].$$

This algebraic condition for a closed [open] mapping is a direct analogue of the set-theoretic condition. In some cases the algebraic condition implies the set-theoretic one, for example, in simplicial complexes.

The resultant $C \to C''$ of one closed [open] mapping $C \to C'$ followed by another $C' \to C''$ is closed [open] (we assume that the numbers $[x_p : x''_p]$ exist). The inverse $C' \to C$ of a closed [open] mapping $C \to C'$ is open [closed]. The dual $C^* \to C'^*$ of a closed [open] mapping $C \to C'$ is open [closed] (assuming of course that $[x^*_{n-p} : x'^*_{n-p}] = [x_p : x'_p]$).

A closed [open] mapping $C \to C'$ which is one-many, or many-one, can be "broken in two" by the interpolation of a complex C° provided $C \to C'$ does not increase [decrease] dimensions, i.e.

$$\dim x \underset{[\leqq]}{\geqq} \dim x' \qquad \text{if} \qquad x \to x'.$$

To establish this we merely have to assign appropriate algebraic properties to the cell space S° of §4. We give dimensions to the cells of S° by the rule

$$\dim x^\circ = \dim \bar{x},$$

where $\bar{x}$ denotes the cell of Y' or Z of which x° is a copy. Due to the above proviso there can be no violation of the fundamental requirement that $p < q$ if $x^\circ_p \prec x^\circ_q$. We assign incidence numbers and correspondence numbers by the rules:

$$[x^\circ_{p-1} : x^\circ_p] = [\bar{x}_{p-1} : \bar{x}_p]$$

$$[x_p : x^\circ_p] = [x_p : \bar{x}_p], \qquad [x^\circ_p : x'_p] = [\bar{x}_p : x'_p].$$

Then S° is a complex C° and $C \to C^\circ$, $C^\circ \to C$ are closed [open] mappings which have $C \to C'$ as their resultant.

9. **Chains.** A *chain* of dimension p is a function $g(x_p)$ which is defined over

the p-cells of a complex C and whose values are elements of an Abelian group G (which we write additively). Using incidence numbers and correspondence numbers we are able to associate with a given chain $g(x_p)$ certain other chains:

$$g_{\mp}(x_p) = \sum g(x_p)\,[x_p : x_{p\mp1}] \qquad (p \mp 1)\text{-chain on } C$$
$$g'(x_p) = \sum g(x_p)\,[x_p : x'_p] \qquad p\text{-chain on } C'$$
$$g'_{\mp}(x_p) = \sum g(x_p)\,[x_p : x'_{p\mp1}] \qquad (p \mp 1)\text{-chain on } C'$$
$$g''(x_p) = \sum g(x_p)\,[x_p : x''_p] \qquad p\text{-chain on } C''.$$

These chains exist only if the sums involved have at most a finite number of non-zero terms. The $(p-1)$-chain $g_-(x_p)$ is called the *boundary* of $g(x_p)$ and the $(p+1)$-chain $g_+(x_p)$ is called the *derived*. If $g_{\mp}(x_p) = 0$ the chain $g(x_p)$ is said to be *closed*[13] [*open*]. Due to the condition $[x_{p-1} : x_{p+1}] = 0$ on a complex a boundary [derived] chain is closed [open].

Since it may be undesirable, or even impossible, to consider all chains based on a group G and complexes C, C', etc., we shall restrict ourselves to a class of "admissible" chains which satisfy at least the following requirements: 1) the boundary [derived] $g_{\mp}(x)$ of an admissible chain $g(x)$ exists and is admissible; 2) the image $g'(x)$ of an admissible chain exists and is admissible; 3) the admissible p-chains of a complex form an Abelian group; 4) every finite chain is admissible (i.e. the set $|g(x)|$ of cells x for which $g(x) \neq 0$ is finite); 5) every subchain of an admissible chain is admissible ($\bar{g}(x)$ is a subchain of $g(x)$ if $|\bar{g}(x)| \subset |g(x)|$). It is to be understood in 1) that "boundary" stands throughout or else "derived;" i.e. the class of chains admitted for boundary operations and that admitted for derived operations may well be different, and moreover may be based on different groups G.

If there exists an admissible chain $g(x_{p\pm1})$ such that

$$g_{\mp}(x_{p\pm1}) = g^1(x_p) - g^2(x_p)$$

we write

$$g^1(x_p) \sim_{\mp} g^2(x_p).$$

10. **Subdivision of complexes.** We say that a complex C' is a *subdivision* [*expansion*] of a complex C if there exist mappings

$$C \to C' \qquad \text{one-many and closed [open]}$$
$$C' \to C'' = C \qquad \text{many-one and closed [open]}$$

which combine so that under $C \to C'' = C$ (i.e. $C \to C' \to C'' = C$)

$$x \to x'' = x \qquad \text{mod } Bx \text{ [mod } Dx\text{]}$$
$$[x : x''] = 1 \qquad \text{for } x = x'',$$

[13] A term used by Alexander, loc. cit.[4]

and under $C' \to C''' = C'$ (i.e. $C' \to C'' = C \to C' = C'''$) any admissible

$$(1) \qquad g(x') \sim_{\mp} g''(x') \text{ on } \sum X' \text{ if } g_{\mp}(x') = g''_{\mp}(x').$$

Here $g''(x')$ and $g''_{\mp}(x')$ are the images under $C' \to C''' = C'$ of $g(x')$ and $g_{\mp}(x')$. $\sum X'$ is the minimal set, consisting of images X' under $C \to C'$ of cells x of C, which contains the set $|g(x')|$ of cells x' for which $g(x') \neq 0$; $\sum X'$ is a subcomplex of C' since it is an open subset of the closure $C \sum X'$ [closed subset of the aperture $A \sum X'$]. This definition of subdivision [expansion] is a direct extension of the set-theoretic one in §5 except possibly for condition (1) on chains under $C' \to C''' = C'$. We shall examine this condition.

Condition (1) implies that the image X' on C' of a cell x of C, under $C \to C'$, is "cell-like" in the sense that an admissible

$$(2) \qquad g(x') \sim_{\mp} e'(x) \quad \text{on } X' \quad \text{if } g_{\mp}(x') = 0 \quad \text{on } X',$$

where $e'(x)$ is the image under $C \to C'$ of an "elemental" chain $e(x)$ which vanishes on all cells except at most x; i.e. $e'(x) = 0$ or else a chain arising from x alone. For by (1) there exists an admissible chain $\bar{g}(x')$ on $C' - X'$ such that $\bar{g}_{\mp}(x') = g''_{\mp}(x') - g_{\mp}(x')$. Hence $g_{\mp}(x') + \bar{g}_{\mp}(x') = g''_{\mp}(x')$ which implies that $g(x') + \bar{g}(x') \sim_{\mp}$ some $g''(x')$. This establishes (2). If $C \to C'$ does not increase [decrease] dimensions, $g(x') \sim_{\mp} e'(x) \neq 0$ on X' implies $g(x') = e'(x)$ on X'.

On the other hand (2) may imply (1). This occurs if $C \to C'$ is one-one except for a finite number of cells $x^1, x^2, \cdots, x^k$, which we may suppose numbered so that $x^i \prec x^j$ $[x^i \succ x^j]$ if $i < j$. For then from any admissible $g(x')$ such that $g_{\mp}(x') = g''_{\mp}(x')$ we have by (2) that

$$\begin{aligned}
&g(x') \sim_{\mp} g^1(x') + e'(x^1) \quad \text{on } \textstyle\sum X', && g^1(x') \text{ on } \textstyle\sum X' - X^{1'} \\
&g^1(x') \sim_{\mp} g^2(x') + e'(x^2) \quad \text{on } \textstyle\sum X' - X^{1'}, && g^2(x') \text{ on } \textstyle\sum X' - \sum_1^2 X^{i'} \\
&\qquad\vdots \qquad\qquad\qquad\quad \vdots && \qquad\vdots \\
&g^{k-1}(x') \sim_{\mp} g^k(x') + e'(x^k) \text{ on } \textstyle\sum X' - \sum_1^{k-1} X^{i'}, && g^k(x') \text{ on } \textstyle\sum X' - \sum_1^{k} X^{i'}.
\end{aligned}$$

This yields (1) since $g^k(x')$ is invariant under $C' \to C''' = C'$. Hence (2) is equivalent to (1) for finite complexes. A similar argument based on dimensions rather than individual cells can be formulated if admissible chains satisfy the requirement: 6) $\sum g(x)$ is admissible if it exists and if $\sum g_{\mp}(x)$ exists and is admissible.

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APPROXIMATIONS WITH HELP OF CERTAIN FRACTIONS

BY I. VINOGRADOW

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In a previous paper[1] I have proved the following theorem

Let n be an integer > 1 and ϵ be an arbitrary small positive constant. There exists a number L depending only on n such that for any $Q > 1$ and for any real number α it is possible to satisfy for integral z and m the system of inequalities

$$| \alpha z^n - m | < LQ^{-\rho}$$

$$0 < z < Q$$

where

$$\rho = \frac{n}{n2^{n-1} + 1} - \epsilon.$$

In the present paper I give a better result for the value of ρ. Here I obtain

$$\rho = \frac{1}{15n^2 \log 10n},$$

which for $n > 15$ is far better than the previous value of ρ.

I apply here the same method as in my recent papers.[2] The proof of the theorem indicated above is performed without use of Weyl's approximations while the method elaborated in the paper previously mentioned is based on those approximations. The result of this paper can be improved and the method is applied to more general problems of the same type.

1. **Notation.** n is an integer $\geqq 10$;

$$k = [n \log 10n + 1]; \qquad \lambda = \frac{1}{k}; \qquad \tau = 2k^2;$$

$$\nu = \frac{1}{n}; \qquad \sigma = (1 - \nu)^k; \qquad \delta = \frac{2}{15n^2 \log 10n}.$$

$c, c_1, \cdots$ are constant numbers depending only on n; $\theta, \theta_1, \cdots$ are numbers with modules not exceeding unity. $A \ll B$ or $B \gg A$ where B is positive, means that

$$| A | \leqq cB.$$

If z is a real number, then (z) denotes the distance of z to the next integer.

[1] *Bulletin de l'Académie des Sciences de l'URSS*, 1927, p. 567–578 /Russian/.

[2] *C. R. de l'Académie des Sciences de l'URSS*, 1934, v. II, No. 6; Ibid., 1934, v. III, No. 1; Ibid., 1934, v. IV, No. 5–6.

α is real and

$$\alpha = \frac{a}{q} + \frac{\theta}{q^2}; \qquad (a, q) = 1; \qquad q \geqq 1.$$

$$\Delta = q^{-\delta}; \qquad M = [\Delta^{-1-\lambda}].$$

$$P = P_1 = \left[\left(\frac{q}{2}\right)^{\nu}\right]; \qquad P_2 = \left[\frac{1}{2} M^{-\nu} P_1^{1-\nu}\right]; \cdots; \qquad P_k = \left[\frac{1}{2} M^{-\nu} P_{k-1}^{1-\nu}\right].$$

2. **Some properties of the symbols introduced above.** We have

$$k > 4.6n; \qquad M < q^{\frac{47}{46}\delta} < q^{\frac{\nu^2}{33}}.$$

Then we find

$$\begin{aligned} &P_2 \gg M^{-[1-(1-\nu)]} P^{1-\nu}, \\ &P_3 \gg M^{-[1-(1-\nu)^2]} P^{(1-\nu)^2}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &P_k \gg M^{-[1-(1-\nu)^{k-1}]} P^{(1-\nu)^{k-1}} \gg q^{-\frac{\nu^2}{33} + \nu(1-\nu)^{k-1}}. \end{aligned} \tag{1}$$

But we have

$$\log (1-\nu)^{k-1} > -\left(1 + \frac{\nu}{2-2\nu}\right) \log 10n > -\log 10n - 0.26 > -\log 13n,$$

$$\nu(1-\nu)^{k-1} - \frac{\nu^2}{33} > \frac{\nu^2}{13} - \frac{\nu^2}{33} > \frac{\nu^2}{22}.$$

Therefore if we choose c_0 large enough we obtain for all $q \geqq c_0$,

$$P_k \gg q^{\frac{\nu^2}{22}}.$$

We deduce from (1).

$$P_1 \cdots P_k \gg M^{-k+n-n\sigma} q^{1-\sigma}$$

and find

$$\sigma < (1-\nu)^{n \log 10n} < e^{-\log 10n} = \frac{\nu}{10}.$$

3. **Estimation of the module of a trigonometric polynomial.** The numbers $m_1, \cdots, m_k$ we shall regard for the present as constant positive integers not exceeding M. Let the numbers $x_1, \cdots, x_k, y$ run over the set of values

$$\begin{aligned} x_1 &= P_1, P_1 + 1, \cdots, 2P_1 - 1 \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ x_k &= P_k, P_k + 1, \cdots, 2P_k + 1 \\ y &= 1, 2, \cdots, P. \end{aligned}$$

It is obvious that the numbers

$$\xi = m_1 x_1^n + \cdots + m_k x_k^n$$

are different from each other; none of them exceeds $N = Mqk2^n$, and the number of them is equal to $P_1 \cdots P_k$. We have

$$\sum_y \Big| \sum_\xi e^{2\pi i a y^n \xi} \Big| \ll \Big[P \sum_y \sum_\xi \sum_{\xi_1} e^{2\pi i a y^n (\xi - \xi_1)} \Big]^{\frac{1}{2}}$$

$$\ll \Big[PP_1 \cdots P_k \sum_{u=-N}^{N} \Big| \sum_y e^{2\pi i a y^n u} \Big| \Big]^{\frac{1}{2}}.$$

$$\sum_u \Big| \sum_y e^{2\pi i a y^n u} \Big| \ll \Big[Mq \sum_y \sum_{y_1} \Big| \sum_u e^{2\pi i a (y^n - y_1^n) u} \Big| \Big]^{\frac{1}{2}}$$

$$\ll \Big[Mq \sum_h \min \Big(Mq, \frac{1}{(\alpha h)} \Big) \Big]^{\frac{1}{2}}; \; h = y^n - y_1^n .$$

But

$$-\frac{q}{2} < h < \frac{q}{2}; \qquad \alpha h = \frac{ah}{q} + \frac{\theta h}{q^2} = \frac{ah}{q} + \frac{\theta'}{2q}.$$

The number of solutions of the Diophantine equation

$$y^n - y_1^n = h$$

is $\leqq P$.

Let s denote the absolute least residue of ah modulo q; in the case when $h \gtrless 0$ we have

$$\Big(\frac{ah}{q} + \frac{\theta'}{2q} \Big) \gg \frac{s}{q} - \frac{1}{2q} \gg \frac{s}{q},$$

the same s recurring $\ll P$ times. Therefore it appears that

$$\sum_u \Big| \sum_y e^{2\pi i a y^n u} \Big| \ll \Big[Mq \Big(PMq + P \sum_{s=1}^{q} q/s \Big) \Big]^{\frac{1}{2}} \ll MqP^{\frac{1}{2}},$$

$$\sum_y \Big| \sum_\xi e^{2\pi i a y^n \xi} \Big| \ll P^{\frac{1}{2}} [P_1 \cdots P_k Mq]^{\frac{1}{2}}.$$

4. **A peculiar periodic function.** Let β be a real number. There exists a periodic function $\psi(z)$ with the period 1 possessing the following properties:

$0 \leqq \psi(z) \leqq 1$ in the intervals $m + \beta - \Delta \leqq z \leqq m + \beta + \Delta$ where m is any integer,

$\psi(z) = 0$ in the other intervals;

$\psi(z)$ can be developed in a Fourier series:

$\psi(z) = \Delta + \varphi(z)$; $\varphi(z) = \sum_{m=1}^{\infty} (A_m \cos 2\pi m z + B_m \sin 2\pi m z)$, where

$$A_m \ll \Delta; \qquad B_m \ll \Delta, \text{ if } m \leqq \Delta^{-1},$$

$$A_m \ll \frac{1}{\Delta^r m^{r+1}}; \qquad B_m \ll \frac{1}{\Delta^r m^{r+1}}, \text{ if } m > \Delta^{-1}.$$

5. **The proof of the main theorem.** Put

$$S_{y,s} = \sum_{x_s} \varphi(\alpha y^n x_s^n),$$

$$S = \sum_y \left| S_{y,1} \cdots S_{y,k} \right|.$$

It is easy to see that in order to estimate the module of the sum S we have to estimate the module of a sum of the type

$$R = \sum_{m_1} \cdots \sum_{m_k} D_{m_1} \cdots D_{m_k} T,$$

$$T = \sum_y \left| \sum_\xi e^{2\pi i \alpha y^n \xi} \right|,$$

where for every given system of values $m_1, \cdots, m_k$ the number ξ runs over the set of values

$$\xi = m_1 x_1^n + m_2 x_2^n + \cdots + m_k x_k^n,$$

the numbers x_s running over the set of values described in 3.

The numbers D_m satisfy the conditions

$$D_m = \Delta, \text{ if } m \leqq \Delta^{-1},$$

$$D_m = \frac{1}{\Delta^r m^{r+1}}, \text{ if } m > \Delta^{-1}.$$

If no m_s exceeds M, we have by 3,

$$T \ll P^{\frac{1}{2}} [P_1 \cdots P_k M q]^{\frac{1}{2}}.$$

Since for D_m we have the bounds indicated above we find that the sum of all terms of R corresponding to this case is

$$\ll P^{\frac{1}{2}} [P_1 \cdots P_k M q]^{\frac{1}{2}}.$$

If at least one of the numbers m_s exceeds M we apply the evident estimation

$$T \ll P P_1 \cdots P_k;$$

as

$$\sum_{m>M} D_m = \sum_{m>M} \frac{1}{\Delta^r m^{r+1}} \ll \frac{1}{(\Delta M)^r} = \Delta^{2k},$$

it is easy to see that the sum of the terms of R corresponding to this case is

$$\ll P P_1 \cdots P_k \Delta^{2k}.$$

Thus we have

$$R \ll P^{\frac{1}{2}}[P_1 \cdots P_k M q]^{\frac{1}{2}} + PP_1 \cdots P_k\Delta^{2k}$$

$$\ll PP_1 \cdots P_k\Delta^k\left(\Delta^{-k}P^{-\frac{1}{2}}\left[\frac{Mq}{P_1 \cdots P_k}\right]^{\frac{1}{4}} + \Delta^k\right)$$

$$\ll PP_1 \cdots P_k\Delta^k\left(\Delta^{-k}M^{\frac{k-n+n\sigma+1}{2}}q^{\frac{\sigma}{2}-\frac{\nu}{4}} + \Delta^k\right)$$

$$\ll PP_1 \cdots P_k\Delta^k(q^{\frac{1}{2}\delta\{2k+(1+\lambda)(k-0.8n)\}-0.2\nu} + q^{-k\delta}).$$

But

$$\tfrac{1}{2}\delta\{2k + (1 + \lambda)(k - 0.8n)\} - 0.2\nu$$

$$< \frac{3n \log 10n - 0.4n}{15n^2 \log 10n} - 0.2\nu = -\frac{\delta n}{5}.$$

Therefore

$$R \ll PP_1 \cdots P_k\Delta^k q^{-\mu}; \qquad \mu = -\frac{\delta n}{5},$$

and

$$S \ll PP_1 \cdots P_k\Delta^k q^{-\mu}.$$

Hence there exists at least one value of y such that

$$S_{y,1} \cdots S_{y,k} \ll P_1 \cdots P_k\Delta^k q^{-\mu}.$$

Therefore there exists at least one value of s such that

$$S_{y,s} \ll \Delta P_s q^{-c_1}, \qquad c_1 = \frac{\mu}{k};$$

hence we have

$$\sum_{x_s} \psi(\alpha y^n x_s^n) - \Delta P_s \ll \Delta P_s q^{-c_1}.$$

Thus we see that there exists a $c_2 \geqq c_0$ such that for all $q \geqq c_2$ we have

$$\sum_{x_s} \psi(\alpha y^n x_s^n) \gg \Delta P_s.$$

Therefore there exists at least one value of $x_s = x$ such that

$$\psi(\alpha x^n y^n) > 0,$$

and this is possible only when

$$\alpha x^n y^n = m + \beta + \theta_2\Delta,$$

where m is integral. As

$$1 \leqq xy \leqq q^{2\nu}$$

we can formulate the result as follows.

THEOREM. *Let β be real. It is possible to find a c' such that in representing the real number α in the form*

$$\alpha = \frac{a}{q} + \frac{\theta}{q^2}; \qquad (a, q) = 1,$$

it is possible to satisfy the system of inequalities

$$| \alpha z^n - \beta - m | < c' q^{-\frac{1}{2}},$$
$$0 < z < q^{2\nu}$$

for integral z and m.

REMARK. This theorem holds and its proof differs little from the above, if in the product $z = xy$ the numbers x and y are regarded as primes.

6. **Corollaries.** Put in particular $\beta = 0$. For any $Q > 1$ we can represent the number α in the form

$$\alpha = \frac{a}{q} + \frac{\theta''}{q\tau}; \qquad (a, q) = 1; \qquad 0 < q \leqq \tau; \qquad \tau = Q^{\frac{n}{2}}.$$

If $q > [Q]^{\frac{1}{2}}$ then by the theorem we have

$$| \alpha z^n - m | < c' Q^{-\rho}; \qquad \rho = \frac{\delta}{2} = \frac{1}{15n^2 \log 10n}, \tag{2}$$
$$0 < z < Q.$$

If $q \leqq [Q]^{\frac{1}{2}}$ then we have

$$\alpha q^n = \left(\frac{a}{q} + \frac{\theta''}{q\tau}\right) q^n = aq^{n-1} + \theta_3 Q^{-\frac{1}{2}}.$$

If we put

$$z = q; \qquad m = aq^{n-1},$$

we see that the inequalities (2) may be satisfied also in this case and we obtain therefore the result mentioned at the beginning of this paper.

We can also write the inequalities (2) in the form

$$\left| \alpha - \frac{m}{z^n} \right| < \frac{c'}{z^n Q^\rho}, \qquad 0 < z < Q;$$

which gives the approximation to the number α by a certain fraction with a perfect n^{th} power in the denominator.

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ÜBER DIE BESTE ANNÄHERUNG VON FUNKTIONEN EINER GEGEBENEN FUNKTIONENKLASSE

BY A. KOLMOGOROFF

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1. **Allgemeine Fragestellung.** Wir setzen voraus, dass eine bestimmte Entfernungsdefinition für den in Betracht kommenden Funktionen erklärt ist. Betrachtet man das Problem der Annäherung einer Funktion f durch lineare Formen

$$\varphi = c_1\varphi_1 + c_2\varphi_2 + \cdots + c_n\varphi_n$$

mit festen Funktionen $\varphi_1, \varphi_2, \cdots, \varphi_n$, so entsteht die Aufgabe (das Tchebycheffsche Problem) mit Hilfe einer geeigneten Wahl der Koeffizienten $c_1, c_2, \cdots, c_n$, die Entfernung $\rho(f, \varphi)$ möglichst klein zu machen. Ohne die Existenz und Eindeutigkeitsfrage zu diskutieren bezeichnen wir mit $E_n(f)$ die Untere Grenze von $\rho(f, \varphi)$. Für eine Klasse F von Funktionen f bezeichnen wir weiter mit $E_n(F)$ die obere Grenze von $E_n(f)$ für alle f aus F. Die Grösse $E_n(F)$ ist durch die Klasse F und die Funktionen $\varphi_1, \varphi_2, \cdots, \varphi_n$, bestimmt. Wir Stellen jetzt eine neue Aufgabe: *bei gegebenen F und n durch die Wahl von Funktionen $\varphi_1, \varphi_2, \cdots, \varphi_n$, das Minimum von $E_n(F)$ zu erreichen.* Unabhängig von der Existenz dieses Minimums bezeichnen wir mit $D_n(F)$ die untere Grenze von $E_n(F)$.

Jeder Funktionenklasse F entspricht also eine Folge ganz bestimmter nicht negativer (eventuell unendlicher) Grössen

$$D_1(F) \geqq D_2(F) \geqq \cdots \geqq D_n(F) \geqq \cdots .$$

Wenn $E_n(F)$ wirklich das Minimum $D_n(F)$ erreicht, so kann man auch die Eindeutigkeitsfrage stellen und zwar in folgenden Sinne: sind die Funktionen $\varphi_1, \varphi_2, \cdots, \varphi_n$, welche das Minimum $D_n(F)$ realisieren, *bis auf eine lineare Transformation* eindeutig bestimmit, oder nicht?

2. **Spezialisierung.** Wir betrachten weiter nur die Funktionen einer reellen Veränderlichen, welche auf der Strecke (0, 1) definiert sind. Als Entfernung $\rho(f, \varphi)$ führen wir

$$\rho(f, \varphi) = \left[\int_0^1 (f - \varphi)^2\, dx\right]^{\frac{1}{2}}$$

ein. In diesen Voraussetzungen gelang es mir für die folgenden Funktionenklassen einfache Resultate zu erhalten:

$F_p (p \geqq 1)$ besteht aus allen p-fach differentierbaren Funktionen f mit

$$\int_0^1 \{f^{(p)}\}^2\, dx \leqq 1 .$$

$F_p^*(p \geqq 1)$ besteht aus allen Funktionen von F_p, welche den Periodizitätsbedingungen $f(0) = f(1), f'(0) = f'(1), \cdots, f^{(p-1)}(0) = f^{(p-1)}(1)$, genügen.

3. **Ergebnisse.** Im Falle F_p^* ist

$$D_{2m-1}(F_p^*) = D_{2m}(F_p^*) = \left(\frac{1}{2\pi m}\right)^p, \qquad m = 1, 2, 3, \cdots;$$

das Minimum $D_n(F_p^*)$ ist dabei immer erreichbar; die besten Funktionen φ_1, $\varphi_2, \cdots, \varphi_n$, sind im Falle $n = 2m + 1$ bis auf eine lineare Transformation eindeutig bestimmt und zwar sind es

$$1, \sqrt{2}\sin 2\pi kx, \qquad \sqrt{2}\cos 2\pi kx, \qquad k = 1, 2, \cdots, m.$$

Im Falle F_1 ist

$$D_n(F_1) = \frac{1}{\pi n}, \qquad n = 1, 2, 3, \cdots;$$

die besten Funktionen $\varphi_1, \varphi_2, \cdots, \varphi_n$, sind dabei bei jedem n bis auf eine lineare Transformation eindeutig bestimmt, es sind

$$1, \quad \sqrt{2}\cos \pi kx, \qquad k = 1, 2, \cdots, n-1$$

So einfach lassen sich die Resultate im Falle F_p bei einem beliebigen p nicht ausdrücken: sie hängen von einem speziellen, sich mit p ändernden, orthonormalen System von Funktionen

$$U_{p1}, U_{p2}, \cdots, U_{pr}, \cdots,$$

welches im 4 näher bestimmt wird, ab. Es wird bewiesen, das im Falle F_p

$$D_n(F_p) = +\infty, \qquad n = 1, 2, \cdots, p-1,$$

$$D_n(F_p) = \frac{1}{\sqrt{\lambda_{n-p+1}^{(p)}}}, \qquad n = p, p+1, p+2, \cdots,$$

gilt, wobei $\lambda_k^{(p)}$ die im 4 definierten Eigenwerte sind. Assymptotisch gilt dabei (bei festem p)

$$D_n(F_p) = \left(\frac{1}{\pi n}\right)^p + O\left(\frac{1}{n}\right)^{p+1}.$$

Die besten Funktionen $\varphi_1, \varphi_2, \cdots, \varphi_n$, sind im Falle F_p für $n \geqq p$ bis auf eine lineare Transformation eindeutig bestimmt, es sind die n ersten Funktionen U_{pk}, $k = 1, 2, \cdots, n$. Für $p = 1$ ist dieses Ergebniss mit dem oben gesagten im Einklang, da es

$$U_{11} = 1, \qquad U_{1n} = \sqrt{2}\cos \pi(n-1)x, \qquad n = 1, 2, 3, \cdots,$$

gilt.

4. **Definition und Eigenschaften der Funktionen U_{pn}.** Für $n = 1, 2, \cdots, p$, setzen wir U_{pn} gleich dem mit

$$\int_0^1 U_{pn}^2 \, dx = 1$$

normierten Legendreschen Polynome $n - 1$-es Grades. Für $n = p + k, k > 0$, definieren wir U_{pn} als die ebenso normierte Lösung des Systems

$$\text{(A)} \qquad \begin{cases} (-1)\, y^{(2p)} - \lambda y = 0, \\ y^{(p)}_{x=0,1} = y^{(p+1)}_{x=0,1} = \cdots = y^{(2p-1)}_{x=0,1} = 0, \end{cases}$$

welche dem k-ten vom Null verschiedenen Eigenwert $\lambda_k^{(p)}$ des Systems (A) entspricht (Selbstverständlich denkt man sich dabei $\lambda_k^{(p)}$ nach ihrem Grösse geordnet). Diese Definition setzt voraus, dass alle vom Null verschiedenen Eigenwerte von (A) positiv und einfach sind, was tatsächlich auch zutrifft.[1]

Unter den Eigenschaften des Systems $\{U_{pn}\}$ (bei einem festen p) wollen wir die folgenden hervorzuheben:

1. Dieses System ist orthonormal und vollständig.
2. Die p-fachen Ableitungen

$$u_{pn} = \frac{d^p}{dx^p} U_{pn}$$

bilden ein orthogonales (obwohl nicht normiertes) Funktionensystem.

3. Für jede p-fach differentierbare Funktion f mit

$$\int_0^1 \{f^{(p)}\}^2 \, dx < +\infty$$

folgt aus

$$f \sim \sum a_n U_{pn}$$

die Gültigkeit der Entwicklung

$$f^{(p)} \sim \sum a_n u_{pn}.$$

Es ist bemerkenswert, dass *die Eigenschaften* 1–3 *das System* $\{U_{pn}\}$, *bis auf der Ersetzung der Legendreschen Polynomen durch irgendwelche andere* p *orthonormale Polynome vom Grade* $< p$, *eindeutig bestimmen.*

Es sei noch zum Schluss erwähnt, dass bei $n \leqq p$ die Funktionen u_{pn} identisch verschwinden und dass bei $n = p + k, k > 0$, man u_{pn} als diejenige Lösung des Systems

$$\text{(B)} \qquad \begin{cases} (-1)^p\, y^{(2p)} - \lambda y = 0, \\ y_{x=0,1} = y'_{x=0,1} = \cdots = y^{(p-1)}_{x=0,1} = 0, \end{cases}$$

[1] Vgl. M. Krein, "Sur une classe des opérateurs differentiels," im Druck in *Bull. de l'Acad. des Sc. de l'URSS* (1935). Herrn M. Krein bin ich für die entsprechende briefliche Mitteilung dankbar.

erhält, welche dem k-ten vom Null verschiedenen Eigenwert $\lambda_k^{(p)}$ entspricht (die Eigenwerte von (A) und (B) sind offenbar identisch). Statt der gewöhnlichen Normierungsbedingung braucht man hier

$$\int_0^1 u_{pn}^2 \, dx = \lambda_k^{(p)} .$$

5. **Geometrische Andeutung zum Beweise.** Zuerst machen wir einige Bemerkungen zur allgemeinen Fragestellung. Die Menge aller Funktionen $\varphi = c_1\varphi_1 + c_2\varphi_2 + \cdots + c_n\varphi_n$ mit festen Funktionen $\varphi_1, \varphi_2, \cdots, \varphi_n$, bildet einen n-dimensionalen linearen Unterraum Φ_n des Raumes R aller in Betracht kommenden Funktionen. $E_n(f)$ ist die Entfernung des Punktes f von der Menge Φ_n. Die Grösse $E_n(F)$ ist also ein natürliches Mass der Abweichung der Menge F von dem linearen Raum Φ_n. Die untere Grenze $D_n(F)$ der Grössen $E_n(F)$ kann man, folglich, als *die n-te Breite* der Menge F zu bezeichnen.

Unsere speziellen Betrachtungen beziehen sich auf den Hilbertschen Raum H aller quadratisch integrierbaren Funktionen. Die Mengen F_p und F_p^* lassen sich als elliptische Zylinder dieses Raumes auffassen. Es ist geometrisch fast slbstverständlich, dass man in diesem Falle das Minimum $D_n(F)$ der Grösse $E_n(F)$ erreicht, wenn man den n-dimensionalen linearen Raum Φ_n betrachtet, welche von den n grössten Hauptachsen des Zylinders F aufgespannt ist. Diese Hauptachsen bestimmt man leicht nach den klassischen Regeln der Variationsrechnung. Insbesondere stimmen im Falle F_p die Richtungen der Hauptachsen mit den oben definierten Funktionen U_{pk} überein. Die entsprechenden Halbachsenlängen sind bei $k \leqq p$ unendlich und bei $k = p + i$, $i > 0$, den Grössen $1/\sqrt{\lambda_i^{(p)}}$ gleich.

KLJASMA BEI MOSKAU.

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ON A CERTAIN TOPOLOGY FOR RINGS OF OPERATORS

BY J. v. NEUMANN

(Received June 5, 1935)

1. The notation to be used in this paper agrees with the one which is described in §1.1 of the following paper *On rings of operators.*

The quotations refer to the bibliography of that paper, quotations from the paper itself will be marked RO.

We will make free use of the results of Chapters I and II in RO, excepting however Lemma 2.3.7 in §2.3. This Lemma is based on the results of the present paper, but it will not be used in the other parts of those sections.

Our subject is the introduction of a new topology in the ring of operators **B** (cf. (18), p. 370) of a space $\mathfrak{H}$ (cf. RO, §1.1, point (b)), which is more appropriate for use under certain conditions than the three topologies of **B** defined in (18), pp. 378–388. Apart from its independent interest, we need it for application in RO, Lemma 2.3.7 in §2.3, and in Chapter III.

This topology is technically superior to the three above mentioned ones in this: For the "uniform" topology in **B** the results of our §4 are true, but those of our §3 are not. For the "strong" as well as for the "weak" topology in **B**, §3 is true, but §4 is not. (We will give an example for this in §5.) The "strongest" topology however, as we are going to define it, will have all these properties.

2. Consider a space $\mathfrak{H}$ satisfying the postulates A, B, C, E of (16) (pp. 64–66), that is, a Hilbert space or a finite dimensional Euclidean space (cf. (16), §1, point (b)). (Of course only the first case is interesting. In the second it is easy to see that the "weak" and the "uniform" topology coincide, and therefore all others do too.) Form the ring of all everywhere defined, bounded, linear and closed operators in $\mathfrak{H}$, **B**. Define the *strongest topology* in **B** as follows:

Strongest topology in **B**. If $A_0 \in \mathbf{B}$, $\epsilon > 0$, and if $f_1, f_2, \cdots$ is a sequence from $\mathfrak{H}$ with a finite $\sum_1^\infty \| f_i \|^2$, then let $U(A_0; f_1, f_2, \cdots, \epsilon)$ be the set of all $A \in \mathbf{B}$ with $\sum_1^\infty \| (A - A_0) f_i \|^2 < \epsilon$. All $U(A_0; f_1, f_2, \cdots, \epsilon)$ of this sort, for a given A_0, are the neighborhoods of A_0.

It is easy to show that this is a topology in the sense of Fréchet-Hausdorff, fulfilling his postulates $\alpha) - \delta)$ (cf. (18), p. 378; cf. for reference footnote 28) eod). The countability axioms are not fulfilled, and we will even show in §5 that a condensation point of a set $S \subset \mathbf{B}$ need not be a limit point of some convergent subsequence of S.

The strongest topology is stronger than the strong but weaker than the uniform topology in **B** (cf. (18), pp. 381–384). In fact: Every $U_3(A_0, f_1, \cdots, f_\alpha, \epsilon)$

coincides with $U(A_0; f, \cdots, f_s, 0, 0, \cdots, \epsilon)$ and every $U(A_0, f_1, f_2, \cdots, \epsilon)$ contains $U_\delta(A, \epsilon/\sum_1^\infty \| f_i \|^2)$. That it is really different from the uniform topology follows from its violating Hausdorff's first countability axiom; as to the strong topology we will give an example in §5.

Thus every strongly, and a fortiori every weakly closed subset S of **B** is strongest closed too. (Note that if a topology becomes stronger, the notion of closedness with respect to it becomes broader.)

Every strongest convergent sequence of operators is strongly convergent too, and therefore uniformly bounded (cf. (18), p. 382).

Every strongly convergent sequence of operators is strongest convergent also: Let the sequence be $A_1, A_2, \cdots$; it must be uniformly bounded, say $\| A_n f \| \leqq C \| f \|$, and let the limit be A. Let $f_1, f_2, \cdots$ be a sequence with a finite $\sum_1^\infty \| f_i \|^2$. Choose a j with $\sum_{i=j+1}^{\infty} \| f_i \|^2 < \frac{\epsilon}{8C^2}$. If $i = 1, \cdots, j$ every $A_n f_i$ converges strongly to $A f_i$. So for any sufficiently great n

$$\| (A_n - A) f_i \| \leqq \left(\sqrt{\frac{\epsilon}{2j}} \right)^{\frac{1}{2}}$$

for $i = 1, \cdots, j$. Then

$$\sum_{i=1}^{\infty} \| (A_n - A) f_i \|^2 = \sum_{i=1}^{j} \| (A_n - A) f_i \|^2 + \sum_{i=j+1}^{\infty} \| (A_n - A) f_i \|^2$$

$$\leqq j \left(\sqrt{\frac{\epsilon}{2j}} \right)^2 + 4C^2 \left(\sum_{i=j+1}^{\infty} \| f_i \|^2 \right) \leqq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon ;$$

that is $A_n \in U(A; f_1, f_2, \cdots, \epsilon)$ proving our statement. Thus the entire difference between strong and strongest topology is due to the difference between condensation points and limits of convergent sequences. But, as we saw, this difference is inherent to such topologies in **B**; and for the considerations which we are going to make in §3, it is decisive.

3. We wish to prove:

If a subset **M** of **B** has the property that $A, B \in \mathbf{M}$ imply αA, A^*, $A + B$, $AB \in \mathbf{M}$ and if **M** is strongest closed, then it is weakly closed too. Thus it is a ring. (Cf. (18), p. 388, Definition 1.)

Note that every ring **M** does obviously have the above properties, therefore they are characteristic for a ring.

We will prove the above statement by reconsidering the proof of Theorem 5 in (18), pp. 393–396.

Consider the operator E_0 defined for the ring **R(M)** in (18), p. 393. Consider an arbitrary sequence $f_1^0, f_2^0, \cdots$ in $\mathfrak{H}$ with a finite $\sum_1^\infty \| f_i^0 \|$. Form the space $\infty \otimes \mathfrak{H}$ of all sequences $f_1, f_2, \cdots$ in $\mathfrak{H}$ with a finite $\sum_1^\infty \| f_i \|^2$. $\infty \otimes \mathfrak{H}$ is a space of the same type as $\mathfrak{H}$, if we define the inner product by

$$(< f_1, f_2, \cdots >, < g_1, g_2, \cdots >) = \sum_{i=1}^{\infty} (f_i, g_i)$$

(the sum is absolutely convergent). Cf. RO, §2.4.

Let correspond to every $A \in \mathbf{B}$ the point $< Af_1^0, Af_2^0, \cdots >$ in $\infty \otimes \mathfrak{H}$ (as A is bounded, $\| Af \| \leqq C \| f \|$, therefore, $\sum_1^\infty \| Af_i \|^2 \leqq C^2 \sum_1^\infty \| f_i^0 \|^2$ is finite), and call it the image of A. Let $\overline{\mathfrak{L}}$ be the set of the images of all $A \in \mathbf{M}$, and $\overline{\mathfrak{M}}$ the (strong) closure of $\overline{\mathfrak{L}}$; $\overline{\mathfrak{L}}$ is linear and $\overline{\mathfrak{M}}$ is linear and closed. Let $\bar{E}$ be the projection of $\overline{\mathfrak{M}}$.

$\bar{E}$ can be written as an operator matrix $< E_{t,s} >$, $t, s = 1, 2, \cdots$ (cf. RO, §2.4, Definition 2.4.2).

If $\bar{f} \in \overline{\mathfrak{L}}$, then $\bar{f} = < Af_1^0, Af_2^0, \cdots >, A \in \mathbf{M}$. Thus if $B \in \mathbf{M}$, then for $B^{(2)} = < \delta_{t,s} B >$ (cf. loc. cit., Lemma 2.4.4) $B^{(2)}\bar{f} = < BAf^0, BAf_2^0, \cdots > \in \overline{\mathfrak{L}}$. Thus $B^{(2)}$ maps $\overline{\mathfrak{L}}$ on part of $\overline{\mathfrak{L}}$, and therefore $\overline{\mathfrak{M}}$ on part of $\overline{\mathfrak{M}}$. Every $\bar{E}\bar{g} \in \overline{\mathfrak{M}}$, thus $B^{(2)}\bar{E}\bar{g} \in \overline{\mathfrak{M}}$, $\bar{E}B^{(2)}\bar{E}\bar{g} = B^{(2)}\bar{E}\bar{g}$, that is $\bar{E}B^{(2)}\bar{E} = B^{(2)}\bar{E}$. Replace $B \in \mathbf{M}$ by $B^* \in \mathbf{M}$, then $B^{*(2)} = B^{(2)*}$; applying now* to the above equation, we obtain $\bar{E}B^{(2)}\bar{E} = \bar{E}B^{(2)}$. So $B^{(2)}\bar{E} = \bar{E}B^{(2)}$, $< \delta_{t,s}B > < E_{t,s} > = < E_{t,s} > < \delta_{t,s}B >$; $BE_{t,s} = E_{t,s}B$ (cf. loc. cit., Lemma 2.4.5).

Thus every $E_{t,s}$ commutes with every $B \in \mathbf{M}$, that is $E_{t,s} \in \mathbf{M}'$. Now $\bar{f} = < f_1, f_2, \cdots > \in \overline{\mathfrak{M}}$ is characterized by $\bar{E}\bar{f} = \bar{f}$, that is by

$$f_t = \sum_{s=1}^{\infty} E_{t,s} f_s, \qquad \text{for} \qquad t = 1, 2, \cdots .$$

If $\bar{f}$ is the image of an $A \in \mathbf{B}$ (not necessarily $A \in \mathbf{M}$!), then we have $f_t \leftarrow Af_t^0$ and our condition becomes $Af_t^0 - \sum_1^\infty E_{t,s} Af_s^0 = 0$. If in particular $A \in \mathbf{M}''$, then it commutes with every $E_{t,s}$ ($\in \mathbf{M}'$) and so we can write

$$Aw_t^0 = 0 \qquad \text{where} \qquad w_t^0 = f_t^0 - \sum_{s=1}^{\infty} E_{t,s} f_s^0 .$$

If $A \in \mathbf{M}$ then $A^* \in \mathbf{M}$, thus the images of A, A^* belong to $\overline{\mathfrak{M}}$. Thus $Aw_t^0 = A^*w_t^0 = 0$; therefore $E_0 w_t^0 = 0$ (cf. (18), p. 393). Now if only $A \in \mathbf{M}''$ (remember that $\mathbf{M} \subseteq \mathbf{M}''$!) and $AE^0 = A$ then

$$Aw_t^0 = AE_0 w_t^0 = 0 .$$

Thus the image of A belongs to $\overline{\mathfrak{M}}$, and so for every $\epsilon > 0$ a $B \in \mathbf{M}$ exists, the image of which has a distance $< \epsilon$ from the image of A: $\sum_{i=1}^{\infty} \| (B - A)f_i^0 \|^2 < \epsilon$. In other words $B \in U(A, f_1^0, f_2^0, \cdots, \epsilon)$.

So if $A \in \mathbf{M}''$, $AE_0 = A$, then for every sequence $f_1, f_2, \cdots$ with a finite $\sum_1^\infty \| f_i^0 \|^2$ and for every $\epsilon > 0$ a $B \in \mathbf{M}$ with $B \in U(A; f_1^0, f_2^0, \cdots, \epsilon)$ exists. Therefore A is a strongest condensation point of $\mathbf{M}$, and so $A \in \mathbf{M}$. Conversely every $A \in \mathbf{M}$ satisfies $A \in \mathbf{M}''$, $AE_0 = A$ (cf. (18), p. 394). Thus $\mathbf{M}$ is the common part of $\mathbf{M}''$ with the set of the A with $AE_0 = A$. These sets are obviously weakly closed (cf. (18), p. 388), and so $\mathbf{M}$ is too. (In fact, Theorem 6 in (18) shows at once, that it is precisely $\mathbf{R}(\mathbf{M})$.) This completes the proof.

Note that we have proved Theorem 6 in (18) over again: Consider an arbitrary $\mathbf{M}$, and use $\mathbf{R}(\mathbf{M})$ in our considerations.

4. Our next objective is to prove this:

Consider a direct product $\mathfrak{H} = \prod_1^n \otimes \mathfrak{H}_i$ (cf. RO, §2.1, Theorem I), and the correspondence $A_i \leftrightarrow A_i^{(i)}$ between all $A_i \in \mathbf{B}_i$ and some $A_i^{(i)} \in \mathbf{B}\otimes$, the set of the latter ones being precisely $\mathbf{B}_i^{(i)}$ (cf. RO, §2.3, Lemma 2.3.2 and Definition 2.3.1). $A_i \leftrightarrow A_i^{(i)}$ is a one-to-one mapping of $\mathbf{B}_i$ on $\mathbf{B}_i^{(i)}$. We claim: It is continuous both ways in the sense of the strongest topology in $\mathbf{B}_i$ resp. $\mathbf{B}\otimes$.

For the proof, note first that $\prod_1^n \otimes \mathfrak{H}_j$ is isomorphic, (RO, §2.2, Lemmas 2.2.1, 2.2.3), to $(\prod_{j=1, j\neq i}^n \otimes \mathfrak{H}_j) \otimes \mathfrak{H}_i$. Therefore we may assume $n = 2$, $i = 2$ and use the apparatus of RO, §2.4. Then A_i, $A_i^{(i)}$ become $A \in \mathbf{B}_2$ and $A^{(2)} = < \delta_{t,s} A >_{t,s=1,2}$. Denote the complete normalized orthogonal set in $\mathfrak{H}_1$, used there (loc. cit.) again by $\varphi_1, \varphi_2, \cdots$.

Now $B \in U(A; f_1, f_2, \cdots, \epsilon)$ (in $\mathbf{B}_2$) is obviously equivalent to

$$B^{(2)} \in U(A^{(2)}; < f_1, 0, 0, \cdots, >, < f_2, 0, 0, \cdots > \cdots, \epsilon >),$$

(in $\mathbf{B}\otimes$) and $B^{(2)} \in U(A^{(2)}; < g_{1,1}, g_{1,2}, \cdots >, < g_{2,1}, g_{2,2}, \cdots >, \cdots, \epsilon)$ (in $\mathbf{B}\otimes$) is equivalent to $B \in U(A; f_1, f_2, \cdots, \epsilon)$ (in $\mathbf{B}_2$), if $f_1, f_2, \cdots$ is some simple sequential arrangement of the double sequence $g_{1,1}, g_{1,2}, \cdots, g_{2,1}, g_{2,2}, \cdots, \cdots$. The auxiliary conditions are satisfied because

$$\sum_{i=1}^{\infty} || < f_i, 0, 0, \cdots > ||^2 = \sum_{i=1}^{\infty} || f_i ||^2$$

$$\sum_{i=1}^{\infty} || < g_{i,1}, g_{i,2}, \cdots > ||^2 = \sum_{i,j=1}^{\infty} || g_{i,j} ||^2 = \sum_{k=1}^{\infty} || f_k ||^2 .$$

Thus $A \leftrightarrow A^{(2)}$ is bicontinuous between $\mathbf{B}_2$ and $\mathbf{B}_2^{(2)} \subset \mathbf{B} \otimes$ in the strongest sense. This completes the proof.

5. We now give the examples mentioned in §§1–2.

The first example coincides with that one in (18), p. 383. Let $\mathfrak{H}$ be a Hilbert space, $\varphi_1, \varphi_2, \cdots$, complete normalized orthogonal set in it, $A_{m,n} = P_{[\varphi m]} + m P_{[\varphi n]}$ the set of all $A_{m,n}$, $m, n = 1, 2, \cdots, m < n$.

If a sequence $f_1, f_2, \cdots$ with a finite $\sum_{i=1}^{\infty} || f_i ||^2$ is given, then

$$\sum_{p=1}^{\infty} \left(\sum_{i=1}^{\infty} | (f_i, \varphi_p) |^2 \right) = \sum_{i=1}^{\infty} \left(\sum_{p=1}^{\infty} | (f_i, \varphi_p) |^2 \right) = \sum_{i=1}^{\infty} || f_i ||^2$$

is finite, and so $\lim_{p\to\infty} \sum_{i=1}^{\infty} | (f_i, \varphi_p) |^2 = 0$. Choose an m with $\sum_{i=1}^{\infty} | (f_i, \varphi_m) |^2 \leqq \epsilon/2$ and an $n > m$ with $\sum_{i=1}^{\infty} | (f_i, \varphi_n) |^2 \leqq \epsilon/2m^2$. Then $\sum_{i=1}^{\infty} || A_{m,n} f_i ||^2 = \sum_{i=1}^{\infty} (| (f_i, \varphi_m) |^2 + m^2 | (f_i, \varphi_n) |^2) \leqq \epsilon/2 + m^2 \epsilon/2m^2 = \epsilon$. Thus $A_{m,n} \in U(0; f_1, f_2, \cdots, \epsilon)$. So 0 is a strongest condensation point of S; it is a fortiori a strong and a weak one.

If a subsequence A_{m_ν, n_ν}, $\nu = 1, 2, \cdots$ from S converged weakly to 0, it would be uniformly bounded (cf. (18), p. 382). As $|| A_{m,n} \varphi_n || = m$ this implies that the m_ν are bounded. Thus for some $\bar{m}$ we have infinitely often $m_\nu = \bar{m}$, $(A_{m_\nu, n_\nu} \varphi_{\bar{m}}, \varphi_{\bar{m}}) = 1$, $A_{m_\nu, n_\nu} \notin U_3(0; \varphi_{\bar{m}}, 1)$. Thus A_{m_ν, n_ν}, $\nu = 1, 2, \cdots$ cannot converge weakly to 0, a fortiori not strongly and not strongest.

We now pass to the second example. Let now S be the set of all $256n^2(1 - P_{\mathfrak{M}})$ where $n = 1, 2, \cdots$ and where $\mathfrak{M}$ is any closed linear set with $\leqq n$ dimensions.

If any $f_1, \cdots, f_n$ are given, then $256\ n^2(1 - P_{[f_1, f_2 \cdots, f_n]})\ f_i = 0$ for $i = 1, \cdots, n$ so $256\ n^2(1 - P_{[f_1, \cdots, f_n]}) \in U_4(0; f_1, \cdots, f_n, \epsilon)$. Thus 0 is a strong condensation point of S. But it is not a strongest one: Put $f_i^0 = 1/2i\ \varphi_i$, $i = 1, 2, \cdots$. Then $\sum_1^\infty \| f_i^0 \|^2 = \frac{1}{4} \sum_1^\infty 1/i^2 < 1$. Consider now an $A \in S$ that is an $A = 256\, n^2\ (1 - P_{\mathfrak{M}})$ where $\mathfrak{M}$ has $\leqq n$ dimensions. Then $\mathfrak{M} = \{\psi_1, \cdots \psi_k\}$ where $\psi_1, \cdots, \psi_k$ are normalized, orthogonal and $k \leqq n$; thus

$$\sum_{i=1}^{\infty} \| P_{\mathfrak{M}}\, \varphi_i \|^2 = \sum_{i=1}^{\infty} \left(\sum_{\nu=1}^{k} \| P_{[\psi_\nu]}\, \varphi_i \|^2 \right) = \sum_{i=1}^{\infty} \sum_{\nu=1}^{k} | (\varphi_i, \psi_\nu) |^2$$

$$= \sum_{\nu=1}^{k} \left(\sum_{i=1}^{\infty} | (\varphi_i, \psi_\nu) |^2 \right) = \sum_{\nu=1}^{k} 1 = k \leqq n\,.$$

So $\| P_{\mathfrak{M}}\, \varphi_i \| > \frac{1}{2}$ cannot occur $\geqq 4n$ times, and therefore an $i' \leqq 4n$ with $\| P_{\mathfrak{M}}\, \varphi_{i'} \| \leqq \frac{1}{2}$ exists. Then $\| (1 - P_{\mathfrak{M}})\, f_{i'}^0 \|^2 = 1/4i'^2 \| (1 - P_{\mathfrak{M}})\, \varphi_{i'} \|^2 = 1/4i'^2 \| \varphi_{i'} - P_{\mathfrak{M}}\, \varphi_{i'} \|^2 \geqq 1/4i''^2\ (1 - \frac{1}{2})^2 = 1/16i'^2 \geqq 1/256\ n^2$ and so $\sum_1^\infty (Af_i^0, f_i^0) = 256\ n^2\ (\sum_1^\infty ((1 - P_{\mathfrak{M}})\, f_i^0, f_i^0)) = 256\ n^2 \sum_1^\infty \| (1 - P_{\mathfrak{M}})\, f_i^0 \|^2 \geqq 256\, n^2 \| (1 - P_{\mathfrak{M}})\, f_{i'}^0 \|^2 \geqq 1$.

So we have $\sum_1^\infty (Af_i^0, f_i^0) \geqq 1$ and as

$$\sum_1^\infty (Af_i^0, f_i^0) \leqq \sum_1^\infty \| Af_i^0 \| \cdot \| f_i^0 \| \leqq \sqrt{\sum_1^\infty \| Af_i^0 \|^2 \cdot \left(\sum_1^\infty \| f_i^0 \|^2 \right)}$$

$$\leqq \sqrt{\sum_1^\infty \| Af_i^0 \|^2}$$

we have $\sum_1^\infty \| Af_i^0 \|^2 \geqq 1$ too, thus $A \notin U(0; f_1^0, f_2^0, \cdots, 1)$. Therefore 0 is not a strongest condensation point of S.

Form now $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ witb $\mathfrak{H}_1$ a Hilbert space and $\mathfrak{H}_2 = \mathfrak{H}$ that is $\infty \otimes \mathfrak{H}$ (cf. §§3 and 4). Form the $A \leftrightarrow A^{(2)}$ image of $S \subset \mathbf{B}$: it is $S^{(2)} \subset \mathbf{B}^{(2)} \otimes$ (cf. RO, §2.3). $\varphi = < f_1^0, f_2^0, \cdots >$ is an element $\infty \otimes \mathfrak{H}$ with

$$\| \varphi \|^2 = \sum_1^\infty \| f_i \|^2 < 1$$

$$(A^{(2)}\varphi, \varphi) = (< Af_1^0, Af_2^0, \cdots >, < f_1^0, f_2^0, \cdots >) = \sum_1^\infty (Af_i^0, f_i^0) \geqq 1$$

if $A \in S$, that is if $A^{(2)} \in S^{(2)}$. Thus 0 is not a weak condensation point of $S^{(2)}$. Let $S'^{(2)}$ be the weak closure of $S^{(2)}$, S' its inverse image by $\bar{A} \leftrightarrow A^{(2)}$. Then $0 \notin S'^{(2)}$, and so $0 \notin S'$.

$S'^{(2)}$ is weakly, and a fortiori strongly closed. $S^{(2)} \subset S'^{(2)}$, $S \subset S'$, 0 is a strong condensation point of S, so it is one of S' too, therefore as $0 \notin S'$, S' is not closed strongly and a fortiori not weakly.

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ON RINGS OF OPERATORS

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Introduction

1. The problems discussed in this paper arose naturally in continuation of the work begun in a paper of one of us ((18), chiefly parts I and II). Their solution seems to be essential for the further advance of abstract operator theory in Hilbert space under several aspects. First, the formal calculus with operator-rings leads to them. Second, our attempts to generalise the theory of unitary group-representations essentially beyond their classical frame have always been blocked by the unsolved questions connected with these problems. Third, various aspects of the quantum mechanical formalism suggest strongly the elucidation of this subject. Fourth, the knowledge obtained in these investigations gives an approach to a class of abstract algebras without a finite basis, which seems to differ essentially from all types hitherto investigated.

The results which we shall obtain throw light on an entirely new side of operator theory; they lead to a new notion of linear dimensionality (which, under certain conditions, has a continuous range of numerical values); and indicate a way out of the paradoxes of unbounded operator theory.

In the four following §§ of this Introduction we will give a somewhat more detailed outline of the four aspects of our problems, which were enumerated above.

2. We consider a linear space $\mathfrak{H}$ with an inner product (that is a Hilbert space or a finite dimensional Euclidean space, cf. (b) in §1.1), and in it the ring $\mathbf{B}$ of all bounded operators (cf. (*f*) in 1.1). Subsets $\mathbf{M}$ of $\mathbf{B}$ for which $A, B \in \mathbf{M}$ implies $\alpha A, A^*, A + B, AB \in \mathbf{M}$ (α is any complex number, A^* is the adjoint of A, cf. the notation in §1.1), and which are closed in a suitable topology of $\mathbf{B}$ are called rings. (All these notions are discussed in detail in (18).) We will chiefly consider rings $\mathbf{M}$ which contain the unit operator 1: $1 \in \mathbf{M}$. If $\mathbf{M}, \mathbf{N}$ are given rings, denote the smallest ring containing $\mathbf{M}$, $\mathbf{N}$ by $\mathbf{R}(\mathbf{M}, \mathbf{N})$, and the greatest ring contained in $\mathbf{M}$ and $\mathbf{N}$ by $\mathbf{M} \cdot \mathbf{N}$ (this, by the way, happens to be the set theoretical intersection of $\mathbf{M}$ and $\mathbf{N}$).

The operations $\mathbf{R}(\mathbf{M}, \mathbf{N})$ and $\mathbf{M} \cdot \mathbf{N}$ obey both the commutative and the associative law, therefore each of them could be analogised with addition or with multiplication. Owing to $\mathbf{R}(\mathbf{M}, \mathbf{M}) = \mathbf{M} \cdot \mathbf{M} = \mathbf{M}$ the analogy is closer with the "Boolean algebras" (as they occur in set theory or in logics), than with algebra proper. As the analogue of the distributive law, which connects addition and

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multiplication, does not hold in general, this analogy is not perfect. (The distributive law would be, according to how we identify $\mathbf{R}(\mathbf{M}, \mathbf{N})$ and $\mathbf{M} \cdot \mathbf{N}$ with addition and multiplication, either $\mathbf{R}(\mathbf{M} \cdot \mathbf{N}, \mathbf{M} \cdot \mathbf{P}) = \mathbf{M} \cdot \mathbf{R}(\mathbf{N}, \mathbf{P})$ or $\mathbf{R}(\mathbf{M}, \mathbf{N}) \cdot \mathbf{R}(\mathbf{M}, \mathbf{P}) = \mathbf{R}(\mathbf{M}, \mathbf{N} \cdot \mathbf{P})$. None of these equations holds in general.) Thus it is somewhat arbitrary how we carry out these identifications. We choose to make $\mathbf{R}(\mathbf{M}, \mathbf{N})$ correspond to addition and $\mathbf{M} \cdot \mathbf{N}$ to multiplication.

In this notation the rings $\mathbf{M}$ (and similarly the rings with $1 \in \mathbf{M}$) form a lattice. (Cf. (1), (9), (12a).) We use the terminology of (9), and therefore write in this § $\mathbf{M} \frown \mathbf{N}$ for $\mathbf{R}(\mathbf{M}, \mathbf{N})$ and $\mathbf{M} \smile \mathbf{N}$ for $\mathbf{M} \cdot \mathbf{N}$. One verifies immediately the lattice postulates I–IV (cf. (1), p. 422) for these operations.

The lattice R of all rings with $1 \in \mathbf{M}$ contains a smallest element: the set $(\alpha 1)$ of all operators of the form $\alpha 1$; and a greatest element: the set $\mathbf{B}$ of all bounded operators in $\mathfrak{H}$. We write in this §, $\mathbf{0}$ and $\mathbf{1}$ for $(\alpha 1)$ and $\mathbf{B}$.

Now R has a property, which brings it nearer to the Boolean algebras than lattices usually are: there exists an operation $\mathbf{M}'$ in R which dualises addition and multiplication; that is, for which

$$(\mathrm{D}_1) \qquad (\mathbf{M} \smile \mathbf{N})' = \mathbf{M}' \frown \mathbf{N}'$$

$$(\mathrm{D}_2) \qquad (\mathbf{M} \frown \mathbf{N})' = \mathbf{M}' \smile \mathbf{N}'$$

hold. To this end we define $\mathbf{M}'$ as the set of those A which commute with all $B \in \mathbf{M}$ (cf. (18), p. 374); then $\mathbf{M}' \in R$ and

$$(\mathrm{D}_3) \qquad \mathbf{M} = \mathbf{M}''$$

(cf. (18), p. 397). Now (D_1) is obvious; and (D_2) follows from (D_1) by replacing $\mathbf{M}, \mathbf{N}$ by $\mathbf{M}', \mathbf{N}'$, applying $'$ to the equation, and using (D_3).

In the analogy with Boolean algebras (or logics) in which $\mathbf{M} \smile \mathbf{N}$ corresponds to the sum (resp. "or") and $\mathbf{M} \frown \mathbf{N}$ to the product (resp. "and"), $\mathbf{M}'$ should correspond to the complement (resp. "no").

Another lattice of such a structure consists of all closed linear manifolds of $\mathfrak{H}$. Denoting it by M we define: If $\mathfrak{M}$, $\mathfrak{N}$ are two closed linear manifolds in $\mathfrak{H}$, then $\mathfrak{M} \smile \mathfrak{N}$ is the smallest closed linear manifold containing them (cf. (d) in §1.1, where it is denoted by $[\mathfrak{M}, \mathfrak{N}]$); $\mathfrak{M} \frown \mathfrak{N}$ is the greatest closed linear manifold contained in them (their set theoretical intersection, $\mathfrak{M} \cdot \mathfrak{N}$); $\mathbf{0}$ is the smallest element of M: the set (0) consisting of 0 alone; $\mathbf{1}$ is the greatest element of M: $\mathfrak{H}$. (We use these notations in this § only.) Then we may define $\mathfrak{M}'$ as the set of all elements of $\mathfrak{H}$ which are orthogonal to all elements of $\mathfrak{M}$: $\mathfrak{H} - \mathfrak{M}$ (cf. (16), p. 74). One verifies again the lattice postulates I–IV (cf. (1), p. 422) for M, while the distributive law (postulate VI, eod., p. 453) does not hold in general. (Postulate V, eod., p. 445, holds, thus M is a B-lattice, cf. eod.) (D_1)–(D_3) too hold.

Following the analogy with the complement in Boolean algebras (and "no"

in logics) further, we are led to expect

$$(D_4) \qquad \mathbf{M} \smile \mathbf{M}' = \mathbf{1}$$

$$(D_5) \qquad \mathbf{M} \frown \mathbf{M}' = \mathbf{0}.$$

(In the lattice M (D_4), (D_5) hold. In fact there is a quite intimate connection between M and a certain type of logics: The probability-logics of quantum mechanics. Cf. (20), pp. 130–134.) These equations, however, are not true in general in R. For instance, if $\mathbf{M}$ is Abelian (cf. (18), p. 374), then $\mathbf{M} \subset \mathbf{M}'$, and thus $\mathbf{M} \smile \mathbf{M}' = \mathbf{M}'$, $\mathbf{M} \frown \mathbf{M}' = \mathbf{M}$. For this reason it is of interest to find out, for which particular elements $\mathbf{M}$ of R (D_4), (D_5) hold. As $'$ dualises (D_4) and (D_5) (apply $'$, and use (D_1)–(D_3)), they are equivalent. Thus it suffices to discuss one of them. In our original notations they read:

$$(\overline{D}_4) \qquad \mathbf{R}(\mathbf{M}, \mathbf{M}') = \mathbf{B}$$

$$(\overline{D}_5) \qquad \mathbf{M} \cdot \mathbf{M}' = (\alpha 1).$$

The theory of these equations will be the chief subject of this paper (cf. §3.1).

3. Let $\mathfrak{H}$ be the n-dimensional Euclidean space $\mathfrak{E}_n$, $n = 1, 2, \cdots$. In this case it is easy to detail the meaning of $(\overline{D}_5)$.

Assume $\mathbf{M} \cdot \mathbf{M}' = (\alpha 1)$. Let $\mathbf{M}^U$ be the set of all unitary elements of $\mathbf{M}$, then $\mathbf{M}$ is the ring generated by $\mathbf{M}^U$ (cf. (18), p. 392). Thus $\mathbf{M}' = (\mathbf{M}^U)'$; and as $\mathbf{M}^U$ is a group of unitary matrices, $\mathbf{M}$ consists of all linear aggregates of elements of $\mathbf{M}^U$. Now apply the theory of unitary group representations to $\mathbf{M}^U$.

As $\mathbf{M}^U$ consists of unitary matrices, it is completely reducible (cf. (14), pp. 11–12, footnote 1). Introduce a coordinate system in $\mathfrak{H} = \mathfrak{E}_n$, in which $\mathbf{M}^U$ appears completely reduced. Let the irreducible parts of $\mathbf{M}^U$ correspond to the sets of coordinates $p_1 + p_2 + \cdots + p_{s-1} + 1, \cdots, p_1 + p_2 + \cdots + p_{s-1} + p_s$ for $s = 1, \cdots, r$, where $r = 1, 2, \cdots$; $p_1, \cdots, p_r = 1, 2, \cdots$; $p_1 + \cdots + p_r = n$, and denote them by $\mathbf{I}_1, \cdots, \mathbf{I}_r$. Denote the number of those which are equivalent to $\mathbf{I}_1$ by $q(= 1, 2, \cdots,)$, and arrange the $\mathbf{I}_s$ so that these should be $\mathbf{I}_1, \cdots, \mathbf{I}_q$. By a further coordinate transformation we can even make $\mathbf{I}_1, \cdots, \mathbf{I}_q$ identical. Besides $p_1 = \cdots = p_q = p$. Thus every element of $\mathbf{M}^U$ looks like this: It consists of r $(\geqq q)$ matrices succeeding each other along the main diagonal, the q first ones being identical and of degree p. The same is true for their linear aggregates, the elements of $\mathbf{M}$.

The matrices which occur in the q first diagonal minors form an irreducible system, inequivalent to those which occur in the $r - q$ other diagonal minors. So the theorems of Burnside, Frobenius, and I. Schur (cf. (3), (8); or (14), p. 412) apply. Therefore these matrices are perfectly arbitrary, and independent of the $r - q$ other ones. Thus we can prescribe the q first ones to be equal to the unit matrix, and the $r - q$ others to vanish. So an element E_0 of $\mathbf{M}$ results, which commutes obviously with all elements of $\mathbf{M}$; thus $E_0 \in \mathbf{M} \cdot \mathbf{M}'$. As

$\mathbf{M}\cdot\mathbf{M}' = (\alpha 1)$, we have $E_0 = \alpha 1$. This clearly necessitates $\alpha = 1$, and so the $r - q$ vanishing diagonal minors in E_0 cannot be present. Therefore $r - q = 0$, $q = r$, $n = p_1 + \cdots + p_q = p \cdot q$.

If we write the general element A of $\mathbf{M}$ as a matrix: $A = \{a_{\mu,\nu}\}$, $\mu, \nu = 1, \cdots, n$ then our discussions have characterised A as follows: Put $\mu = p(s-1) + \sigma$, $\nu = p(t-1) + \tau$, where $s, t = 1, \cdots q$; $\sigma, \tau = 1, \cdots, p$ (remember $n = pq$), then

$$a_{\mu,\nu} \begin{cases} = 0 & \text{if } s \neq t \\ = \begin{cases} \text{a function of} \\ \sigma, \tau \text{ alone} \end{cases} & \text{if } s = t \end{cases}.$$

If we replace the index $\mu (= 1, \cdots, n)$ by the two indices $\sigma (= 1, \cdots, p)$ and $s (= 1, \cdots, q)$, and similarly ν by τ and t, then we have:

$A = \{a_{\sigma,s,\tau,t}\}_{\substack{\sigma,\tau=1,\cdots,p \\ s,t=1,\cdots,q}}$ with

$$(S) \qquad a_{\sigma,s,\tau,t} = \delta_{s,t} b_{\sigma,\tau}$$

where $\delta_{s,t}$ is the Weierstrass-Kronecker symbol, and $b_{\sigma,\tau}$ is arbitrary.

Conversely: If $\mathbf{M}$ consists of all $\{\delta_{s,t}\, b_{\sigma,\tau}\}_{\substack{\sigma,\tau=1,\cdots,p \\ s,t=1,\cdots,q}}$

then it is a ring, $\mathbf{M}'$ consists of all $\{\delta_{\sigma,\tau} c_{s,t}\}_{\substack{\sigma,\tau=1,\cdots,p \\ s,t=1,\cdots,q}}$

and so $\mathbf{M}\cdot\mathbf{M}'$ of all $\{\delta_{\sigma,\tau}\delta_{s,t}\alpha\} = \alpha 1$. Thus (S) gives the general solution of $(\overline{\mathrm{D}}_5)$ for $\mathfrak{H} = \mathfrak{E}_n$.

In other words: The general solution of $(\overline{\mathrm{D}}_5)$ obtains, by replacing the one coordinate index $\mu = 1, \cdots, n$ in $\mathfrak{H} = \mathfrak{E}_n$ by two independent coordinate indices $\sigma = 1, \cdots, p$ and $s = 1, \cdots, q$ $(n = pq)$, and collecting in $\mathbf{M}$ all linear operators A which operate on σ alone, while $\mathbf{M}'$ consists of all those which operate on s alone.

This is the effect of the application of the classical Burnside-Frobenius-I. Schur theory of unitary group representations. Thus the solving of $(\overline{\mathrm{D}}_5)$ connects directly with the fundamental facts about unitary representations.

If the analogous situation held in Hilbert space, we would be led to expect that if $\mathfrak{H}$ is Hilbert space, the solutions of $(\overline{\mathrm{D}}_5)$ can be brought in the following form: $\mathfrak{H}$ is isomorphic to the functional space of all functions $f(x, y)$ in two independent variables x, y, ($\iint |f(x,y)|^2\, dx\, dy$ finite); the ranges of x and of y are continuous or discrete, in the latter case $\int \cdots dx$ resp. $\int \cdots dy$ is meant to indicate a sum. (Cf. (16), pp. 69 and 108–111.) $\mathbf{M}$ consists of all operators which operate on x alone, and $\mathbf{M}'$ of all those which operate on y alone. (Cf. §3.2, where this is more precisely discussed.)

The essentially different character of Hilbert space, compared with the spaces $\mathfrak{E}_n$, $n = 1, 2, \cdots$, is reflected in the fact that this is not so. We will see that if

$\mathfrak{H}$ is Hilbert space, $(\overline{D}_5)$ has further solutions. (Cf. in particular §§8.6 and 13.2.) Their properties seem to be of great importance for the general theory of operators.

The general importance of the solutions of $(\overline{D}_5)$ for operator theory follows from this fact too, that their knowledge allows a characterisation and classification of all rings of operators. This will be discussed in a subsequent paper of the second author.

4. Another interpretation of $(\overline{D}_5)$ is suggested by quantum mechanics. The operators of $\mathfrak{H}$ correspond there to all observable quantities which occur in a mechanical system $\mathfrak{S}$. (Cf. (6), pp. 55–60, and (20), p. 167. We restrict ourselves to bounded operators, which correspond to those observables which have a bounded range. Thus $\mathbf{B}$ corresponds to the totality of these observables.) Now if $\mathfrak{S}$ can be decomposed into two parts $\mathfrak{S}_1$, $\mathfrak{S}_2$ and if we denote the set of the operators which correspond to observables situated entirely in $\mathfrak{S}_1$ or in $\mathfrak{S}_2$ by $\mathbf{M}_1$ resp. $\mathbf{M}_2$, then we see:

(1) $\mathbf{M}_1$, $\mathbf{M}_2$ are rings, and 1 (which corresponds to the "constant" observable 1) belongs to both $\mathbf{M}_1$, $\mathbf{M}_2$.
(2) If $A \in \mathbf{M}_1$, $B \in \mathbf{M}_2$ then the measurements of the observables of A and B do not interfere (being in different parts of $\mathfrak{S}$); therefore A, B commute (cf. (6), pp. 11–14 and 76, or (20), pp. 117–121). Thus $\mathbf{M}_2 \subset \mathbf{M}_1'$.
(3) As $\mathfrak{S}$ is the sum of $\mathfrak{S}_1$, $\mathfrak{S}_2$ therefore $\mathbf{R}(\mathbf{M}_1, \mathbf{M}_2) = \mathbf{B}$.

(1)–(3) describe the problem of "factorising" $\mathbf{B}$ which is discussed in more detail in §3.1; it leads to our old problem: As $\mathbf{M}_1' \supset \mathbf{M}_2$ therefore $\mathbf{R}(\mathbf{M}_1, \mathbf{M}_1') \supset \mathbf{R}(\mathbf{M}_1, \mathbf{M}_2) = \mathbf{B}$ so $\mathbf{R}(\mathbf{M}_1, \mathbf{M}_1') = \mathbf{B}$, that is precisely $(\overline{D}_4)$, which, as we know, is equivalent to $(\overline{D}_5)$. Conversely: If $\mathbf{M}$ fulfills $(\overline{D}_4)$ (that is $\overline{D}_5$), then $\mathbf{M}_1 = \mathbf{M}$, $\mathbf{M}_2 = \mathbf{M}'$ satisfy (1)–(3). (Cf. §3.1 for more details.)

Thus our problem of solving $(\overline{D}_5)$ corresponds to the quantum mechanical problem of dividing a system $\mathfrak{S}$ into two subsystems $\mathfrak{S}_1$, $\mathfrak{S}_2$; and in particular the solutions $\mathbf{M}$ of $(\overline{D}_5)$ correspond to the complete rings of all observables of suitable quantum mechanical systems.

This interpretation of $(\overline{D}_5)$ suggests of course strongly the surmise formulated at the end of §2.2: It should be possible to describe $\mathfrak{H}$ as (isomorphic to) the space of all two variable functions $f(x, y)$, $(\iint |f(x, y)|^2 \, dx \, dy$ finite), $\mathbf{M}$ operating on x only, and $\mathbf{M}'$ on y only. In this case $\mathfrak{S}_1$, $\mathfrak{S}_2$ would be explicitly given: $\mathfrak{S}_1$ being described by the coordinate x, and $\mathfrak{S}_2$ by the coordinate y.

The fact that the surmise of §2.2 is not true, is therefore the more remarkable; particularly so because certain features of the "exceptional" rings $\mathbf{M}$ seem to make them even better suited for quantum mechanical purposes than the customary $\mathbf{B}$. We will now discuss these properties of $\mathbf{M}$.

5. The full system of solutions of $(\overline{D}_5)$ will be discussed in §§8.3–8.4. While we refer to those sections for a complete discussion, we would like to call atten-

tion here to the following: If the surmise formulated at the end of §2.2 holds, then **M** is obviously isomorphic to the ring of all operators on x. That is, according to whether x has a finite or infinite range, **M** is isomorphic to the ring **B** of either a finite dimensional Euclidean space $\mathfrak{E}_n$, $n = 1, 2, \cdots$, or of a Hilbert space $\mathfrak{H}'$. In the systematic notation used in §8.4 these possibilities are labeled as cases (I_n), $n = 1, 2, \cdots$, respectively, (I_∞). (In the last case the range of x may be either continuous or discontinuous, but it is well known, that this does not affect the character of the Hilbert space $\mathfrak{H}'$ at all.) As we mentioned, these cases do not exhaust all possibilities for **M**: further cases, labeled (II_1), (II_∞), (III_∞) may occur. (All of them, except (III_∞) certainly exist; while the existence of (III_∞) is as yet undecided. Cf. §13.3.)

The cases (I_n), $n = 1, 2, \cdots$, are of course the simplest ones, and they are the ones on which our notions as to what operators should look like have been developed. This is true for general operator theory as well as for quantum mechanics. It is therefore of particular importance to compare the other cases (I_∞)–(III_∞) under the following aspects: Which one has the most properties in common with (I_n), the ring of all matrices of n rows and n columns, and which one can be considered as the limiting case of (I_n) for $n \to \infty$? The investigations of operators in Hilbert space have always been carried on with the idea that the **B** of Hilbert space, that is (I_∞), is the natural limiting case. We think however that our results indicate that there is more point in assigning this rôle to (II_1). There are two chief reasons for this assertion: The existence of a trace, and the behavior of unbounded operators. Chaps. XV and XVI of this paper have been devoted to the purpose of deriving these common features of (I_n) and (II_1). In what follows we will make a few qualitative remarks on the subject.

The cases (I_n) and (II_1) are characterised by this property: It is possible to define for all Hermitian operators $A \in \mathbf{M}$ a function $T(A)$ with finite real values, which has the formal properties of the trace (cf. §§15.4–15.5), and there exists only one such function $T(A)$. In the cases (I_n), where the operators $A \in \mathbf{M}$ correspond to matrices $\{a_{\mu,\nu}\}_{\mu,\nu=1,\cdots,n}$ obviously $T(A) = (1/n) \sum_1^n a_{\mu,\mu}$ (the "normalising factor" $1/n$ is necessary, because we require $T(1) = 1$), and it is well known, that in case (I_∞) (that is: for all bounded infinite matrices $\{a_{\mu,\nu}\}_{\mu,\nu=1,2,\cdots}$) no such function $T(A)$ can be defined. (The expression which is formed in analogy to $(1/n) \sum_1^n a_{\mu,\mu}$ will not converge in general.) Now we proved, as mentioned above, that the only further case where such a $T(A)$ exists and is unique, is (II_1).

Considering the immediate applicability of $T(A)$ to quantum mechanics (it is the "a priori" expectation value of the observable A, which is correctly normalised here, but cannot be in (I_∞), cf. (20), pp. 165, 169), it is significant that its existence and uniqueness is a characteristic of (II_1).

A more general problem which will be discussed by the second named author in a forthcoming paper arises now from the following motive: It would be desirable to characterise those abstract algebras, in which one and only one function

$T(A)$ with the properties of the trace (as discussed above) exists. (They should be abstract, that is no connection with the operator rings should be postulated.) The result is, that all algebras which meet these requirements are isomorphic to an operator ring $\mathbf{M}$ in a (Euclidean or Hilbert) space $\mathfrak{H}$, which solves $(\bar{D}_5)$ and belongs to cases (I_n), $n = 1, 2, \cdots$, or (II_1).

Returning to case (II_1) let us point out, that the analogy with (I_n) goes so far, that theorems of the following type hold: Every system of linear equations (which can of course be written as one operatorial equation $Af = 0$ where $A \in \mathbf{M}$ is the "coefficient matrix," and $f \in \mathfrak{H}$ represents the variables) has a rank. In case (I_n) this is a number $0, 1, \cdots, n-1, n$, but we replace it by its n-th part: $0, 1/n, \cdots, (n-1)/n, 1$. Now in case (II_1) it is any number $\geqq 0, \leqq 1$. Two adjoint systems of equations ($Af = 0$ and $A^*f = 0$, cf. §16.1) have the same rank. (Observe the analogy with (I_n), and the contrast with (I_∞)!) Similarly a dimensionality for subspaces of $\mathfrak{H}$ (linear, closed sets) can be defined, which has the values $0, 1/n, \cdots, (n-1)/n, 1$ (in the customary normalisation $0, 1, \cdots, n-1, n$) in case (I_n), and all values $\geqq 0, \leqq 1$ in case (II_1).

With this notion of dimensionality even the "minimax principle" (cf. (5), pp. 26–29) can be proved, which thus labels the proper values of all Hermitian operators (in $\mathbf{M}$), even in continuous spectra! So it makes sense to speak about the "proper value No. α of the operator A (from below)" for any $\alpha \geqq 0, \leqq 1$, even for irrational ones! (cf. §15.2).

6. Let us now consider the unbounded operators. While $\mathbf{M}$ consists prima facie of bounded operators only, there is a natural way to extend it, so as to include unbounded ones too: An arbitrary (not necessarily bounded) operator X is said to belong to $\mathbf{M}$, if it is invariant under all unitary transformations $U' \in \mathbf{M}'$; we denote the set of all those linear, closed operators X with an everywhere dense domain which belong to $\mathbf{M}$, by $U(\mathbf{M})$. (Cf. §§4.2 and 16.4, as to the notions of linearity and closure cf. (16), p. 70. The bounded elements of $U(\mathbf{M})$ form precisely $\mathbf{M}$.)

While in general Hermitian operators possess a resolution of unity if and only if they are hypermaximal (cf. (16), p. 92, or (15), Theorem 9.3, p. 339) this is much simpler in cases (I_n) and (II_1): Here every Hermitian operator $A \in U(\mathbf{M})$ is hypermaximal, and has therefore a (unique) resolution of unity. (In case (I_n) this is due to the trivial reason, that every linear operator is bounded in that case. But in case (II_1) unbounded operators exist, in the same way as they do in (I_∞)!)

The operators in $U(\mathbf{B})$ ($\mathfrak{H}$ a Hilbert space) behave very pathologically from the point of view of operator algebra: Thus if $A, B, \in U(\mathbf{B})$, then $A + B$ and AB cannot be formed in general, and the entire mechanism of replacing operators by their extension leads to very paradoxic results. (Cf. (17), pp. 230–234, where these conditions are discussed in detail.) This applies, of course, to every $U(\mathbf{M})$, $\mathbf{M}$ in case (I_∞). In cases (I_n) and (II_1) again it is not so: The

algebra of $U(\mathbf{M})$ works without any difficulties, and if A is an extension of B, $A, B \in U(\mathbf{M})$, then $A = B$. (Cf. for details §§16.3 and 16.4.)

7. A detailed account of the content of this paper is given in the table of contents which follows. The main results are summed up in the Theorems I–XV, while the essential problems are formulated as Problems. It will be pointed out after each problem to what extent it is solved and where the solutions are to be found. The auxiliary considerations, which lead to the Theorems, are grouped into Lemmas.

The notations are explained in §1.1, where the chief references too are given. These, as well as all other quotations, refer to the bibliography, which follows after the table of contents.

Various continuations of the investigations begun in this paper, which we propose to undertake in several subsequent papers, are indicated throughout the text. (Cf. also note at the end of this paper.)

Here however we would like to mention that very similar methods to those used in this paper permit us to discuss the corresponding problems in the non-separable analogues of Hilbert space (cf. (11), (13)). The analogy to Cantor's theory of alephs, which we stress in Chapters VI and VII, becomes then even more apparent. This subject, too, will be treated at a later occasion.

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Part I: Preparatory Considerations

Chapter I: Notations

1.1. In what follows we will have to assume that the reader is familiar with the unitary–orthogonal–geometrical discussion of the elementary properties of Hilbert space, as contained in the treatise (16) (particularly pp. 63–78) or in the remarkable exposition of the subject in (15) Chapter I. Besides we will have to make frequent use of some other papers of one of us, namely (18), (21) and (22), which is a variant to a theorem of (18).

Certain notations will be used on this account subsequently, without further references to their meaning. They are as follows:

(a) $x \in S$ means that x is an element of the set S, $S \subset T$ or $T \supset S$ means that S is a subset of T, (including $S = T$). The set of all elements x with a certain property $\epsilon(x)$, will be denoted by $(x;\ \epsilon(x))$; the set of all $f(x)$ with x having the property $\epsilon(x)$, by $(f(x);\ \epsilon(x))$; the set theoretical sum of the sets $S, T, \cdots$, plus the elements $x_0, y_0, \cdots$ by $(S, T, \cdots, x_0, y_0, \cdots)$. The set theoretical product (common part) of the sets $S, T, \cdots$ is denoted by $S \cdot T \cdot \cdots$

A symbol η, such that $x \eta S$ has a meaning similar to that of $x \in S$, will be defined in §4.2, Definition 4.2.1.

(b) A linear space with a linear product, and which is separable and complete, will be denoted by $\mathfrak{H}$. (We will use affixes and suffixes if more than one such space occurs.) In other words: $\mathfrak{H}$ is a space in which operations $\alpha \cdot f$, $f + g$, (f, g) satisfying the conditions A, B, C, E, of (16) p. 64–66 are given. Condition D (loc. cit.) is explicitly excepted. Thus $\mathfrak{H}$ is either a Hilbert space or a finite dimensional Euclidean space accordingly as D is fulfilled or not. We only consider 1, 2, $\cdots$, dimensional Euclidean spaces, excluding explicitly the 0-dimensional case where $\mathfrak{H} = (0)$.

(c) We denote complex or real numbers by $\alpha, \epsilon, a, x, C, D$; integers by $i, j, k, m, n, p, q, s, t, u, v$ and N. Elements of $\mathfrak{H}$ are denoted by f, g, φ, ψ, sometimes (in direct products) by Φ, Ψ.

(d) Arbitrary subsets of $\mathfrak{H}$ are denoted by $\mathfrak{S}$, linear subsets by $\mathfrak{A}$, linear and closed subsets by $\mathfrak{E}, \mathfrak{F}, \mathfrak{M}, \mathfrak{N}, \mathfrak{P}, \mathfrak{Q}$. As the latter are again spaces of the type of $\mathfrak{H}$, their symbols sometimes replace $\mathfrak{H}$.

The smallest linear and the smallest closed linear set containing certain sets

or elements are denoted by $\{ \cdots \}$ and $[\cdots]$ respectively (instead of $(\cdots)$, cf. (a)). The set of all elements of $\mathfrak{N}$ which are orthogonal to $\mathfrak{M}$ is denoted by $\mathfrak{N} - \mathfrak{M}$.

(e) Operators in an $\mathfrak{H}$ will be denoted by A, B, X, Y. For certain special kinds of operators, the following letters will be used: E, F, G and P for projection operators, U, V, and W for unitary and partially isometric operators (cf. (16), p. 70).

(f) The ring of all bounded, everywhere defined, linear and closed operators in $\mathfrak{H}$ is $\mathbf{B}$, its subrings are $\mathbf{M}$, $\mathbf{N}$, (cf. (18), p. 388).

Arbitrary subsets of $\mathbf{B}$ are S, the smallest ring containing S is $\mathbf{R}(S)$, the ring of all A which commute with X and X^* for every $X \in S$ is S'.

(g) As discussed in (18), pp. 378–388, two different topologies can be used in $\mathfrak{H}$, and four in $\mathbf{B}$, "Strong" and "Weak" in $\mathfrak{H}$ and $\mathbf{B}$, "uniform" in $\mathbf{B}$ and finally there is the "strongest" topology in $\mathbf{B}$. If nothing in particular is said, we always mean the "strong" topology in $\mathfrak{H}$; otherwise the topology to be used will be specified. The properties of these topologies are not all of the customary type and they are of some importance. Details are given in (18) and (22) loc. cit.

(h) In part IV, a group $\mathfrak{G}$ and a space S, will be considered, where $\mathfrak{G}$ consists of one-to-one mappings of S on itself. We will denote the elements of $\mathfrak{G}$ by $a, b,$ and c; those of S by x, y; the unit, the inverse induced by $a \in \mathfrak{G}$ in S are $1, a^{-1}$. These notations will however only be used in Part IV.

The results of operator theory which we will use most frequently (apart from the general theory of Hilbert space and the spectral form of hypermaximal operators) are these: Theorems 1, 5, 9 in (18), Theorems 5, 6, in (19), and Theorem 7 in (21).

Chapter II: Direct Products

2.1. An operation which generates a new space $\mathfrak{H}$ with the help of n given ones, $\mathfrak{H}_1, \cdots, \mathfrak{H}_n$ is the *direct multiplication*. As it is one of the essential elements of the situations which we are going to discuss, and as a general abstract treatment (valid for Hilbert spaces too and not using special coordinate systems) of this notion does not occur in the literature, we will give a systematic discussion.

Definition 2.1.1. Let $n = 1, 2, \cdots$ spaces $\mathfrak{H}_1, \cdots, \mathfrak{H}_n$ be given. Consider all functionals $\Phi(f_1, \cdots, f_n)$, which are defined for all systems

$$f_i \in \mathfrak{H}_i, i = 1, \cdots, n,$$

and have complex values, and which are conjugate linear in each f_i:

(i) $\Phi(\cdots, \alpha f_i, \cdots) = \bar{\alpha} \Phi(\cdots, f_i, \cdots)$.

(ii) $\Phi(\cdots, f_i + g_i, \cdots) = \Phi(\cdots, f_i, \cdots) + \Phi (\cdots, g_i, \cdots)$.

Call their set $\prod_{i=1}^{n} \otimes \mathfrak{H}_i$.

Definition 2.1.2. If a system $f_i^0 \in \mathfrak{H}_i$, $i = 1, \cdots, n$, is given, form the functional

$$\Phi(f_1, \cdots, f_n) = \prod_{i=1}^{n} (f_i^0, f_i) .$$

Clearly $\Phi \in \prod_1^n \otimes \mathfrak{H}_i$. Define $\Phi = \prod_1^n \otimes f_i^0 = f_1^0 \otimes \cdots \otimes f_n^0$.

Definition 2.1.3. Consider all finite linear aggregates of the form

$$\Phi = \sum_{\nu=1}' \prod_{i=1}^{n} \otimes f_{i,\nu}^0, \qquad p = 0, 1, \cdots, \ f_{i,\nu}^0 \in \mathfrak{H}_i .$$

Call their set $\prod_{i=1}^{n}{}' \otimes \mathfrak{H}_i$. (Clearly $\prod_{i=1}^{n}{}' \otimes \mathfrak{H}_i \subset \prod_{i=1}^{n} \otimes \mathfrak{H}_i$.)

Lemma 2.1.1. *If* $\Phi, \Psi \in \prod_{i=1}^{n}{}' \otimes \mathfrak{H}_i$, *that is*

$$\Phi = \sum_{\nu=1}^{p} \prod_{i=1}^{n} \otimes f_{i,\nu}^0 ; \qquad \Psi = \sum_{\mu=1}^{q} \prod_{i=1}^{n} \otimes g_{i,\mu}^0$$

then form

$$(\Phi, \Psi) = \sum_{\nu=1}^{p} \sum_{\mu=1}^{q} \prod_{i=1}^{n} (f_{i,\nu}^0, g_{i,\mu}^0) .$$

This expression depends on Φ, Ψ *only, but not on the particular decomposition used.*

Proof: Owing to the linearity it suffices to consider the case when $\Phi = 0$ or $\Psi = 0$; and owing to the symmetry in Φ, Ψ, we may assume $\Psi = 0$. We will show that each addend of the sum $\sum_{\nu=1}^{p}$ in (Φ, Ψ) vanishes separately. Indeed:

$$\sum_{\mu=1}^{q} \prod_{i=1}^{n} (f_{i,\nu}^0, g_{i,\mu}^0) = \Psi(f_{i,1}^0, \cdots, f_{i,n}^0) = 0 .$$

Lemma 2.1.2. If $\Phi \in \prod_{i=1}^{n}{}' \otimes \mathfrak{H}_i$, *then* $(\Phi, \Phi) > 0$ *for* $\Phi \neq 0$.

Proof: Consider first any $\Phi \in \prod_{i=1}^{n}{}' \otimes \mathfrak{H}_i$. Then

$$(\Phi, \Phi) = \sum_{\mu,\nu=1}^{p} \prod_{i=1}^{n} (f_{i,\mu}^0, f_{i,\nu}^0) .$$

For any complex $x_1, \cdots, x_p$,

$$\sum_{\mu,\nu=1}^{p} (f_{i,\mu}^0, f_{i,\nu}^0) x_\mu \bar{x}_\nu = \Big(\sum_1^p x_\mu f_{i,\mu}^0, \sum_{\nu=1}^{p} x_\nu f_{i,\nu}^0\Big) = \Big\| \sum_{\mu=1}^{p} x_\mu f_{i,\mu}^0 \Big\|^2 \geqq 0 ;$$

therefore each matrix $((f_{i,\mu}^0, f_{i,\nu}^0))_{\mu,\nu=1,\cdots p}$ (p dimensional!) is semidefinite. Thus it is the sum of (at most p) semidefinite matrices of rank 1, that is of the form $((\alpha_\mu^i \bar{\alpha}_\nu^i))_{\mu,\nu=1,\cdots p}$. Thus (Φ, Φ) is a sum of terms

$$\sum_{\mu,\nu=1}^{p} \prod_{i=1}^{n} \alpha_\mu^i \bar{\alpha}_\nu^i = \sum_{\mu,\nu=1}^{p} \prod_{i=1}^{n} \alpha_\mu^i \cdot \overline{\prod_{i=1}^{n} \alpha_\nu^i}$$
$$= \sum_{\mu=1}^{p} \prod_{i=1}^{n} \alpha_\mu^i \cdot \overline{\sum_{\nu=1}^{p} \prod_{i=1}^{n} \alpha_\nu^i} = \Big| \sum_{\mu=1}^{p} \prod_{i=1}^{n} \alpha_\mu^i \Big|^2 \geqq 0 .$$

So we have $(\Phi, \Phi) \geqq 0$.

From this, Schwarz's inequality

$$|(\Phi, \Psi)| \leqq \sqrt{(\Phi, \Phi) \cdot (\Psi, \Psi)}$$

follows literally as in (16), p. 64. Thus $(\Phi, \Phi) = 0$ implies $(\Phi, \Psi) = 0$ for every $\Psi \in \prod_{i=1}^{n}{}' \otimes \mathfrak{H}_i$. Put $\Psi = \prod_{i=1}^{n} \otimes f_i$, then

$$\Phi(f_1, \cdots, f_n) = (\Phi, \prod_{i=1}^{n} \otimes f_i) = 0 ,$$

and so $\Phi = 0$. Thus $\Phi \neq 0$ implies $(\Phi, \Phi) > 0$.

Lemma 2.1.3. *With the above definition of* (Φ, Ψ), $\prod_{i=1}^{n}{}' \otimes \mathfrak{H}_i$ *is a linear space with an inner product, that is, it satisfies the conditions* A, B, *in* (16), *p.* 64.

Thus it can be metrized by defining

$$\textit{Distance } (\Phi, \Psi) = \| \Phi - \Psi \|, \textit{ where } \| \Phi \| = \sqrt{(\Phi, \Phi)}.$$

Proof: This follows immediately from Lemmas 2.1.1 and 2.1.2, remembering (16), pp. 64–65.

Lemma 2.1.4. *We have* $\| \prod_{i=1}^{n} \otimes f_i \| = \prod_{i=1}^{n} \| f_i \|$, *and for every* $\Phi \in \prod_{i=1}^{n}{}' \otimes \mathfrak{H}_i$,

$$\Phi(f_1, \cdots, f_n) = (\Phi, \prod_{i=1}^{n} \otimes f_i)$$

$$| \Phi(f_1, \cdots, f_n) | \leqq \| \Phi \| \cdot \prod_{i=1}^{n} \| f_i \|.$$

Proof: The first equation follows directly from the definition of $\| \cdots \|$ in $\prod_{i=1}^{n}{}' \otimes \mathfrak{H}_i$; while the second is obvious, and the third results from Schwarz's inequality:

$$|\Phi(f_1, \cdots, f_n)| = |(\Phi, \prod_{i=1}^{n} \otimes f_i)| \leqq \|\Phi\| \cdot \| \prod_{i=1}^{n} \otimes f_i \| = \|\Phi\| \cdot \prod_{i=1}^{n} \|f_i\|.$$

Definition 2.1.4. Consider those functionals $\Phi \in \prod_{i=1}^{n} \otimes \mathfrak{H}_i$, for which a sequence $\Phi_1, \Phi_2, \cdots \in \prod_{i=1}^{n}{}' \otimes \mathfrak{H}_i$ exists, so that

(i) $\lim_{r \to \infty} \Phi_r(f_1, \cdots, f_n) = \Phi(f_1, \cdots, f_n)$ for all $f_i \in \mathfrak{H}_i$.

(ii) $\lim_{r, s \to \infty} \| \Phi_r - \Phi_s \| = 0$.

We write for this briefly $\Phi \sim (\Phi_1, \Phi_2, \cdots)$. Call their set $\prod_{i=1}^{n} \otimes \mathfrak{H}_i = \mathfrak{H}_1 \otimes \cdots \otimes \mathfrak{H}_n$, the *direct product of* $\mathfrak{H}_1, \cdots, \mathfrak{H}_n$.

Lemma 2.1.5. *Every sequence* $\Phi_1, \Phi_2, \cdots$ *from* $\prod_{i=1}^{n}{}' \otimes \mathfrak{H}_i$ *satisfying* (ii) *in Definition* 2.1.4. *belongs to precisely one* $\Phi \in \prod_{i=1}^{n} \otimes \mathfrak{H}_i$ *in the sense of* (i), (ii) *eod.*

If $\Phi \sim (\Phi_1, \Phi_2, \cdots)$, *then all other sequences with* $\Phi \sim (\Psi_1, \Psi_2, \cdots)$ *are characterised by* $\lim_{r \to \infty} \| \Phi_r - \Psi_r \| = 0$.

Proof: We have (by Lemma 2.1.4)

$$| \Phi_r(f_1, \cdots, f_n) - \Phi_s(f_1, \cdots, f_n) | \leqq \| \Phi_r - \Phi_s \| \prod_{i=1}^{n} \| f_i \|.$$

Thus $\lim_{r, s \to \infty} | \Phi_r(f_1, \cdots, f_n) - \Phi_s(f_1, \cdots, f_n) | = 0$, so $\lim_{r \to \infty} \Phi_r(f_1, \cdots, f_n)$ exists. Call it $\Phi(f_1, \cdots, f_n)$. Clearly $\Phi \in \prod_{i=1}^{n} \otimes \mathfrak{H}_i$. Thus (i), (ii) hold; therefore $\Phi \in \prod_{1}^{n} \otimes \mathfrak{H}_i$, $\Phi \sim (\Phi_1, \Phi_2, \cdots)$. Φ is unique by (i).

As to the second statement, we can assume, owing to the linearity, that $\Phi = \Phi_1 = \Phi_2 = \cdots = 0$. If $\lim_{r \to \infty} \| \Psi_r \| = 0$ then we find, as above, $0 = \lim_{r \to \infty} \Psi_r(f_1, \cdots, f_n)$. And $\| \Psi_r - \Psi_s \| \leqq \| \Psi_r \| + \| \Psi_s \|$, thus $\lim_{r, s \to \infty} \| \Psi_r - \Psi_s \| = 0$. Thus $0 \sim (\Psi_1, \Psi_2, \cdots)$.

Assume conversely $0 \sim (\Psi_1, \Psi_2, \cdots)$. If we did not have $\lim_{r \to \infty} \| \Psi_r \| = 0$, we would have $\| \Psi_r \| \geqq a > 0$ for a fixed $a > 0$ and infinitely many r's. Considering a suitable subsequence, we obtain it for all r's. Now $\lim_{r \to \infty} \Psi_r(f_1, \cdots f_n) = 0$ implies $\lim_{r \to \infty} (\Psi_r, \Psi^0) = 0$ for any fixed $\Psi^0 \in \prod_{i=1}^{n}{}' \otimes \mathfrak{H}_i$. Put $\Psi^0 = \Psi_{r_0}$,

where r_0 is so great that $r, s \geqq r_0$ imply $\| \Psi_r - \Psi_s \| \leqq a/2$. Then we have for $r \geqq r_0$

$$| (\Psi_r, \Psi_{r_0}) | \geqq | (\Psi_{r_0}, \Psi_{r_0}) | - | (\Psi_{r_0} - \Psi_r, \Psi_{r_0}) | \geqq \| \Psi_{r_0} \|^2 - \| \Psi_{r_0} - \Psi_r \| \cdot \| \Psi_{r_0} \|$$

$$\geqq \| \Psi_{r_0} \|^2 - \frac{a}{2} \| \Psi_{r_0} \| = \left(\| \Psi_{r_0} \| - \frac{a}{2} \right) \| \Psi_{r_0} \| \geqq \left(a - \frac{a}{2} \right) a = \frac{a^2}{2}$$

contradicting $\lim_{r \to \infty} (\Psi_r, \Psi_{r_0}) = 0$.

Lemma 2.1.6. *If* $\Phi, \Psi \in \prod_{i=1}^{n} \otimes \mathfrak{H}_i$, *that is*

$$\Phi \sim (\Phi_1, \Phi_2, \cdots), \qquad \Psi \sim (\Psi_1, \Psi_2, \cdots), \qquad \Psi_r, \Phi_r \in \prod_{i=1}^{n}{}' \otimes \mathfrak{H}_i$$

then $\lim_{r \to \infty} (\Phi_r, \Psi_r)$ *exists. Denote it by* (Φ, Ψ). *This quantity depends on* Φ, Ψ *only, and not on the particular representation used. If* $\Phi, \Psi \in \prod_{i=1}^{n}{}' \otimes \mathfrak{H}_i$, *it agrees with the previous definition.*

Proof: $\lim_{r, s \to \infty} \| \Phi_r - \Phi_s \| = 0$, $\lim_{r, s \to \infty} \| \Psi_r - \Psi_s \| = 0$ imply by Schwarz's inequality $\lim_{r, s \to \infty} | (\Phi_r, \Psi_r) - (\Phi_s, \Psi_s) | = 0$, thus $\lim_{r \to \infty} (\Phi_r, \Psi_r)$ exists. If further $\Phi \sim (\Phi_1', \Phi_2', \cdots)$, $\Psi \sim (\Psi_1', \Psi_2', \cdots)$, then we have by Lemma 2.1.5. $\lim_{r \to \infty} \| \Phi_r - \Phi_r' \| = 0$, $\lim_{r \to \infty} \| \Psi_r - \Psi_r' \| = 0$ and so

$$\lim_{r \to \infty} (\Phi_r, \Psi_r) = \lim_{r \to \infty} (\Phi_r', \Psi_r') .$$

If $\Phi, \Psi \in \prod_{i=1}^{n}{}' \otimes \mathfrak{H}_i$, then $\Phi \sim (\Phi, \Phi, \cdots)$, $\Psi \sim (\Psi, \Psi, \cdots)$ show that the new (Φ, Ψ) agrees with the old one.

Lemma 2.1.7. *If* $\Phi \in \prod_{i=1}^{n} \otimes \mathfrak{H}_i$, *then* $(\Phi, \Phi) > 0$ *for* $\Phi \neq 0$.

Proof: $(\Phi, \Phi) \geqq 0$ follows by continuity from Lemma 2.1.2. If $(\Phi, \Phi) = 0$, then for $\Phi \sim (\Phi_1, \Phi_2, \cdots)$, $\Phi_r \in \prod_{i=1}^{n}{}' \otimes \mathfrak{H}_i$, $\lim_{r \to \infty} (\Phi_r, \Phi_r) = 0$,

$$\lim_{r \to \infty} \| \Phi_r \| = 0 ,$$

and thus by Lemma 2.1.5 $\Phi = 0$. So $\Phi \neq 0$ implies $(\Phi, \Phi) > 0$.

Lemma 2.1.8. *With the above definition of* (Φ, Ψ), $\prod_{i=1}^{n} \otimes \mathfrak{H}_i$ *is a linear space with an inner product, that is it satisfies the conditions* A, B *in* (16), *p.* 64. *Thus it can be metrized by defining.*

Distance $(\Phi, \Psi) = \| \Phi - \Psi \|$, *where* $\| \Phi \| = (\Phi, \Phi)^{\frac{1}{2}}$.

Proof: This follows immediately from Lemmas 2.1.6 and 2.1.7, remembering (16) pp. 64–65.

Lemma 2.1.9. $\prod_{i=1}^{n} \otimes f_i$ *is a continuous function of the* $f_i \in \mathfrak{H}_i$. (We use now the metric, strong, topology in all these spaces.)

Proof: Put $a = \max_{i=1, \cdots, n} \| f_i^0 \|$, and assume that all $\| f_i - f_i^0 \| \leqq \epsilon$ for an $\epsilon > 0$. Then

$$\| \prod_{i=1}^{n} \otimes f_i - \prod_{i=1}^{n} \otimes f_i^0 \| = \| \sum_{j=1}^{n} \left(\prod_{i=1}^{j} \otimes f_i \otimes \prod_{i=j+1}^{n} \otimes f_i^0 - \prod_{i=1}^{j-1} \otimes f_i \otimes \prod_{i=j}^{n} \otimes f_i^0 \right) \|$$

$$= \| \sum_{j=1}^{n} \prod_{i=1}^{j-1} \otimes f_i \otimes (f_j - f_j^0) \otimes \prod_{i=j+1}^{n} \otimes f_i^0 \|$$

$$\leqq \sum_{j=1}^{n} \| \prod_{i=1}^{j-1} \otimes f_i \otimes (f_j - f_j^0) \otimes \prod_{i=j+1}^{n} \otimes f_i^0 \|$$

$$\leqq \sum_{j=1}^{n} \prod_{i=1}^{j-1} \| f_i \| \cdot \| f_j - f_j^0 \| \cdot \prod_{j+1}^{n} \| f_i^0 \|$$
$$\leqq \sum_{j=1}^{n} (a + \epsilon)^{j-1} \epsilon a^{n-j} = (a + \epsilon)^n - a^n .$$

Lemma 2.1.10. *Lemma* 2.1.4 *holds for* $\Phi \in \prod_{i=1}^{n} \otimes \mathfrak{H}_i$ *too.*

Proof: Follows by continuity.

Lemma 2.1.11. *If* $\Phi \in \prod_{i=1}^{n} \otimes \mathfrak{H}_i$, $\Phi_1, \Phi_2, \cdots \in \prod_{i=1}^{n}{}' \otimes \mathfrak{H}_i$ *then*

$$\Phi \sim (\Phi_1, \Phi_2, \cdots)$$

is equivalent to $\lim_{r\to\infty} \| \Phi - \Phi_r \| = 0$.

Proof: If $\lim_{r\to\infty} \| \Phi - \Phi_r \| = 0$, then Lemma 2.1.10 gives Definition 2.1.4, (i), and $\| \Phi_r - \Phi_s \| \leqq \| \Phi - \Phi_r \| + \| \Phi - \Phi_s \|$ gives Definition 2.1.4, (ii). Thus $\Phi \sim (\Phi_1, \Phi_2, \cdots)$. Conversely, if $\Phi \sim (\Phi_1, \Phi_2, \cdots)$, then by definition, $\| \Phi - \Phi_s \| = \lim_{r\to\infty} \| \Phi_r - \Phi_s \|$. Thus $\lim_{r,\, s\to\infty} \| \Phi_r - \Phi_s \| = 0$ implies $\lim_{s\to\infty} \| \Phi - \Phi_s \| = 0$.

Lemma 2.1.12. $\prod_{i=1}^{n}{}' \otimes \mathfrak{H}_i$, $\prod_{i=1}^{n} \otimes \mathfrak{H}_i$ *are both separable spaces, that is they satisfy condition* D *in* (16), *p.* 65.

Proof: $\prod_{i=1}^{n}{}' \otimes \mathfrak{H}_i$ is dense in $\prod_{i=1}^{n} \otimes \mathfrak{H}_i$ by Lemma 2.1.11, so it suffices to show that $\prod_{i=1}^{n}{}' \otimes \mathfrak{H}_i$ is separable. By Definition 2.1.3 the separability of the set of all $\prod_{i=1}^{n} \otimes f_i$, $f_i \in \mathfrak{H}_i$, suffices; and this follows, by Lemma 2.1.9 from the separability of the spaces $\mathfrak{H}_1, \cdots, \mathfrak{H}_n$.

Lemma 2.1.13. $\prod_{i=1}^{n} \otimes \mathfrak{H}_i$ *is topologically complete, that is it satisfies condition* E *in* (16), *p.* 65.

Proof: Assume $\Phi_1, \Phi_2, \cdots \in \prod_{i=1}^{n} \otimes \mathfrak{H}_i$, $\lim_{r,\, s\to\infty} \| \Phi_r - \Phi_s \| = 0$. For each Φ_r choose a $\Phi_r^0 \in \prod_{i=1}^{n}{}' \otimes \mathfrak{H}_i$ with $\| \Phi_r - \Phi_r^0 \| \leqq 1/r$; thus $\lim_{r\to\infty} \| \Phi_r - \Phi_r^0 \| = 0$. So $\lim_{r,\, s\to\infty} \| \Phi_r^0 - \Phi_s^0 \| = 0$, and for $\Phi \sim (\Phi_1^0, \Phi_2^0, \cdots)$, $\lim_{r\to\infty} \| \Phi - \Phi_r^0 \| = 0$ (by Lemma 2.1.11), and thus $\lim_{r\to\infty} \| \Phi - \Phi_r \| = 0$, too.

This discussion can be summed up as follows:

Theorem I. The direct product of $\mathfrak{H}_1, \cdots, \mathfrak{H}_n$, $\prod_{i=1}^{n} \otimes \mathfrak{H}_i$, *arises by the topological completion of the set* $\prod_{i=1}^{n}{}' \otimes \mathfrak{H}_i$ *of all finite linear aggregates of expressions* $\prod_{i=1}^{n} \otimes f_i^0$, $f_i^0 \in \mathfrak{H}_i$. *It is a set of functionals* $\Phi = \Phi(f_1, \cdots, f_n)$ *in the sense of Definition* 2.1.1, *thus* $\prod_{i=1}^{n} \otimes \mathfrak{H}_i \subset \prod_{i=1}^{n} \otimes \mathfrak{H}_i$ *and it is a space satisfying conditions* A, B, C, E *of* (16), *pp.* 64–65 (cf. our §1, (a)). *Further essential properties are given in Lemmas* 2.1.1, 2.1.4, 2.1.9, 2.1.10, 2.1.11.

If $n = 1$, $\prod_{i=1}^{n} \otimes \mathfrak{H}_i$ coincides not only in our notation with $\mathfrak{H}_1$, but in reality too.

In this case we may write f_1^0 for $\prod_{i=1}^{n} f_i^0$, or even better f^0. The linear aggregates of f^0's are f^0's again, so $\prod_{i=1}^{n}{}' \otimes \mathfrak{H}_i$ coincides with $\mathfrak{H}_1$. Thus already $\prod_{i=1}^{n}{}' \otimes \mathfrak{H}_i$ is complete, and $\prod_{i=1}^{n} \otimes \mathfrak{H}_i$ contains no further elements. Note that in this correspondence of the one-variable functionals Φ to the $f^0 \in \mathfrak{H}$, $\Phi(f)$ becomes simply (f^0, f). A well known theorem of M. Frèchet and F. Riesz implies therefore, that in this case the $\Phi \in \prod_{i=1}^{n} \otimes \mathfrak{H}_i$ are characterised by $\| \Phi(f) \| \leqq C \cdot \| f \|$ for a suitable constant C. (Cf. for instance (16), p. 94.)

This latter point is remarkable, because for $n > 1$ the condition

$$| \Phi(f_1, \cdots, f_n) | \leqq C \cdot \prod_{i=1}^{n} \| f_i \|$$

is necessary (cf. Lemma 2.1.10) but not sufficient for the $\Phi \in \prod_{i=1}^{n} \otimes \mathfrak{H}_i$, as we will show after Lemma 2.2.2.

2.2. It is of interest to discuss the relationship between $\prod_{i=1}^{n} \otimes \mathfrak{H}_i$ and a set of complete normalised orthogonal sets in each $\mathfrak{H}_i$.

Lemma 2.2.1. *Let a complete normalised orthogonal set $\varphi_{i,1}, \varphi_{i,2} \cdots$ be given in each $\mathfrak{H}_i$, $i = 1, \cdots n$. Then the $\prod_{i=1}^{n} \otimes \varphi_{i,t_i}$, $t_1, t_2, \cdots t_n, = 1, 2, \cdots$ form a complete normalised orthogonal set in $\prod_{i=1}^{n} \otimes \mathfrak{H}_i$.*

Proof: By definition

$$(\prod_{i=1}^{n} \otimes \varphi_{i,t_i}, \prod_{i=1}^{n} \otimes \varphi_{i,u_i}) = \prod_{i=1}^{n} (\varphi_{i,t_i}, \varphi_{i,u_i}) = \prod_{i=1}^{n} \delta_{t_i,u_i}$$

proving the normalised and orthogonal character. The linear aggregates of the $\varphi_{i,t}$ are dense in $\mathfrak{H}_i$, thus those of the $\prod_{i=1}^{n} \otimes \varphi_{i,t_i}$ are dense in $\prod_{i=1}^{n'} \otimes \mathfrak{H}_i$ (by Lemma 2.1.9), and therefore in $\prod_{i=1}^{n} \otimes \mathfrak{H}_i$. This proves the completeness too.

Lemma 2.2.2. *In order that an n-variable functional $\Phi \in \prod_{1}^{n} \otimes \mathfrak{H}_i$* (cf. Definition 2.1.1) *should belong to $\prod_{i=1}^{n} \otimes \mathfrak{H}_i$*

$$| \Phi(f_1, \cdots, f_n) | \leqq C \prod_{i=1}^{n} \| f_i \|$$

(for a suitable constant C) is necessary. If this is the case, Φ is completely characterised by the numerical values

$$\Phi(\varphi_{1,t_1}, \cdots, \varphi_{n,t_n}) = a_{t_1,\ldots,t_n} \text{ for all } t_1, \cdots, t_n = 1, 2.$$

The finiteness of $\sum_{t_1,\ldots,t_n=1}^{\infty} | a_{t_1,\ldots,t_n} |^2$ is then characteristic for $\Phi \in \prod_{i=1}^{n} \otimes \mathfrak{H}_i$. With this restriction the $a_{t_1,\ldots t_n}$ can even be prescribed arbitrarily, and a $\Phi \in \prod_{i=1}^{n} \otimes \mathfrak{H}$ with these $a_{t_1,\ldots,t_n}$ will exist.

Proof: The necessity of the first condition follows from Lemma 2.1.10. It implies, owing to the linearity, that $\Phi(f_1, \cdots, f_n)$ is continuous in the $f_1, \cdots f_n$. Thus the values $\Phi(\varphi_{1,t_1}, \cdots, \phi_{n,t_n})$ determine $\Phi(f_1, \cdots, f_n)$ if every f_i is a limit of linear aggregates of $\varphi_{i,1}, \varphi_{i,2}, \cdots$; that is always.

By Lemma 2.1.10, $(\Phi, \prod_{i=1}^{n} \otimes \varphi_{i,t_i}) = \Phi(\varphi_{1,t_1}, \cdots, \varphi_{n,t_n}) = a_{t_1,\ldots,t_n}$ therefore $\sum_{t_1,\ldots,t_n=1}^{\infty} | a_{t_1,\ldots t_n} |^2$ must be finite, by Lemma 2.2.1 and (16), p. 66. Conversely if a system $a_{t_1,\ldots t_n}$ with a finite $\sum_{t_1,\ldots,t_n=1}^{\infty} | a_{t_1,\ldots t_n} |^2$ is given, then $\Phi = \sum_{t_1,\ldots,t_n=1}^{\infty} a_{t_1,t_2,\ldots,t_n} \prod_{i=1}^{n} \otimes \varphi_{i,t_i}$ exists and is $\in \prod_{i=1}^{n} \otimes \mathfrak{H}_i$ by Lemma 2.2.1 and (16), p. 67, and for the same reason

$$\Phi(\varphi_{1,t_1}, \cdots \varphi_{n,t_n}) = (\Phi, \prod_{i=1}^{n} \otimes \varphi_{i,t_i}) = a_{t_1,\ldots,t_n}.$$

We now see that in the case $n = 2$, the preliminary condition on Φ,

$$| \Phi(f_1, f_2) | \leqq C \| f_1 \| \cdot \| f_2 \|,$$

means that the matrix $(a_{t_1,t_2})_{t_1,t_2=1,2,\ldots}$ associated with Φ is bounded in the sense of Hilbert's theory; while $\Phi \in \prod_{i=1}^{n} \otimes \mathfrak{H}_i$ means the finiteness of

$$\sum_{t_1,t_2=1}^{\infty} | a_{t_1,t_2} |^2,$$

that is that the matrix belongs to E. Schmidt's class of matrices. Thus while

the preliminary condition is already characteristic for $n = 1$, it is not for $n = 2$. If $\mathfrak{H}_1 = \mathfrak{H}_2 = \mathfrak{H}$ is a Hilbert space, and I a conjugation (passing to the complex conjugate, (cf. (15), p. 357) in $\mathfrak{H}$, then $\Phi(f_1, f_2) = (If_1, f_2)$ is an example for the opposite; because with $\varphi_{1,t} = \varphi_{2,t}$ we obtain $a_{t_1,t_2} = \delta_{t_1,t_2}$.

Lemma 2.2.3. *If in the notation of Lemma* 2.2.2 $\Phi, \Psi \epsilon \prod_{i=1}^{n} \otimes \mathfrak{H}_i$ *corresponds to* $a_{t_1,\cdots,t_n}$ *resp.* $b_{t_1,\cdots,t_n}$ *then*

$$(\Phi, \Psi) = \sum_{t_1,\cdots,t_n=1}^{\infty} a_{t_1,\cdots,t_n} \cdot \overline{b_{t_1,\cdots,t_n}},$$

the series being absolutely convergent.

Proof: The last formula of the proof of Lemma 2.2.2, together with Lemma 2.2.1 and (16), p. 66, give this.

Lemmas 2.2.2 and 2.2.3 show that the dimension of $\prod_{i=1}^{n} \otimes \mathfrak{H}_i$ is the product of the dimensions of all $\mathfrak{H}_i$ that is: 0 if at least one of them is 0; ∞ if none of them is 0 but at least one ∞; the finite product if all are finite. From now on we assume that all $\mathfrak{H}_i$ have dimensions $\neq 0$, that is that $\mathfrak{H}_i \neq (0)$.

2.3. We define now certain operators in $\prod_{i=1}^{n} \otimes \mathfrak{H}_i$. We will denote the ring of all bounded, everywhere defined, linear and closed operators in $\mathfrak{H}_i$ by $\mathbf{B}_i$, $i = 1, \cdots, n$ the same in $\prod_{i=1}^{n} \otimes \mathfrak{H}_i$ by $\mathbf{B} \otimes$. (Cf. (17), p. 370.)

Lemma 2.3.1. *If an operator* $A_i \epsilon \mathbf{B}_i$ *and a* $\Phi \epsilon \prod_{i=1}^{n} \otimes \mathfrak{H}_i$ *is given, then a unique* $\Phi^* \epsilon \prod_{1}^{n} \otimes \mathfrak{H}_i$ *with*

$$\Phi^*(f_1, \cdots, f_i, \cdots f_n) = \Phi(f_1, \cdots, A_i^* f_i, \cdots, f_n)$$

exists. (A_i^* *is the adjoint of* A_i). *If* $\| A_i f_i \| \leqq D \| f_i \|$ *for all* $f_i \epsilon \mathfrak{H}_i$, *then* $\| \Phi^* \| \leqq D \| \Phi \|$.

Proof: We may assume $i = n$. The uniqueness of Φ^* and its existence in $\prod_{i=1}^{n} \otimes \mathfrak{H}_i$ are obvious.

A C with $| \Phi(f_1, \cdots, f_n) |^2 \leqq C^2 \prod_{i=1}^{n} \| f_i \|^2$ exists, therefore there exists for any given $f_1, \cdots, f_{n-1}$ an $f^0 = f^0(f_1, \cdots, f_{n-1}) \epsilon \mathfrak{H}_n$ with $\Phi(f_1, \cdots, f_n) = (f^0, f_n)$, and $\| f^0 \| \leqq C \prod_{i=1}^{n-1} \| f_i \|$ (cf. (16), p. 94). Now

$$\Phi^*(f_1, \cdots, f_n) = \Phi(f_1, \cdots, A_n^* f_n) = (f^0, A_n^* f_n) = (A_n f^0, f_n).$$

Furthermore A_n is bounded, therefore a D with $\| A_n f \| \leqq D \| f \|$ exists. So we have:

$$| \Phi^*(f_1, \cdots, f_n) | = | (A_n f^0, f_n) | \leqq \| A_n f^0 \| \cdot \| f_n \|$$

$$\leqq D \| f^0 \| \cdot \| f_n \| \leqq CD \prod_{i=1}^{n} \| f_i \|$$

$$\sum_{t_1,\cdots,t_n=1}^{\infty} | \Phi^*(\varphi_{1,t_1}, \cdots \varphi_{n,t_n}) |^2 = \sum_{t_1,\cdots,t_n=1}^{\infty} | (A_n f^0(\varphi_{1,t_1}, \cdots), \varphi_{n,t_n}) |^2$$

$$= \sum_{t_1,\cdots,t_{n-1}=1}^{\infty} \| A_n f^0(\varphi_{1,t_1}, \cdots, \varphi_{n-1,t_{n-1}}) \|^2$$

$$\leqq D^2 \sum_{t_1,\cdots,t_{n-1}=1}^{\infty} \| f^0(\varphi_{1,t_1}, \cdots, \varphi_{n-1,t_{n-1}}) \|^2$$

$$\leqq D^2 \sum_{t_1,\cdots,t_n=1}^{\infty} | (f^0(\varphi_{1,t_1}, \cdots, \varphi_{n-1,t_{n-1}}), \varphi_{n,t_n}) |^2$$

$$= D^2 \sum_{t_1,\cdots,t_n=1}^{\infty} | \Phi(\varphi_{1,t_1}, \cdots, \varphi_{n,t_n}) |^2.$$

Thus $\Phi \in \prod_{i=1}^{n} \otimes \mathfrak{H}_i$ implies $\Phi^* \in \prod_{i=1}^{n} \otimes \mathfrak{H}_i$ (by Lemma 2.2.2), and we have $\| \Phi^* \| \leqq D \| \Phi \|$ (by Lemma 2.2.3).

Lemma 2.3.2. *If we define an operator* $A_i^{(i)}$ *by* $A_i^{(i)} \Phi = \Phi^*$ *in the sense of Lemma* 2.3.1, *then* $A_i^{(i)} \in \mathbf{B} \otimes$.

Proof: $A_i^{(i)}$ is everywhere defined and linear by Lemma 2.3.1; and if we choose D with $\| A_i f \| \leqq D \| f \|$, then $\| A_i^{(i)} \Phi \| \leqq D \| \Phi \|$ by the same Lemma. Thus $A_i^{(i)}$ is bounded and therefore it is closed, too.

Lemma 2.3.3. $A_i^{(i)} \prod_{j=1}^{n} \otimes f_j = \prod_{j=1}^{i-1} f_j \otimes A_i f_i \otimes \prod_{j=i+1}^{n} f_j$.

Proof: Follows immediately from the definition.

Lemma 2.3.4. *The correspondence* $A_i \sim A_i^{(i)}$ *is a (one to one) isomorphism for the operations* αA_i, A_i^*, $A_i + B_i$, *and* $A_i B_i$.

Proof: This is obvious for αA_i and $A_i + B_i$; for A_i^* and $A_i B_i$ it follows by applying these operators to the $\prod_{j=1}^{n} \otimes f_j$ as the limits of linear aggregates of $\prod_{j=1}^{n} \otimes f_j$ cover all $\prod_{j=1}^{n} \otimes \mathfrak{H}_j$.

That $A_i = B_i$ implies $A_i^{(i)} = B_i^{(i)}$, is clear; the converse is again seen by considering the $\prod_{j=1}^{n} \otimes f_j$.

Definition 2.3.1. Call the set of all $A_i^{(i)}$, if A_i runs over all $\mathbf{B}_i$, $\mathbf{B}_i^{(i)}$.

Lemma 2.3.5. $\mathbf{B}_i^{(i)} = (\mathbf{B}_j^{(j)}, j \gtrless i)'$; *that is:* $\mathbf{B}_i^{(i)}$ *is the set of all operators in* $\mathbf{B} \otimes$ *which commute with all elements of all* $\mathbf{B}_j^{(j)}$*s* $j \gtrless i$.

Proof: It is obvious that for $j \gtrless i$, $A_i^{(i)} A_j^{(j)} \Phi = A_j^{(j)} A_i^{(i)} \Phi$ if $\Phi = \prod_{k=1}^{n} \otimes f_k$. Thus this holds for all $\Phi \in \prod_{k=1}^{n} \otimes \mathfrak{H}_k$, $A_j^{(j)}$ and $A_i^{(i)}$ commute. In other words: $\mathbf{B}_i^{(i)} \subset (\mathbf{B}_j^{(j)}, j \gtrless i)'$.

Assume now $A^0 \in (\mathbf{B}_j^{(j)}, j \gtrless i)'$. Then A^0 commutes with every $A_j^{(j)}$, $j \gtrless i$.

Introduce again the complete normalised orthogonal systems $\varphi_{j,1}, \varphi_{j,2}, \cdots$ in $\mathfrak{H}_j$, $j = 1, \cdots, n$. The projection $P_{[\varphi_{j,t}]}$ (cf. (16), p. 74) belongs to $\mathfrak{H}_j$, it transforms $\varphi_{j,u}$ into $\delta_{u,t} \varphi_{j,t}$. Thus $P_{[\varphi_{j,t}]}^{[j]} \in \mathbf{B}_j^{(j)}$ transforms $\prod_{k=1}^{n} \otimes \varphi_{k,t_k}$ into $\delta_{t_j,t} \prod_{k=1}^{n} \otimes \varphi_{k,t_k}$; and as this is a complete normalised orthogonal system, this means that $P_{[\varphi_{j,t}]}^{(j)}$ is the projection of $[\prod_{k=1}^{n} \varphi_{k,t_k}, t_j = t]$ (the smallest linear and closed set containing all $\prod_{k=1}^{n} \varphi_{k,t_k}$ with $t_j = t$). Now A^0 commutes with $P_{[\varphi_{j,t}]}^{[j]}$. Therefore $P_{[\varphi_{j,t}]}^{[j]} \Phi = \Phi$ implies $P_{[\varphi_{j,s}]}^{[j]} A^0 \Phi = A^0 \Phi$. The former is the case for $\Phi = \prod_{k=1}^{i-1} \otimes \varphi_{k,t_k} \otimes f_i \otimes \prod_{k=i+1}^{n} \otimes \varphi_{k,t_k}$ and $t = t_j$. Thus $A^0\Phi$ when written as a $\prod_{k=1}^{n} \otimes \varphi_{k,u_k}$ series, $u_1, \cdots u_n = 1, 2, \cdots$ has terms with $u_j = t_j$ only. As this holds for every $j \gtrless i$ we have

$$A^0 \Phi = \prod_{k=1}^{i-1} \otimes \varphi_{k,t_k} \otimes f_i^* \otimes \prod_{k=i+1}^{n} \otimes \varphi_{k,t_k}.$$

Thus f_i^* is determined uniquely by f_i and the t_k, $k \gtrless i$. Consideration of the operator $P_{[\varphi_{j,r}+\varphi_{j,s}]}^{(j)}$, $j \gtrless i$, $t \gtrless s$, shows however, that $t_j = t$ and $t_j = s$ give the same. So the value of t_j does not matter. Thus $f_i^* = A_i f_i$ for a fixed operator A_i in $\mathfrak{H}_i$. It is clear that A_i is everywhere defined and linear; and $\| A^0 \Phi \| \leqq D \| \Phi \|$ implies $\| A_i f_i \| \leqq D \| f_i \|$ so that A_i is bounded and therefore closed. So $A_i \in \mathbf{B}_i$. The definition of A_i gives $A^0 \Phi = A_i^{(i)} \Phi$ for all $\Phi = \prod_{i=1}^{n} \otimes \varphi_{i t_i}$, thus for all Φ. So we have $A^0 = A_i^{(i)} \in \mathbf{B}_i^{(i)}$.

This proves $\mathbf{B}_i^{(i)} \supset (B_j^{(j)}; j \gtrless i)'$, completing the proof of $\mathbf{B}_i^{(i)} = (\mathbf{B}_j^{(j)}; j \gtrless i)'$.

Lemma 2.3.6. $\mathbf{B}_i^{(i)}$ *is a ring and contains the operator* 1.

Proof: This follows from (18), p. 397, because $\mathbf{B}_i^{(i)}$ has the form S' (by Lemma 2.3.5).

Lemma 2.3.7. *Let* S_i *be a set* $\subset \mathbf{B}_i$, *and* $S_i^{(i)}$ *the set of all* $A_i^{(i)}$, $A_i \in S_i^{(i)} \subset (\mathbf{B} \otimes)$. *Then* S_i *is a ring if and only if* $S_i^{(i)}$ *is one.*

Proof: Define a ring by its characteristics established in (22), §3. The only notions entering there are the operations αA, A^*, $A + B$, AB, and the strongest topology, as defined in (22), §2. Now all these are invariant under $A_i \sim A_i^{(i)}$: the algebraic operations by Lemma 2.3.4, and the strongest topology by (22), §4. Besides closure in $\mathbf{B}_i^{(i)}$ and in $\mathbf{B} \otimes$ is the same thing, because $\mathbf{B}_i^{(i)}$ is a ring by Lemma 2.3.6, thus weakly closed in $\mathbf{B}\otimes$, and a fortiori in the strongest sense. This completes the proof.

Lemma 2.3.8. $\mathbf{R}(\mathbf{B}_1^{(1)}, \cdots, \mathbf{B}_n^{(n)}) = \mathbf{B} \otimes$.

Proof: We could prove this by a direct construction, but the following indirect proof is quicker: The same considerations as the ones we made in the proof of Lemma 2.3.5 (but now without the exceptional i) show that

$$(\mathbf{B}_j^{(j)}; j = 1, \cdots, n)' = (\alpha 1) .$$

So $\mathbf{R}(\mathbf{B}_1^{(1)}, \cdots, \mathbf{B}_n^{(n)})' = (\mathbf{B}_1^{(1)}, \cdots, \mathbf{B}_n^{(n)})' = (\alpha 1)$, and by (18), p. 397, $\mathbf{R}(\mathbf{B}_1^{(1)}, \cdots, \mathbf{B}_n^{(n)}) = (\alpha 1)' = \mathbf{B} \otimes$.

We will frequently use the above quoted Theorem 8 from (18), p. 397: That if $\mathbf{M}$ is a ring which contains 1, then $\mathbf{M} = \mathbf{M}''$. No particular references will therefore be made to it.

The results of the foregone discussion can be summed up as follows:

Theorem II. *The sets* $\mathbf{B}_i^{(i)}$, $i = 1, \cdots, n$, *defined in Definition* 2.3.1, *are rings which contain* 1; $\mathbf{B}_i^{(i)}$ *being isomorphic to* $\mathbf{B}_i$. *Any two* $\mathbf{B}_i^{(i)}$ *commute, and* $\mathbf{R}(\mathbf{B}_1^{(1)}, \cdots, \mathbf{B}_n^{(n)}) = \mathbf{B} \otimes$.

Further essential properties are given in Lemmas 2.3.5, *and* 2.3.7.

2.4. The asymmetric aspect of the case $n = 2$ is of importance.

Form $\prod_{i=1}^{2} \otimes \mathfrak{H}_i = \mathfrak{H}_1 \otimes \mathfrak{H}_2$, and introduce a (finite or infinite) complete normalised orthogonal system in $\mathfrak{H}_1$: $\varphi_1, \varphi_2 \cdots$. Now define

Definition 2.4.1. If $\Phi \in \mathfrak{H}_1 \otimes \mathfrak{H}_2$, form the $f^0(\phi_t) \in \mathfrak{H}_2$, $t = 1, 2, \cdots$ as in the proof of Lemma 2.3.1. Denote them by $f^{(t)}$, $t = 1, 2, \cdots$ resp., and write $\Phi \sim < f^{(1)}, f^{(2)}, \cdots > = < f^{(t)} >_{t=1, 2 \cdots}$.

Lemma 2.4.1. *The* $f^{(t)}$, $t = 1, 2, \cdots$ *determine* $\Phi \sim < f^{(1)}, f^{(2)}, \cdots >$ *uniquely. If dimension* $\mathfrak{H}_1$, *is finite, the* $f^{(t)}$ *can be prescribed arbitrarily; if it is infinite, then the only restriction is that* $\sum_{t=1}^{\infty} \| f^{(t)} \|^2$ *must be finite.*

If $\Phi \sim < f^{(1)}, f^{(2)}, \cdots >$, $\Psi \sim < g^{(1)}, g^{(2)}, \cdots >$, *then* $(\Phi, \Psi) = \sum_t (f^{(t)}, g^{(t)})$. *The sum is absolutely convergent, if infinite at all.*

Proof: Let $\psi_1, \psi_2, \cdots$ be a complete normalised orthogonal system in $\mathfrak{H}_2$; form, as in Lemma 2.2.2, $\Phi\ (\varphi_{t_1}, \psi_{t_2}) = (f^{(t_1)}, \psi_{t_2}) = a_{t_1, t_2}$. Φ is uniquely determined by the a_{t_1, t_2} and so by the $f^{(t_1)}$. For given a_{t_1, t_2}'s Φ exists if $\sum_{t_1, t_2} | a_{t_1, t_2} |^2$ is finite; but if the a_{t_1, t_2} are defined by the $f^{(t_1)}$: $a_{t_1, t_2} = (f^{(t_1)}, \psi_{t_2})$

then $\sum_{t_2} | a_{t_1, t_2} |^2 = \| f^{(t_1)} \|^2$, therefore the condition is, that $\sum_{t_1} \| f^{(t_1)} \|^2$ is finite.

If the dimension of $\mathfrak{H}_1$, is finite, the number of the t_1's is, and the condition is void. If the dimension of $\mathfrak{H}_1$, is infinite, we obtain the condition: $\sum_{t_1=1}^{\infty} \| f^{(t_1)} \|^2$ is finite. Finally Lemma 2.2.3 gives:

$$(\Phi, \Psi) = \sum_{t_1, t_2} (f^{(t_1)}, \psi_{t_2}) (g^{(t_1)}, \psi_{t_2}) = \sum_{t_1} (f^{(t_1)}, g^{(t_1)}) ,$$

all sums being absolutely convergent.

The operators in $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ can be expressed in a simple normal form, in which only operators from $\mathfrak{H}_2$ occur:

Definition 2.4.2. If A^0 is an operator in $\mathbf{B} \otimes$ (for $\mathfrak{H}_1 \otimes \mathfrak{H}_2$), then form $A^0 < 0, \cdots, 0, f, 0, \cdots > = < f_1^*, f_2^*, \cdots >$ (the f stands in place no. t). Every f_s^* defines an operator $A_{t,s}$ by $f_s^* = A_{t,s} f_t$. Write $A^0 \sim < A_{t,s} >_{t,s=1,2\cdots}$.

Lemma 2.4.2. *All $A_{t,s}$ belong to $\mathbf{B}_2$ (for $\mathfrak{H}_2$), they determine A^0 uniquely. If dimension of $\mathfrak{H}_1$ is finite, the $A_{t,s}$ can be prescribed arbitrarily; if it is infinite, they cannot.*

Proof: $A_{t,s}$ is clearly everywhere defined and linear. A^0 is bounded, so a C with $\| A^0 \Phi \| \leqq C \| \Phi \|$ exists. Then

$$\| A_{t,s} f \|^2 \leqq \sum_{s'} \| A_{t,s'} f \|^2 = \| A^0 < 0, 0, \cdots, 0, f, 0, \cdots > \|^2$$

$$\leqq C^2 \| < 0, \cdots, 0, f, 0 \cdots > \|^2 = C^2 \| f \|^2, \| A_{t,s} f \| \leqq C \| f \| .$$

Thus $A_{t,s}$ is bounded and closed too. It is clear from Definition 2.4.3, that the $A_{t,s}$ determine A^0 uniquely for all $\Phi = < 0, \cdots, 0, f, 0, \cdots >$, and therefore for all Φ.

If dimension of $\mathfrak{H}_1$, is finite, say p, then the formula in Definition 2.4.2 certainly defines an operator A^0, which is everywhere defined and linear. Each $A_{t,s}$ is bounded, say $\| A_{t,s} f \| \leqq C_{t,s} \| f \|$. Then

$$A^0 < f_1, \cdots, f_p > = < \sum_{t=1}^{p} A_{t,1} f_t, \cdots, \sum_{t=1}^{p} A_{t,p} f_t > .$$

$$\| A^0 \Phi \|^2 = \| A^0 < f_1, \cdots, f_p > \|^2 = \sum_{s=1}^{p} \| \sum_{t=1}^{p} A_{t,s} f_t \|^2$$

$$\leqq \sum_{s=1}^{p} p \sum_{t=1}^{p} \| A_{t,s} f_t \|^2$$

$$\leqq \sum_{s=1}^{p} p \sum_{t=1}^{p} C_{t,s} \| f_t \|^2 \leqq C^2 \sum_{t=1}^{p} \| f_t \|^2 = C^2 \| \Phi \|^2 ,$$

where $C^2 = p \max_{t=1, \cdots, p} \sum_{s=1}^{p} C_{t,s}^2$, and so $\| A^0 \Phi \| \leqq C \| \Phi \|$. Thus A^0 is bounded and closed too.

It would be easy to formulate the restrictions arising if the dimension of $\mathfrak{H}_1$ is infinite, but they are of a very implicit type. In fact, even if $\mathfrak{H}_2$ is one dimensional, they express that the numerical matrix $< a_{t,s} >_{t,s=1,2,\cdots}$ is bounded in the sense of Hilbert's theory.

The rules of computation for this representation $A^0 \sim < A_{t,s} >_{t,s=1,2,\cdots}$ are very similar to those of common matrix-algebra.

Lemma 2.4.3. *If A^0, B^0, $C^0 \in \mathbf{B} \otimes$, and $A^0 \sim < A_{t,s} >_{t,s=1,2\cdots}$*

$B^0 \sim < B_{t,s} >_{t,s=1,2,\cdots}$, $C^0 \sim < C_{t,s} >_{t,s=1,2,\cdots}$ *then the following rules of computation hold:*

(i) *If* $C^0 = \alpha A^0$, *then* $C_{t,s} = \alpha A_{t,s}$.

(ii) *If* $C^* = (A^0)^*$, *then* $C_{t,s} = A^*_{s,t}$.

(iii) *If* $C^0 = A^0 + B^0$, *then* $C_{t,s} = A_{t,s} + B_{t,s}$.

(iv) *If* $C^0 = A^0 B^0$, *then* $C_{t,s} = \sum_{n=1}^{\infty} A_{n,s} B_{t,n}$. *The* $\sum_{n=1}^{\infty}$ *converges in the sense of the strong topology of* $\mathbf{B}_2$, *irrespectively of the order in which the* $n = 1, 2, \cdots$ *are gone through.*

Proof: (i), (iii) are obvious. Denoting $< 0, \cdots, 0, f, 0, \cdots >$ (the f stands in place no. t) by $f^{(t)}$, we see: $(A^0 f^{(t)}, g^{(s)}) = (A_{t,s} f, g)$. Therefore in case (ii) we have on one hand $(C^0 f^{(t)}, g^{(s)}) = (C_{t,s} f, g)$ and on the other $(C^0 f^{(t)}, g^{(s)}) = \overline{(A^0 g^{(s)}, f^{(t)})} = \overline{(A_{s,t} g, f)} = (A^*_{s,t} f, g)$ thus proving $C_{t,s} = A_{s,t}{}^*$.

Consider finally case (iv). We have $C^0 f^{(t)} = < C_{t,s} f >_{s=1,2,\cdots}$.

On the other hand $C^0 f^{(t)} = A^0 B^0 f^{(t)} = A^0 < B_{t,n} f >_{n=1,2,\cdots} = A^0 \sum_{n=1}^{\infty} (B_{t,n} f)^{(n)}$ where the $\sum_{n=1}^{\infty}$ is strongly convergent, irrespectively of the order in which the $n = 1, 2, \cdots$ are gone through. So we have further (as A^0 is continuous):

$$C^0 f^{(t)} = \sum_{n=1}^{\infty} A^0 (B_{t,n} f)^{(n)} = \sum_{n=1}^{\infty} < A_{n,s} B_{t,n} f >_{s=1,2,\cdots}$$

$$= < \sum_{n=1}^{\infty} A_{n,s} B_{t,n} f >_{s=1,2,\cdots}$$

with the same kind of convergence. This proves $C_{t,s} = \sum_{n=1}^{\infty} A_{n,s} B_{t,n}$. We now investigate the correspondences $A_i \sim A_i^{(i)}$.

Lemma 2.4.4. *If* $A \in \mathbf{B}_2$, *then* $A^{(2)} \in \mathbf{B} \otimes$ *is* $\sim < \delta_{t,s} A >_{t,s=1,2,\cdots}$.

Proof: Obviously $\varphi_t \otimes f \sim < 0, \cdots, 0, f, 0, \cdots >$ (f in place no. t), therefore by Lemma 2.3.3 $A^{(2)} < 0, \cdots, 0, f, 0, \cdots > = < 0, \cdots, 0, Af, 0, \cdots >$. Thus $A^{(2)}_{t,s} = 0$ if $s \neq t$ and $= A$ if $s = t$, in other words $A_{t,s} = \delta_{t,s} A$.

Lemma 2.4.5. *If* S *is a set* $\subset \mathbf{B}_2$, *then the set* $S^{(2)}$ *of all* $A^{(2)}$, $A^{(2)} \in S$, *consists of all* $A^0 \sim < \delta_{t,s} A >_{t,s=1,2,\cdots}$, $A \in S$ (*every* $A \in S$ *generates an* A^0!); *and the set* $S^{(2)\prime}$ *consists of all* $A^0 \sim < A_{t,s} >_{t,s=1,2,\cdots}$, $A_{t,s} \in S'$.

Proof: The first statement follows immediately from Lemma 2.4.4. As to the second, observe that

$$< \delta_{t,s} A > < A_{t,s} > = < A A_{t,s} >; \quad < A_{t,s} > < \delta_{t,s} A > = < A_{t,s} A >;$$

thus $< A_{t,s} >$ commutes with $A^{(2)} \sim < \delta_{t,s} A >$ if and only if every $A_{t,s}$ commutes with A. Furthermore $A^{(2)*} = A^{*(2)}$ (by Lemma 2.3.4, $< (\delta_{s,t} A)^* > = < \delta_{t,s} A^* >$. These facts together establish the equivalence of $< A_{t,s} > \in S^{(2)\prime}$ and of all $A_{t,s} \in S'$.

Lemma 2.4.6. $\mathbf{B}_2^{(2)}$ *is the set of all* $A^0 \sim < \delta_{t,s} A >_{t,s=1,2,\cdots}$, $A \in \mathbf{B}_2$ (*every* $A \in \mathbf{B}_2$ *generates an* A^0!); $\mathbf{B}_1^{(1)}$ *is the set of all* $A^0 \sim < \alpha_{t,s} 1 >_{t,s=1,2,\cdots}$.

Proof: Follows from Lemma 2.4.5 by putting $S = \mathbf{B}_2$, considering that $\mathbf{B}_2^{(2)\prime} = \mathbf{B}_1^{(1)}$ (by Lemma 2.3.5).

In those applications in which the aspect of this § will be used, the important object will always be $\mathfrak{H}_2$, while $\mathfrak{H}_1$ will only play a rôle in so far as its dimension may be prescribed. If dimension of $\mathfrak{H}_1$ is $N = 1, 2, \cdots, \infty$, and if we write $\mathfrak{H}$ for $\mathfrak{H}_2$, we will use the abbreviated notation $N \otimes \mathfrak{H}$ for $\mathfrak{H}_1 \otimes \mathfrak{H}_2$.

Chapter III: Factorisations of B

3.1. We now formulate the two notions which play the central rôle in all our discussions (chiefly the second one). We consider a space $\mathfrak{H}$ and the ring $\mathbf{B}$ of everywhere defined, bounded, linear and closed operators in $\mathfrak{H}$.

Definition 3.1.1. A system of $n(=1, 2, \cdots)$ subrings $\mathbf{M}_1, \cdots, \mathbf{M}_n \neq (0)$ of $\mathbf{B}$ is called a *factorisation*, if

(i) $\mathbf{M}_i \subset \mathbf{M}_j'$, that is every element of $\mathbf{M}_i$ commutes with every element of $\mathbf{M}_j$, for $i \neq j$,

(ii) $\mathbf{R}(\mathbf{M}_1, \cdots, \mathbf{M}_n) = \mathbf{B}$.

Definition 3.1.2. A subring $\mathbf{M}$ of $\mathbf{B}$ is called a *factor*, if

$$\mathbf{M} \cdot \mathbf{M}' = (\alpha 1),$$

that is, if the center of $\mathbf{M}$ (the set of all those elements of $\mathbf{M}$ which commute with every element of $\mathbf{M}$) consists of the numerical multiples of 1 alone.

The connection between these two notions is given by the following Lemmas:

Lemma 3.1.1. *If* $\mathbf{M}_1, \cdots, \mathbf{M}_n$ *is a factorisation, then every* $\mathbf{M}_i$ *is a factor.*

Proof: We have

$$\mathbf{M}_i \cdot \mathbf{M}_i' \subset (\prod_{j=1, j\neq i}^{n} \mathbf{M}_j') \cdot \mathbf{M}_i' = \prod_{j=1}^{n} \mathbf{M}_j' = (\mathbf{M}_1, \cdots, \mathbf{M}_n)'$$
$$= (\mathbf{R}(\mathbf{M}_1, \cdots, \mathbf{M}_n))' = \mathbf{B}' = (\alpha 1).$$

By (18), p. 393, $\mathbf{M}_i \cdot \mathbf{M}_i'$ contains a projection E_0; thus $E_0 = \alpha 1$, that is $E_0 = 0$ or 1. $E_0 = 0$ would imply $\mathbf{M}_i = (0)$ (cf. eod.: if $A \in \mathbf{M}_i$ then $AE_0 = A$), thus $E_0 = 1$. Therefore $\mathbf{M}_i \cdot \mathbf{M}_i' = (\alpha 1)$.

Lemma 3.1.2. $\mathbf{M}$ *is a factor if and only if* $\mathbf{M}, \mathbf{M}'$ *is a factorisation.*

Proof: If $\mathbf{M}, \mathbf{M}'$ is a factorisation then $\mathbf{M}$ is a factor by Lemma 3.1.1. If conversely $\mathbf{M}$ is a factor, then $1 \in \mathbf{M} \cdot \mathbf{M}'$, and so $\mathbf{M}, \mathbf{M}' \neq (0)$; further $1 \in \mathbf{M} \subset \mathbf{R}(\mathbf{M}, \mathbf{M}')$, and so $\mathbf{M} = \mathbf{M}''$ and $\mathbf{R}(\mathbf{M}, \mathbf{M}') = (\mathbf{R}(\mathbf{M}, \mathbf{M}'))''$. Now $(\mathbf{R}(\mathbf{M}, \mathbf{M}'))' = (\mathbf{M}, \mathbf{M}')' = \mathbf{M}' \cdot \mathbf{M}'' = \mathbf{M}' \cdot \mathbf{M} = (\alpha 1)$. $\mathbf{R}(\mathbf{M}, \mathbf{M}') = (\alpha 1)' = \mathbf{B}$. Finally $\mathbf{M}$ and $\mathbf{M}'$ commute. Thus $\mathbf{M}, \mathbf{M}'$ is a factorisation.

These Lemmas prove that if $\mathbf{M}$ is a factor, then $\mathbf{M}'$ is one too. If $\mathbf{M}_1$ is a factor then it is a ring and $1 \in \mathbf{M}_1$; thus $\mathbf{M}_1 = \mathbf{M}_1''$; therefore $\mathbf{M}_1' = \mathbf{M}_2$ implies $\mathbf{M}_2' = \mathbf{M}_1$. Therefore we can define:

Definition 3.1.3. If $\mathbf{M}_1, \mathbf{M}_2$ is a factorisation, then $\mathbf{M}_1$ and $\mathbf{M}_2$ are called *coupled factors* if $\mathbf{M}_1' = \mathbf{M}_2$ or (which is equivalent to it) $\mathbf{M}_2' = \mathbf{M}_1$. The following problem arises immediately:

Problem 1. *Is every factorisation* $\mathbf{M}_1, \mathbf{M}_2$ *made up of coupled factors?*

We will see that the answer is affirmative in a certain special case (cf. Lemma 3.2.4) but negative in general (cf. §13.4). This problem is of some interest in the case $n > 2$ too, because if $\mathbf{M}_1, \cdots, \mathbf{M}_n$ is a factorisation, then $\mathbf{R}(\mathbf{M}_1, \cdots, \mathbf{M}_k)$, $\mathbf{R}(\mathbf{M}_{k+1}, \cdots, \mathbf{M}_n)$ where $1 \leqq k \leqq n-1$, is one too.

3.2. A simple example of a factorisation is this one: If $\mathfrak{H} = \prod_{i=1}^{n} \otimes \mathfrak{H}_i = \mathfrak{H}_1 \otimes \cdots \otimes \mathfrak{H}_n$, then $\mathbf{B}_1^{(1)}, \cdots, \mathbf{B}_n^{(n)}$ form a factorisation by Theorem II. Thus every $\mathbf{B}_i^{(i)}$ is a factor. This suggests the following definitions:

Definition 3.2.1. A factorisation $\mathbf{M}_1, \cdots, \mathbf{M}_n$ is a *direct factorisation*, if $\mathfrak{H}$ is isomorphic to a direct product $\mathfrak{H}_1 \otimes \cdots \otimes \mathfrak{H}_n$, so that $\mathbf{M}_1, \cdots, \mathbf{M}_n$ becomes $\mathbf{B}_1^{(1)}, \cdots, \mathbf{B}_n^{(n)}$ resp.

Definition 3.2.2. A factor $\mathbf{M}$ is a *direct factor*, if $\mathfrak{H}$ is isomorphic to a direct product $\mathfrak{H}_1 \otimes \cdots \otimes \mathfrak{H}_n$, so that $\mathbf{M}$ becomes a $\mathbf{B}_i^{(i)}$, $i = 1, \cdots, n$.

The following remarks lie near:

Lemma 3.2.1. *We can restrict ourselves in Definition* 3.2.1 *to the case where* $n = 2$, $i = 1$ (*or just as well* $i = 2$).

Proof: Lemmas 2.2.2 and 2.2.3 show, that $\prod_{j=1}^{n} \otimes \mathfrak{H}_j$ is isomorphic to $\mathfrak{H}_i \otimes (\prod_{j=1\,(j\neq i)}^{n} \otimes \mathfrak{H}_j)$ and to $(\prod_{j=1(j\neq i)}^{n} \otimes \mathfrak{H}_j) \otimes \mathfrak{H}_i$.

Lemma 3.2.2. *If* $\mathbf{M}_1, \cdots, \mathbf{M}_n$ *is a direct factorisation, then every* $\mathbf{M}_i$ *is a direct factor.*

Proof: Obvious.

Lemma 3.2.3. $\mathbf{M}$ *is a direct factor if and only if* $\mathbf{M}, \mathbf{M}'$ *is a direct factorisation.*

Proof: If $\mathbf{M}, \mathbf{M}'$ is a direct factorisation, then $\mathbf{M}$ is a direct factor by Lemma 3.2.2. If conversely $\mathbf{M}$ is a direct factor, then we may assume by Lemma 3.2.1 that $\mathfrak{H} = \mathfrak{H}_1 \otimes \mathfrak{H}_2$, $\mathbf{M} = \mathbf{B}_1^{(1)}$. Then by Lemma 2.3.5 $\mathbf{M}' = \mathbf{B}_1^{(1)\prime} = \mathbf{B}_2^{(2)}$, proving the statement.

Lemmas 3.2.2 and 3.2.3 are perfect analogs of Lemmas 3.1.1 and 3.1.2. We now solve Problem 1 for direct factors.

Lemma 3.2.4. *If* $\mathbf{M}_1$, $\mathbf{M}_2$ *is a factorisation, and either* $\mathbf{M}_1$ *or* $\mathbf{M}_2$ *is a direct factor, then both are it, they are coupled, and* $\mathbf{M}_1$, $\mathbf{M}_2$ *is a direct factorisation.*

Proof: Owing to the symmetry we may assume that $\mathbf{M}_1$ is a direct factor, and thus that $\mathfrak{H} = \mathfrak{H}_1 \otimes \mathfrak{H}_2$, $\mathbf{M}_1 = \mathbf{B}_1^{(1)}$. Now $\mathbf{M}_2 \subset \mathbf{M}_1' = \mathbf{B}_1^{(1)\prime} = \mathbf{B}_2^{(2)}$. Besides $\mathbf{M}_2' \cdot \mathbf{B}_2^{(2)} = \mathbf{M}_2' \cdot \mathbf{M}_1' = (\mathbf{M}_1, \mathbf{M}_2)' = (\mathbf{R}(\mathbf{M}_1, \mathbf{M}_2))' = \mathbf{B} \otimes' = (\alpha 1)$.

Now consider the mapping $A_2^{(2)} \leftrightarrow A_2$ of $\mathbf{B}_2^{(2)}$ on $\mathbf{B}_2$ (belonging to $\mathfrak{H}_2$). It carries the ring $\mathbf{M}_2 \subset \mathbf{B}_2^{(2)}$ into a ring $\mathbf{N} \subset \mathbf{B}_2$ by Lemma 2.3.4 and the ring $\mathbf{M}_2' \cdot \mathbf{B}_2^{(2)}$ into $\mathbf{N}'$ by Lemma 2.3.7. So $\mathbf{N}' = (\alpha 1)$, and therefore $\mathbf{N} = (\alpha 1)' = \mathbf{B}_2$, $\mathbf{M}_2 = \mathbf{B}_2^{(2)}$. Thus $\mathbf{M}_2 = \mathbf{M}_1'$, $\mathbf{M}_1 = \mathbf{B}_1^{(1)}$, $\mathbf{M}_2 = \mathbf{B}_2^{(2)}$, proving all statements.

We have thus a general picture of the formal properties of factorisations and factors, general, direct, and coupled. A further point which is of interest in this connection will be elucidated by Lemma 5.4.1.

The decisive question is now this:

Problem 2. *Is every factorisation direct? Is every factor direct?*

We will see in §8.4 and §13.2, that the answer to the second and therefore (by Lemma 3.2.2) to both questions is no. We will therefore have to investigate and characterise the non-direct factors.

Chapter IV: Auxiliary Theorems

4.1. The theorem which follows has some interest of its own. We therefore prove it in its general form, although we will mostly need its extremely special Corollary only.

Theorem III. *Let* $\mathbf{M}$ *be a factor and let operators* $A_1, \cdots, A_n \in \mathbf{M}$, $A_1', \cdots, A_n' \in \mathbf{M}'$ *be given. We have*

$$\sum_{i=1}^{n} A_i A_i' = 0,$$

if and only if a matrix $(a_{i,j})_{i,j=1,\cdots,n}$ *exists, such that*

$$\sum_{i=1}^{n} a_{i,j} A_i = 0 \qquad \sum_{j=1}^{n} a_{i,j} A_j' = A_i'.$$

Proof: The condition is obviously sufficient: If it is valid, $\sum_{i,j=1}^{n} a_{i,j} A_i A_j'$ gives, when first summed over i, 0; and when first summed over j, $\sum_{i=1}^{n} A_i A_i'$. In order to prove its necessity, assume $\sum_{i=1}^{n} A_i A_i' = 0$.

Form the space $n \otimes \mathfrak{H}$ of all systems $f_1, \cdots, f_n$ in $\mathfrak{H}$, with the inner product

$$(< f_1, \cdots, f_n >, < g_1, \cdots, g_n >) = \sum_{i=1}^{n} (f_i, g_i)$$

(Cf. §2.4). Let $\mathfrak{M}$ be the set of all those $< f_1, \cdots, f_n >$, for which $\sum_{i=1}^{n} A_i X f_i = 0$ for every $X \in \mathbf{M}$. $\mathfrak{M}$ is obviously linear and closed. Let $\bar{E}$ be the projection of $\mathfrak{M}$. $\bar{E}$ can be written as an operator matrix $< E_{r,s} >_{r,s=1,\cdots,n}$.

Assume now $< f_1, \cdots, f_n > \in \mathfrak{M}$. If $X_0 \in \mathbf{M}$, then for every $X \in \mathbf{M}$ $\sum_{i=1}^{n} A_i X X_0 f_i = 0$, thus $< X_0 f_1, \cdots, X_0 f_n > \in \mathfrak{M}$. If $X_0' \in \mathbf{M}'$, then for every $X \in \mathbf{M}$, $\sum_{i=1}^{n} A_i X X_0' f_i = X_0' \sum_{1}^{n} A_i X f_i = 0$ thus $< X_0' f_1, \cdots, X_0' f > \in \mathfrak{M}$. Thus $\Phi \in \mathfrak{M}$ and $X_0 \in \mathbf{M}$ or $\mathbf{M}'$ imply for $X_0^{(2)} = < \delta_{r,s} X_0 >_{r,s=1,\cdots,n}$, $X_0^{(2)} \Phi \in \mathfrak{M}$. As every $\bar{E}\Phi \in \mathfrak{M}$, we have $X_0^{(2)} \bar{E}\Phi \in \mathfrak{M}$, $\bar{E} X_0^{(2)} \bar{E}\Phi = X_0^{(2)} \bar{E}\Phi$ that is $\bar{E} X_0^{(2)} \bar{E} = X_0^{(2)} \bar{E}$. Replacing X_0 by X_0^*, and thus $X_0^{(2)}$ by $X_2^{*(2)} = X_2^{(2)*}$, and applying * gives $\bar{E} X_0^{(2)} \bar{E} = \bar{E} X_0^{(2)}$. Thus $X_0^{(2)} \bar{E} = \bar{E} X_0^{(2)}$, or in other words $X_0 E_{r,s} = E_{r,s} X_0$. Thus $E_{r,s}$ commutes with every $X_0 \in \mathbf{M}$ or $\mathbf{M}'$, $E_{r,s} \in \mathbf{M}' \cdot \mathbf{M}'' = \mathbf{M}' \cdot \mathbf{M} = (\alpha 1)$, that is $E_{r,s} = a_{r,s} 1$.

Consider $< A_1' f, \cdots, A_n' f >$ for an arbitrary $f \in \mathfrak{H}$. If $X \in \mathbf{M}$, then $\sum_{i=1}^{m} A_i X A_i' f = \sum_{1}^{n} A_i A_i' X f = 0$. Thus $< A_1' f, \cdots, A_n' f > \in \mathfrak{M}$, $\bar{E} < A_1' f, \cdots, A_n' f > = < A_1' f, \cdots, A_n' f >$. That is: $A_i' f = \sum_{j=1}^{n} E_{ij} A_j' f = \sum_{1}^{n} a_{ij} A_j' f$, $A_i' = \sum_{j=1}^{n} a_{ij} A_j'$.

Consider next $< A_1^* f, \cdots, A_n^* f >$ for an arbitrary $f \in \mathfrak{H}$. Further choose an $< f_1, \cdots, f_n > \in \mathfrak{M}$. Then

$$(< A_1^* f, \cdots, A_n^* f >, < f_1, \cdots, f_n >) = \sum_{i=1}^{n} (A_i^* f, f_i) = \sum_{i=1}^{n} (f, A_i f_i)$$
$$= (f, \sum_{i=1}^{n} A_i f_i) = 0.$$

Thus $< A_1^* f, \cdots, A_n^* f >$ is orthogonal to $\mathfrak{M}$, and therefore

$$\bar{E} < A_1^* f, \cdots, A_n^* f > = 0.$$

That is: $0 = \sum_{j=1}^{n} E_{i,j} A_j^* f = \sum_{j=1}^{n} a_{i,j} A_j^* f$, $\sum_{j=1}^{n} a_{i,j} A_j^* = 0$. Applying * gives $\sum_{1}^{n} \bar{a}_{i,j} A_j = 0$. $\bar{E}$ is Hermitian, this implies obviously $E_{r,s}^* = E_{s,r}$, $\bar{a}_{r,s} = a_{s,r}$. Thus the last equation becomes $\sum_{j=1}^{n} a_{j,i} A_j = 0$.

So the proof is completed.

Corollary: For $A \in \mathbf{M}$, $A' \in \mathbf{M}'$ *we have* $AA' = 0$ *if and only if* $A = 0$ *or* $A' = 0$.

Proof: Put $n = 1$, $a_{1,1} = \alpha$: $AA' = 0$ means $\alpha A = 0$, $(1 - \alpha) A' = 0$. This means for $\alpha = 0$, $\alpha = 1$, $\alpha \neq 0, 1$, that $A' = 0$ resp. $A = 0$ resp. both.

4.2. We define

Definition 4.2.1. Let $\mathbf{M}$ be a ring which contains 1. A subset $\mathfrak{S}$ of $\mathfrak{H}$ or an arbitrary operator Z in $\mathfrak{H}$ (not necessarily $\epsilon \mathbf{B}$!) are said to *belong* to the ring $\mathbf{M}$ in symbols $\mathfrak{S} \eta \mathbf{M}$ resp. $Z \eta \mathbf{M}$ (we use the η in order to distinguish this from the element-relation ϵ), if they are invariant under every unitary $U \epsilon \mathbf{M}'$: that is, $U\mathfrak{S} = \mathfrak{S}$ resp. $UZU^{-1} = Z$ for all $U \epsilon \mathbf{M}'$.

The relationship between η and ϵ is easily established.

Lemma 4.2.1. *If $\mathfrak{M}$ is a linear closed set, then $\mathfrak{M} \eta \mathbf{M}$ is equivalent to $P_{\mathfrak{M}} \epsilon \mathbf{M}$* (cf. (16), p. 74). *If $Z \epsilon \mathbf{B}$, then $Z \eta \mathbf{M}$ is equivalent to $Z \epsilon \mathbf{M}$.*

Proof: $\mathfrak{M} \eta \mathbf{M}$ means $UP_{\mathfrak{M}}U^{-1} = P_{U\mathfrak{M}} = P_{\mathfrak{M}}$, $UP_{\mathfrak{M}} = P_{\mathfrak{M}}U$ if U is unitary and $\epsilon \mathbf{M}'$, thus $P_{\mathfrak{M}} \eta \mathbf{M}$. Thus it suffices to prove the second statement.

If $Z \epsilon \mathbf{B}$ then $Z \eta \mathbf{M}$ means $UZ = ZU$ for all unitary $U \epsilon \mathbf{M}'$ that is for $U \epsilon \mathbf{M}'^{U}$ (cf. (18), p. 392). That is: $Z \epsilon (\mathbf{M}'^{U})' = (\mathbf{R}(\mathbf{M}'^{U}))' = \mathbf{M}'' = \mathbf{M}$.

Lemma 4.2.2. *The conditions of Definition* 4.2.1 *can be replaced by $U\mathfrak{S} \subset \mathfrak{S}$ and $UZ \subset ZU$. The latter means that Z commutes with U in the sense of* (18), *p.* 404.

Proof: Replacing U by U^{-1} (which is unitary and $\epsilon \mathbf{M}'$ too, $U^{-1} = U^*$) and multiplying left resp. on both sides with U gives $\mathfrak{S} \subset U\mathfrak{S}$ and $ZU \subset UZ$. Thus $U\mathfrak{S} = \mathfrak{S}$ and $UZU^{-1} = Z$, $UZ = ZU$.

4.3. We will now define a class of operators which is a generalisation of the unitary ones.

Definition 4.3.1. An operator $U \epsilon \mathbf{B}$ is *partially isometric* if it maps a linear closed set $\mathfrak{E}$ isometrically on another linear closed set $\mathfrak{F}$; while it maps $\mathfrak{H} - \mathfrak{E}$ on 0. That is: $f \epsilon \mathfrak{E}$ implies $Uf \epsilon \mathfrak{F}$ and $\| Uf \| = \| f \|$, the range of U is all $\mathfrak{F}$, $f \epsilon \mathfrak{H} - \mathfrak{E}$ implies $Uf = 0$. $\mathfrak{E}$ and $\mathfrak{F}$ are the *initial* resp. *final* sets of U.

A unitary operator U is obviously such a partially isometric one, with $\mathfrak{H}$ as initial and final set.

Lemma 4.3.1. *$P_{\mathfrak{E}} = U^*U$, $P_{\mathfrak{F}} = UU^*$, U^* is partially isometric too, with interchanged initial and final sets: $\mathfrak{F}$, $\mathfrak{E}$; as a mapping of $\mathfrak{F}$ on $\mathfrak{E}$, U^* is the inverse of the mapping of $\mathfrak{E}$ on $\mathfrak{F}$, U.*

Proof: If f, $g \epsilon \mathfrak{E}$, then substitution of $h = \frac{f \pm g}{2}, \frac{f \pm ig}{2}$ into $\| Uh \|^2 = \| h \|^2$ gives $(Uf, Ug) = (f, g)$, that is $(Uf, Ug) = (P_{\mathfrak{E}}f, P_{\mathfrak{E}}g)$ (cf. (16), p. 71). If f or $g \epsilon \mathfrak{H} - \mathfrak{E}$, the latter equation remains valid, as both sides vanish. Owing to the linearity, it holds therefore for all f, g. We can write it as $(U^*Uf, g) = (P_{\mathfrak{E}}f, g)$. So $U^*U = P_{\mathfrak{E}}$. If the character of U^* were already established, substitution of U^* would give $UU^* = P_{\mathfrak{F}}$. So we need only to discuss U^*.

For $f \epsilon \mathfrak{E}$, $U^*Uf = P_{\mathfrak{E}}f = f$, so U^* maps $\mathfrak{F}$ to $\mathfrak{E}$ inversely to U. Thus this mapping is isometric in $\mathfrak{F}$. Assume now $f \epsilon \mathfrak{H} - \mathfrak{F}$. Then $(f, Ug) = 0$ whenever $Ug \epsilon \mathfrak{F}$, but this is the case for $g \epsilon \mathfrak{E}$ and for $g \epsilon \mathfrak{H} - \mathfrak{E}$ (then $Ug = 0$), thus for all g. So $(U^*f, g) = (f, Ug) = 0$, $U^*f = 0$. This completes the proof of U^*'s

being partially isometric with the initial and final sets $\mathfrak{F}$, $\mathfrak{E}$ resp., and its being inverse to U there.

Lemma 4.3.2. *Any of the four following conditions is characteristic for partially isometric operators.* $UU^*U = U$, $U^*UU^* = U^*$, U^*U *is a projection,* UU^* *is a projection.*

Proof: Owing to the symmetry between U and U^* it suffices to consider the first and the third conditions. If $UU^*U = U$ then $U^*UU^*U = U^*U$, $(U^*U)^2 = U^*U$, thus U^*U is a projection; so we need only to prove that the first condition is necessary, and that the third one is sufficient.

If U is partially isometric, we have always $Uf \in \mathfrak{F}$, so $P_{\mathfrak{F}}Uf = Uf$, $P_{\mathfrak{F}}U = U$; but $UU^* = P_{\mathfrak{F}}$, so $UU^*U = U$.

If U^*U is a projection, put $U^*U = P_{\mathfrak{E}}$. Thus for $f \in \mathfrak{E}$, $\| Uf \|^2 = (U^*Uf, f) = (P_{\mathfrak{E}}f, f) = \| f \|^2$, Uf is isometric in $\mathfrak{E}$. The image $\mathfrak{F}$ of $\mathfrak{E}$ is therefore isomorphic to $\mathfrak{E}$, thus a linear closed set (that is: a linear and topologically complete space) too. For $f \in \mathfrak{H} - \mathfrak{E}$, $\| Uf \|^2 = (U^*Uf, f) = (P_{\mathfrak{E}}f, f) = 0$. Thus U is partially isometric.

Lemma 4.3.3. *Let* $U_1, U_2, \cdots$ *be a (finite or infinite) sequence of partially isometric operators, with the resp. initial sets* $\mathfrak{E}_1, \mathfrak{E}_2, \cdots$ *and the resp. final sets* $\mathfrak{F}_1, \mathfrak{F}_2, \cdots$ *; let the* $\mathfrak{E}_i$ *be mutually orthogonal and let the* $\mathfrak{F}_i$ *be mutually orthogonal. Then the sum* $\sum_i U_i$ *is strongly convergent (if infinite at all), it is partially isometric, and its initial and final sets are* $[\mathfrak{E}_i; i = 1, 2, \cdots]$ *and* $[\mathfrak{F}_i; i = 1, 2, \cdots]$.

Proof: The strong convergence of $\sum_i U_i$ means the same thing as that of each $\sum_i U_i f$. As $U_i f_i \in \mathfrak{F}_i$, the addends are mutually orthogonal, thus the above convergence is equivalent to the finiteness of $\sum_i \| U_i f \|^2$ (cf. (16), p. 67). Now $\sum_i \| U_i f \|^2 = \sum_i (U_i^* U_i f, f) = \sum_i (P_{\mathfrak{E}_i} f, f) = (P_{[\mathfrak{E}_i; i=1, 2, \cdots]} f, f)$, and thus finite. Replacing U_i by U_i^*, we see that $\sum_i U_i^*$ is strongly convergent too.

So $(\sum_i U_i)^* (\sum_i U_i) = \sum_i U_i^* \sum_i U_i = \sum_{i,j} U_i^* U_j$. If $i \neq j$, then $U_j f \in \mathfrak{F}_j \subset \mathfrak{H} - \mathfrak{F}_i$, $U_i^* U_j f = 0$, $U_i^* U_j = 0$. So our above expression is equal to $\sum_i U_i^* U_i = \sum_i P_{\mathfrak{E}_i} = P_{[\mathfrak{E}_i; i=1, 2, \cdots]}$. Thus $\sum_i U_i$ is a partially isometric operator by Lemma 4.3.2 and its initial set is $[\mathfrak{E}_i; i = 1, 2, \cdots]$, by Lemma 4.3.1. Replacing U_i by U_i^*, we see that the final set is

$$[\mathfrak{F}_i; i = 1, 2, \cdots].$$

4.4. A construction which is described in (21), p. 307, Theorem 7, will be of importance in all our discussions. This Theorem applies to all linear, closed operators A with an everywhere dense domain, and leads to some operators of which we will consider B and W. B is self adjoint, and definite, and W is obviously partially isometric, its initial and final sets being

$$[\text{Range } B] = \mathfrak{H} - (f; Bf = 0) = \mathfrak{H} - (f, Af = 0) = [\text{Range } A^*]$$

and

$$\mathfrak{H} - (f, A^*f = 0) = [\text{Range } A]$$

resp. We have

$$A = WB, \qquad A^* = BW^*, \qquad B^2 = A^*A$$

Cf. loc. cit. (21).

We define:

Definition 4.4.1. If A is linear, closed, and has an everywhere dense domain, then the above described W, B are its *canonical decomposition.*

We are interested in the relation between rings and the canonical decomposition.

Lemma 4.4.1. *If* $\mathbf{M}$ *is a ring containing* 1, $A \eta \mathbf{M}$, *and* W, B *the canonical decomposition of* A, *then* $W \epsilon \mathbf{M}$, $B \eta \mathbf{M}$.

Proof: W, B are uniquely determined by their properties which follow:

B is self adjoint and definite; $B^2 = A^*A$.

$A = WB$; $Bf = 0$ implies $Wf = 0$.

In fact: The two first properties determine B (cf. (21), p. 303); and the two last ones determine W on Range B and on $(f, Bf = 0)$, therefore on

$$[[\text{Range } B], (f; Bf = 0)] = [\mathfrak{H} - (f, Af = 0), (f, Af = 0)] = \mathfrak{H},$$

that is they determine W completely.

Now if U is unitary and W, B is the canonical decomposition of A, then UWU^{-1}, UBU^{-1} is the one of UAU^{-1}. Thus $UAU^{-1} = A$ implies $UWU^{-1} = W$, $UBU^{-1} = B$. If $A \eta \mathbf{M}$ then let U run over all $\mathbf{M}'$, this gives $W, B \eta \mathbf{M}$. As $W \epsilon \mathbf{B}$, we even have $W \epsilon \mathbf{M}$.

Chapter V: Direct Factors. Ring Isomorphisms

5.1. We define

Definition 5.1.1. If $\mathbf{M}$ is a ring containing 1, denote $[Af; A \epsilon \mathbf{M}']$ by $\mathfrak{M}_f^{\mathbf{M}'}$. Write $E_f^{\mathbf{M}'}$ for $P_{\mathfrak{M}_f^{\mathbf{M}'}}$.

Lemma 5.1.1. $f \epsilon \mathfrak{M}_f^{\mathbf{M}'} \eta \mathbf{M}$; *whenever* $f \epsilon \mathfrak{M} \eta \mathbf{M}$ *then* $\mathfrak{M}_f^{\mathbf{M}'} \subset \mathfrak{M}$.

Proof: $A = 1$ gives $f \epsilon \mathfrak{M}_f^{\mathbf{M}'}$. If $U \epsilon \mathbf{M}'$, then

$$U(Af; A \epsilon \mathbf{M}') = (UAf; A \epsilon \mathbf{M}') \subset (Bf; B \epsilon \mathbf{M}');$$

that is $U\mathfrak{M}_f^{\mathbf{M}'} \subset \mathfrak{M}_f^{\mathbf{M}'}$. Thus $\mathfrak{M}_f^{\mathbf{M}'} \eta \mathbf{M}$ (by Lemma 4.2.2). Conversely: Assume $f \epsilon \mathfrak{M} \eta \mathbf{M}$, and consider those A for which Af and A^*f both $\epsilon \mathfrak{M}$ for all $f \epsilon \mathfrak{M}$. They clearly form a ring (use the strong topology; for instance by (22), §3), and contain all unitary $U \epsilon \mathbf{M}'$, that is all $\mathbf{M}'^U$. Thus they contain all $\mathbf{R}(\mathbf{M}'^U) = \mathbf{M}'$ (cf. (18), p. 392), and so

$$(Af; A \epsilon \mathbf{M}') \subset \mathfrak{M}, [Af; A \epsilon \mathbf{M}'] \subset \mathfrak{M}, \mathfrak{M}_f^{\mathbf{M}'} \subset \mathfrak{M}.$$

Definition 5.1.2. $\mathfrak{M} \eta \mathbf{M}$ is minimal (with respect to $\mathbf{M}$), if $\mathfrak{M} \neq 0$, and $\mathfrak{N} \eta \mathbf{M}$, $\mathfrak{N} \subset \mathfrak{M}$ implies $\mathfrak{N} = (0)$ or $\mathfrak{M}$. Replacing $\mathfrak{M}, \mathfrak{N}$ by $E = P_{\mathfrak{M}}, F = P_{\mathfrak{N}}$ we can say (cf. Lemma 4.2.1 and (16), p. 76): projection $E \epsilon \mathbf{M}$ is minimal (with

respect to $\mathbf{M}$), if $E \gneq 0$, and if for every projection $F \in \mathbf{M}$, $F \leqq E$ implies $F = 0$ or $F = E$.

Lemma 5.1.2. *$\mathfrak{M} \eta \mathbf{M}$ is minimal if and only if $\mathfrak{M} \gneq (0)$, and for every $f \in \mathfrak{M}$, $f \gneq 0$, $\mathfrak{M}_f^{\mathbf{M}'} = \mathfrak{M}$.*

Proof: Assume first that $\mathfrak{M} \eta \mathbf{M}$ is minimal. Then $\mathfrak{M} \gneq (0)$. If $f \in \mathfrak{M}$, $f \gneq 0$, then $\mathfrak{M}_f^{\mathbf{M}'} \eta \mathbf{M}$; as $f \in \mathfrak{M}_f^{\mathbf{M}'}$, $\mathfrak{M}_f^{\mathbf{M}'} \gneq 0$ and as $f \in \mathfrak{M}$, $\mathfrak{M}_f^{\mathbf{M}'} \subset \mathfrak{M}$. Thus $\mathfrak{M}_f^{\mathbf{M}'} = \mathfrak{M}$ by definition.

Assume now conversely that our condition is fulfilled. If $\mathfrak{N} \subset \mathfrak{M}$, $\mathfrak{N} \eta \mathbf{M}$, then either $\mathfrak{N} = (0)$, or an $f \in \mathfrak{N}$, $f \gneq 0$ exists. Then $\mathfrak{M}_f^{\mathbf{M}'} \subset \mathfrak{N}$; but as $f \in \mathfrak{M}$, $f \gneq 0$, $\mathfrak{M}_f^{\mathbf{M}'} = \mathfrak{M}$; so $\mathfrak{M} \subset \mathfrak{N} \subset \mathfrak{M}$, $\mathfrak{N} = \mathfrak{M}$.

Lemma 5.1.3. *If $\mathbf{M}$ is a factor, then $\mathfrak{M}_f^{\mathbf{M}'}$ is minimal if and only if $f \gneq 0$ and $\mathfrak{M}_f^{\mathbf{M}'} \cdot \mathfrak{M}_f^{\mathbf{M}} = (\alpha f)$.*

Proof: Assume first that $f \gneq 0$ and $\mathfrak{M}_f^{\mathbf{M}'} \cdot \mathfrak{M}_f^{\mathbf{M}} = (\alpha f)$. Assume that $\mathfrak{M}_f^{\mathbf{M}'}$ is not minimal. As $f \gneq 0$, $\mathfrak{M}_f^{\mathbf{M}'} \gneq (0)$, there must exist an $\mathfrak{N} \eta \mathbf{M}$ with $\mathfrak{N} \subset \mathfrak{M}_f^{\mathbf{M}'}$, $\mathfrak{N} \gneq 0$, $\mathfrak{M}_f^{\mathbf{M}'}$. Put $\mathfrak{P} = \mathfrak{M}_f^{\mathbf{M}'} - \mathfrak{N}$; then $\mathfrak{N}$, $\mathfrak{P}$ are orthogonal, both $\subset \mathfrak{M}_f^{\mathbf{M}'}$, both $\gneq 0$.

Thus for $F = P_{\mathfrak{N}}$, $G = P_{\mathfrak{P}}$, we have: $F, G \in \mathbf{M}$, $FG = 0$, $F, G \gneq 0$. As $E_f^{\mathbf{M}} = P_{\mathfrak{M}_f^{\mathbf{M}}} \in \mathbf{M}'$ and $\gneq 0$ too, the Corollary to Theorem III gives $P_{\mathfrak{N} \cdot \mathfrak{M}_f^{\mathbf{M}}} = E \cdot P_{\mathfrak{M}_f^{\mathbf{M}}} \gneq 0$, $P_{\mathfrak{P} \cdot \mathfrak{M}_f^{\mathbf{M}}} = F \cdot P_{\mathfrak{M}_f^{\mathbf{M}}} \gneq 0$. So $\mathfrak{N} \cdot \mathfrak{M}_f^{\mathbf{M}}$ and $\mathfrak{P} \cdot \mathfrak{M}_f^{\mathbf{M}}$ are both $\gneq (0)$. But they are $\subset \mathfrak{M}_f^{\mathbf{M}'} \cdot \mathfrak{M}_f^{\mathbf{M}} = (\alpha f)$, so they are both $= (\alpha f)$. Thus $f \in \mathfrak{N} \cdot \mathfrak{M}_f^{\mathbf{M}}$ and $\mathfrak{P} \cdot \mathfrak{M}_f^{\mathbf{M}}$ and a fortiori $f \in \mathfrak{N}$ and $\mathfrak{P}$. As $\mathfrak{N}$, $\mathfrak{P}$ are orthogonal, this would imply $f = 0$, contradicting our original assumption.

Assume now conversely that $\mathfrak{M}_f^{\mathbf{M}'}$ is minimal. As $\mathfrak{M}_f^{\mathbf{M}'} \gneq (0)$, we have at any rate $f \gneq 0$.

Put $E = P_{\mathfrak{M}_f^{\mathbf{M}'}}$, $P_{\mathfrak{M}_f^{\mathbf{M}}} = E'$. As $E \in \mathbf{M}$, $E' \in \mathbf{M}'$, therefore E, E' commute; therefore the E-image of $\mathfrak{M}_f^{\mathbf{M}}$ is $\mathfrak{M}_f^{\mathbf{M}'} \cdot \mathfrak{M}_f^{\mathbf{M}}$. Thus (remember $Ef = f$!) $\mathfrak{M}_f^{\mathbf{M}'} \cdot \mathfrak{M}_f^{\mathbf{M}} = [EAf, A \in \mathbf{M}] = [EAEf, A \in \mathbf{M}] = [Bf, B \in \overline{\mathbf{M}}]$, where $\overline{\mathbf{M}}$ is the set of all $B \in \mathbf{M}$ with $EB = BE = B$ ($\overline{\mathbf{M}}$ is obviously a ring, thus $\overline{\mathbf{M}} = \mathbf{R}(\overline{\mathbf{M}}^p)$ (cf. (18), p. 392). $F' \in \overline{\mathbf{M}}^p$ means: F' is a projection, and $\in \overline{\mathbf{M}}$, and so $EF' = F'E = F'$, $F' \in \mathbf{M}$. So $F' \in \mathbf{M}$, $F' \leqq E$, and as E is minimal, $F' = 0$ or E. So $\overline{\mathbf{M}} = \mathbf{R}(0, E) = (\alpha E)$. Therefore $\mathfrak{M}_f^{\mathbf{M}'} \cdot \mathfrak{M}_f^{\mathbf{M}} = [Bf, B \in (\alpha E)] = [\alpha Ef] = [\alpha f] = (\alpha f)$.

Lemma 5.1.4. *If $\mathbf{M}$ is a factor, then $\mathfrak{M}_f^{\mathbf{M}'}$ is minimal (with respect to $\mathbf{M}$) if and only if $\mathfrak{M}_f^{\mathbf{M}}$ is minimal (with respect to $\mathbf{M}'$).*

Proof: The criterium of Lemma 5.1.3 is obviously symmetric in $\mathbf{M}$ and $\mathbf{M}'$, as $\mathbf{M} = \mathbf{M}''$.

Definition 5.1.3. An $f \gneq 0$ is *minimal* (with respect to $\mathbf{M}$, $\mathbf{M}'$) if its $\mathfrak{M}_f^{\mathbf{M}'}$, or, which is equivalent, its $\mathfrak{M}_f^{\mathbf{M}}$ is minimal.

Lemma 5.1.5. *If $\mathbf{M}$ is a factor, and if f is minimal, then every $g \gneq 0$, $g \in \mathfrak{M}_f^{\mathbf{M}'}$ or $\mathfrak{M}_f^{\mathbf{M}}$ is minimal.*

Proof: $\mathfrak{M}_f^{\mathbf{M}'}$ is minimal, and since $g \gneq 0$, $g \in \mathfrak{M}_f^{\mathbf{M}'}$ implies by Lemma 5.1.2, $\mathfrak{M}_g^{\mathbf{M}'} = \mathfrak{M}_f^{\mathbf{M}'}$, g is too. $g \gneq 0$, $g \in \mathfrak{M}_f^{\mathbf{M}}$ is taken care of by symmetry.

5.2. Various sorts of ring isomorphisms are possible. They are defined as follows:

Definition 5.2.1. Let $\mathfrak{H}_{\mathrm{I}}$ and $\mathfrak{H}_{\mathrm{II}}$ be two spaces, $\mathbf{B}_{\mathrm{I}}$ and $\mathbf{B}_{\mathrm{II}}$ their operator rings, $\mathbf{M}_{\mathrm{I}} \subset \mathbf{B}_{\mathrm{I}}$ and $\mathbf{M}_{\mathrm{II}} \subset \mathbf{B}_{\mathrm{II}}$ rings in $\mathfrak{H}_{\mathrm{I}}$ resp. $\mathfrak{H}_{\mathrm{II}}$ which contain 1. If V is any set of operations and properties defined in both $\mathbf{B}_{\mathrm{I}}$ and $\mathbf{B}_{\mathrm{II}}$, then $\mathbf{M}_{\mathrm{I}}$ and $\mathbf{M}_{\mathrm{II}}$ are called *V-ring-isomorphic*, if a one-to-one mapping of $\mathbf{M}_{\mathrm{I}}$ on $\mathbf{M}_{\mathrm{II}}$ exists which leaves the operations and properties of V invariant.

If $V = (\alpha A,\ A^*,\ A + B,\ AB,$ strongest closure), we speak of full *ring-isomorphism;* if $V = (\alpha A,\ A^*,\ A + B,\ AB)$, of *algebraic ring-isomorphism.*

It is clear that every isomorphism of the spaces $\mathfrak{H}_{\mathrm{I}}$, $\mathfrak{H}_{\mathrm{II}}$ generates full ring isomorphisms (in particular one between $\mathbf{B}_{\mathrm{I}}$ and $\mathbf{B}_{\mathrm{II}}$). But the really interesting cases are such ring isomorphisms between an $\mathbf{M}_{\mathrm{I}}$ and an $\mathbf{M}_{\mathrm{II}}$, which do not originate from spatial isomorphisms.

As our methods are invariant with respect to the spaces $\mathfrak{H}$, all notions we considered are invariant under spatial isomorphisms. But it is not so with ring isomorphisms. For instance the operation $\mathbf{M}'$ is not invariant even under full ring isomorphisms. (Let $\mathfrak{H}_{\mathrm{I}}$, $\mathfrak{H}_{\mathrm{II}}$ be two finite dimensional Euclidean spaces of different dimensions. Put $\mathbf{M}_{\mathrm{I}} = \mathbf{M}_{\mathrm{II}} = (\alpha 1)$, then $\mathbf{M}_{\mathrm{I}}' = \mathbf{B}_{\mathrm{I}}$, $\mathbf{M}_{\mathrm{II}}' = \mathbf{B}_{\mathrm{II}}$; thus $\mathbf{M}_{\mathrm{I}}, \mathbf{M}_{\mathrm{II}}$ are fully ring isomorphic, while $\mathbf{M}_{\mathrm{I}}', \mathbf{M}_{\mathrm{II}}'$ are not.) It is therefore of importance to show that some fundamental notions are invariant under certain types of ring isomorphism.

Lemma 5.2.1. *Every $(A^*,\ AB)$-ring-isomorphism leaves the following notions invariant: To be* 0*; to be* 1*; to be a projection; to be partially isometric; for two operators: to commute; for two projections E, F: $F \leqq E$; for a projection: to be minimal.*

Proof: $A = 0$ means $AX = A$ for all $X \in \mathbf{M}_{\mathrm{I}}$; $A = 1$ means $AX = X$ for all $X \in \mathbf{M}_{\mathrm{I}}$; A is a projection if $A^* = A$, $AA = A$; A is partially isometric if $AA^*A = A$; commutativity means $AB = BA$; $F \leqq E$ means $EF = F$; minimality can be expressed as a combination of the foregoing statements.

Lemma 5.2.2. *Every $(\alpha A,\ A^*,\ AB)$-ring-isomorphism leaves the notion of a factor (as applied to the rings $\mathbf{M}_{\mathrm{I}}$, $\mathbf{M}_{\mathrm{II}}$ themselves) invariant.*

Proof: As the rings $\mathbf{M}$ under consideration contain 1, being a factor means $\mathbf{M} \cdot \mathbf{M}' = (\alpha 1)$, that is: If $A \in \mathbf{M}$ is such that for every $B \in \mathbf{M}$, $AB = BA$, then $A = \alpha 1$. The first half is invariant under $(A^*,\ AB)$-ring-isomorphisms, and so is the 1 (cf. Lemma 5.2.1), while the equation $A = \alpha 1$ is then invariant under (αA)-ring-isomorphisms.

5.3. We are now able to give a simple criterium for direct factors.

Lemma 5.3.1. *The three following conditions are equivalent for a factor* $\mathbf{M}$:

(i) $\mathbf{M}$ *contains a minimal projection E (with respect to* $\mathbf{M}$).

(ii) $\mathbf{M}'$ *contains a minimal projection E' (with respect to* $\mathbf{M}'$).

(iii) *There exists a minimal $f \in \mathfrak{H}$ (with respect to* $\mathbf{M}$, $\mathbf{M}'$).

Proof: (iii) implies (i). (i) implies (iii) by Lemma 5.1.2. Similarly (ii) and (iii) are equivalent.

Lemma 5.3.2. *The conditions of Lemma* 5.3.1. *are fulfilled for every direct factor* $\mathbf{M}$.

Proof: If $\mathbf{M}$ is a direct factor in the $\mathbf{B}$ of $\mathfrak{H}$ then $\mathfrak{H}$ is isomorphic to $\mathfrak{H}_1 \otimes \mathfrak{H}_2$, so that $\mathbf{M}$ becomes $\mathbf{B}_1^{(1)} \cdot \mathbf{B}_1^{(1)}$ is algebraically (even fully) ring-isomorphic to $\mathbf{B}_1$ of $\mathfrak{H}_1$ by Lemma 2.3.4 (resp. (22), §4); and the condition of Lemma 5.3.1, in its form (i), invariant under algebraic ring-isomorphisms by Lemma 5.2.1. Thus we need only to consider $\mathbf{B}_1$ in $\mathfrak{H}_1$. Now if φ is any element $\gtrneqq 0$ of $\mathfrak{H}_1$, then $P_{[\varphi]} \in \mathbf{B}_1$ and obviously minimal.

We begin now to discuss the sufficiency of these conditions.

Lemma 5.3.3. *If a factor* $\mathbf{M}$ *fulfills the conditions of Lemma* 5.3.1, *then every* $\mathfrak{M}\ \eta\ \mathbf{M}$, $\mathfrak{M} \gtrneqq 0$, *contains a minimal* f *with* $\| f \| = 1$.

Proof: A minimal f^0 exists, of course $f^0 \gtrneqq 0$. Thus $E_{f^0}^{\mathbf{M}} \gtrneqq 0$; besides $E_{f_0}^{\mathbf{M}} \in \mathbf{M}'$. Now $P_{\mathfrak{M}} \gtrneqq 0$, $P_{\mathfrak{M}} \in \mathbf{M}$, so by the Corollary of Theorem III.

$$P_{\mathfrak{M}_{f_0}^{\mathbf{M}} \cdot \mathfrak{M}} = P_{\mathfrak{M}} \cdot E_{f_0}^{\mathbf{M}} \gtrneqq 0, \qquad \mathfrak{M}_{f_0}^{\mathbf{M}} \cdot \mathfrak{M} \gtrneqq (0).$$

Thus an $f \in \mathfrak{M}_{f_0}^{\mathbf{M}} \cdot \mathfrak{M}$ with $f \gtrneqq 0$ exists, multiplying it with $1/\| f \|$ makes $\| f \| = 1$.

As $f \in \mathfrak{M}_{f_0}^{\mathbf{M}}$, $f \gtrneqq 0$ this f is minimal by Lemma 5.1.5; and besides $f \in \mathfrak{M}$.

Lemma 5.3.4. *If a factor* $\mathbf{M}$ *fulfills the conditions of Lemma* 5.3.1, *then a finite or infinite normalised orthogonal system* $\varphi_1, \varphi_2 \cdots$ *exists, such that*

(i) *every* φ_n *is minimal*;

(ii) *the* $\mathfrak{M}_{\varphi_1}^{\mathbf{M}'}, \mathfrak{M}_{\varphi_2}^{\mathbf{M}'}, \cdots$ *are mutually orthogonal*;

(iii) $[\mathfrak{M}_{\varphi_1}^{\mathbf{M}'}, \mathfrak{M}_{\varphi_2}^{\mathbf{M}'}, \cdots] = \mathfrak{H}$.

Proof: Let Ω be the first uncountable Cantor ordinal number. Define for all $\alpha < \Omega$, a φ_α as follows: If $\alpha < \Omega$, and all φ_β, $\beta < \alpha$ are already defined then form $\mathfrak{H} - [\mathfrak{M}_{\varphi\beta}^{\mathbf{M}'}; \beta < \alpha]$. As this is obviously $\eta\ \mathbf{M}$, it will contain a minimal f with $\| f \| = 1$ if it is $\gtrneqq (0)$. Let then φ_α be some such f; otherwise let φ_α (and all φ_γ, $\gamma \geqq \alpha$) be undefined.

Let the first α for which φ_α is undefined, be $\bar{\alpha} \leqq \Omega$. If $\bar{\alpha} < \Omega$, we must have $\mathfrak{H} - [\mathfrak{M}_{\varphi\beta}^{\mathbf{M}'}; \beta < \bar{\alpha}] = 0$, $[\mathfrak{M}_{\varphi\beta}^{\mathbf{M}'}; \beta < \bar{\alpha}] = \mathfrak{H}$. Consider now all φ_α, $\alpha < \bar{\alpha}$. Always $\| \varphi_\alpha \| = 1$. If $\alpha, \beta < \bar{\alpha}$ assume $\alpha > \beta$. Then $A', B' \in \mathbf{M}'$ imply $A'^* B' \in \mathbf{M}'$, $A'^* B' \varphi_\beta \in \mathfrak{M}_{\varphi\beta}^{\mathbf{M}'}$ orthogonal to φ_α, thus $B'\varphi_\beta$ orthogonal to $A'\varphi_\alpha$ (because $(A'^* B'\varphi_\beta, \varphi_\alpha) = (B'\varphi_\beta, A'\varphi_\alpha)$). So $(A'\varphi_\alpha;\ A' \in \mathbf{M}')$ is orthogonal to $(B'\varphi_\beta,\ B' \in \mathbf{M}')$, $[A'\varphi_\alpha;\ A' \in \mathbf{M}']$ to $[B'\varphi_\beta;\ B' \in \mathbf{M}']$ and so $\mathfrak{M}_{\varphi\alpha}^{\mathbf{M}'}$ to $\mathfrak{M}_{\varphi\beta}^{\mathbf{M}'}$. This is symmetric in α, β so it holds for $\alpha < \beta$ too, that is whenever $\alpha \gtrneqq \beta$.

Thus $\varphi_\alpha, \varphi_\beta$ are orthogonal if $\alpha \gtrneqq \beta$, that is the φ_α, $\alpha < \bar{\alpha}$, form a normalised orthogonal set. It must therefore be finite or countably infinite (cf. (16), p. 66), thus excluding $\bar{\alpha} = \Omega$. Thus $\bar{\alpha} < \Omega$, and so $[\mathfrak{M}_{\varphi\alpha}^{\mathbf{M}'}; \alpha < \bar{\alpha}] = \mathfrak{H}$.

If $\bar{\alpha}$ is finite, $\mathfrak{H} \gtrneqq (0)$ excludes $\bar{\alpha} = 1$, so $\bar{\alpha} = n + 1$, $n = 1, 2, \cdots$, and α runs over $1, \cdots, n$. If $\bar{\alpha}$ is infinite then $\alpha < \bar{\alpha}$ has the same aleph as $\alpha < \omega$, and may therefore (by a re-indexing of the φ_α) be replaced by it; so we may assume that α runs over $1, 2, \cdots$.

This completes the proof.

Lemma 5.3.5. *If a factor* $\mathbf{M}$ *fulfills the conditions of Lemma* 5.3.1 *then a complete normalised orthogonal set* $f_{m,n}$, $m = 1, 2, \cdots$; $n = 1, 2, \cdots$ *exists (both ranges for* m *and for* n *may be finite or infinite), such that*:

(i) *every* $f_{m,n}$ *is minimal,*

(ii) $\mathfrak{M}^{\mathbf{M}'}_{f_{m,n}} = [f_{m,1}, f_{m,2}, \cdots]$, $\mathfrak{M}^{\mathbf{M}}_{f_{m,n}} = [f_{1,n}, f_{2,n}, \cdots]$.

Proof: If we replace $\mathbf{M}$ by $\mathbf{M}'$, condition (i) of Lemma 5.3.1 becomes (ii), so $\mathbf{M}'$ fulfills it too. Apply therefore Lemma 5.3.4 to $\mathbf{M}$ and to $\mathbf{M}'$, obtaining the two normalised orthogonal systems $\varphi_1, \varphi_2, \cdots$ and $\psi_1, \psi_2, \cdots$. As $E^{\mathbf{M}'}_{\varphi_m} \in \mathbf{M}$ and $\succnsim 0$, $E^{\mathbf{M}}_{\psi_n} \in \mathbf{M}'$ and $\succnsim 0$, we have by the Corollary to Theorem III

$$P_{\mathfrak{M}^{\mathbf{M}'}_{\varphi_m} \cdot \mathfrak{M}^{\mathbf{M}}_{\psi_n}} = E^{\mathbf{M}'}_{\varphi_n} \cdot E^{\mathbf{M}}_{\psi_n} \succnsim 0, \qquad \mathfrak{M}^{\mathbf{M}'}_{\varphi_m} \cdot \mathfrak{M}^{\mathbf{M}}_{\psi_n} \succnsim (0).$$

Choose an $f_{m,n} \in \mathfrak{M}^{\mathbf{M}'}_{\varphi_m} \cdot \mathfrak{M}^{\mathbf{M}}_{\psi_n}$ which is $\succnsim 0$ multiplying it with $1/\| f_{m,n} \|$ makes $\| f_{m,n} \| = 1$.

If $m \succnsim p$, $f_{m,n} \in \mathfrak{M}^{\mathbf{M}'}_{\varphi_m}$, $f_{p,q} \in \mathfrak{M}^{\mathbf{M}'}_{\varphi_p}$, if $n \succnsim q$, $f_{m,n} \in \mathfrak{M}^{\mathbf{M}}_{\psi_n}$, $f_{p,q} \in \mathfrak{M}^{\mathbf{M}}_{\psi_q}$, both imply $(f_{m,n}, f_{p,q}) = 0$. Thus the $f_{m,n}$ form a normalised orthogonal system.

$\mathfrak{M}^{\mathbf{M}'}_{\varphi_m}$ is minimal and $f_{m,n}$ belongs to it and is $\succnsim 0$, therefore Lemma 5.1.5 applies: $f_{m,n}$ is minimal too, and $\mathfrak{M}^{\mathbf{M}'}_{\varphi_m} = \mathfrak{M}^{\mathbf{M}'}_{f_{m,n}}$. Similarly $\mathfrak{M}^{\mathbf{M}}_{\psi_n} = \mathfrak{M}^{\mathbf{M}}_{f_{m,n}}$. As $f_{m,n}$ is minimal, Lemma 5.1.3 gives $\mathfrak{M}^{\mathbf{M}'}_{f_{m,n}} \cdot \mathfrak{M}^{\mathbf{M}}_{f_{m,n}} = (\alpha f_{m,n})$, that is

$$\mathfrak{M}^{\mathbf{M}'}_{\varphi_m} \cdot \mathfrak{M}^{\mathbf{M}}_{\psi_n} = (\alpha f_{m,n}).$$

Thus

$$\sum_{m,n} P_{[f_{m,n}]} = \sum_{m,n} E^{\mathbf{M}'}_{\varphi_m} \cdot E^{\mathbf{M}}_{\psi_n} = \sum_m E^{\mathbf{M}'}_{\varphi_m} \cdot \sum_n E^{\mathbf{M}}_{\psi_n} = 1 \cdot 1 = 1;$$

that is: the normalised orthogonal system of all $f_{m,n}$ is complete.

$f_{m,n} \in \mathfrak{M}^{\mathbf{M}'}_{f_{m,n}} = \mathfrak{M}^{\mathbf{M}'}_{\varphi_m}$, and if $p \succnsim m$, then $f_{p,n} \in \mathfrak{M}^{\mathbf{M}'}_{\varphi_p}$ is orthogonal to $\mathfrak{M}^{\mathbf{M}'}_{\varphi_m}$. This proves $\mathfrak{M}^{\mathbf{M}'}_{\varphi_m} = [f_{m,1}, f_{m,2}, \cdots]$. Similarly $\mathfrak{M}^{\mathbf{M}}_{\psi_n} = [f_{1n}, f_{2n}, \ldots]$.

Thus all parts of our Lemma are proved.

Lemma 5.3.6. *If a factor* $\mathbf{M}$ *fulfills the conditions of Lemma* 5.3.1 *we can choose the* $f_{m,n}$ *of Lemma* 5.3.5 *so that we have*:

There is a unique partially isometric operator $U \in \mathbf{M}$ *which has the initial and final sets* $[f_{m,1}, f_{m,2}, \cdots]$ *resp.* $[f_{p,1}, f_{p,2}, \cdots]$ *and for which* $Uf_{m,n} = f_{p,n}$, $Uf_{r,n} = 0$, *if* $r \succnsim m$. *Denote it by* $U_{m,p}$.

If an $A \in \mathbf{M}$ *has the properties* $P_{[f_{p,1}, f_{p,2}, \ldots]} A = AP_{[f_{m,1}, f_{m,2}, \ldots]} = A$ *then* $A = \alpha U_{m,p}$.

Proof: Assume first $A \in \mathbf{M}$ and $P_{[f_{p,1}, f_{p,2}, \ldots]} A = AP_{[f_{m,1}, f_{m,2}, \ldots]} = A$ and the same for B. Then applying a * gives:

$$P_{[f_{m,1}, f_{m,2}, \ldots]} B^* = B^* P_{[f_{p,1}, f_{p,2}, \ldots]} = B^*$$

and so $P_{[f_{m,1}, f_{m,2}, \ldots]} B^*A = B^*AP_{[f_{m,1}, f_{m,2}, \ldots]} = B^*A$. Now $B^*A \in \mathbf{M}$ and $P_{[f_{m,1}, f_{m,2}, \ldots]} = E^{\mathbf{M}'}_{f_{m,n}}$ is minimal with respect to $\mathbf{M}$, so the argument made at the end of the proof of Lemma 5.1.3 applies $B^*A = \beta P_{[f_{m,1}, f_{m,2}, \ldots]}$.

Put first $A = B$, then $B^*B = \beta_1 P_{[f_{m,1}, f_{m,2}, \cdots]}$. As the left side is Hermitian and definite, $\beta_1 \geqq 0$. If $B \neq 0$, then even $\beta_1 \neq 0$ and we have

$$((\beta_1)^{-\frac{1}{2}}B)^* \cdot ((\beta_1)^{-\frac{1}{2}}B) = P_{[f_{m,1}, f_{m,2}, \cdots]}.$$

Thus $(\beta_1)^{-\frac{1}{2}}B$ is a partially isometric operator with the initial set $[f_{m,1}, f_{m,2}, \cdots]$ (cf. Lemmas 4.3.1 and 4.3.2). Replacing B by B^* we interchange m and p, so $(\beta_2)^{-\frac{1}{2}}B^*$ is partially isometric with the initial set $[f_{p,1}, f_{p,2}, \cdots]$ and $(\beta_2)^{-\frac{1}{2}}B$ with the final set $[f_{p,1}, f_{p,2}, \cdots]$. As $(\beta_1)^{-\frac{1}{2}}B$, $(\beta_2)^{-\frac{1}{2}}B$ are both partially isometric, we have $|(\beta_1)^{-\frac{1}{2}}| = |(\beta_2)^{-\frac{1}{2}}|$, that is $\beta_1 = \beta_2$. So $(\beta_1)^{-\frac{1}{2}}B$ has the initial and final sets $[f_{m,1}, f_{m,2}, \cdots]$ resp. $[f_{p,1}, f_{p,2}, \cdots]$.

Return now to the pair A, B. If $B \neq 0$, then we have

$$A = P_{[f_{p,1}, f_{p,2}, \cdots]} \cdot A = (1/\beta_1)\, BB^*A = (\beta/\beta_1)BP_{[f_{m,1}, f_{m,2}, \cdots]} = (\beta/\beta_1)B.$$

So we see: either $B = 0$ or $A = \alpha B$.

Consider now $f_{m,n}$ and $f_{p,n}$. As $\mathfrak{M}^{\mathbf{M}}_{f_{m,n}} = [f_{1,n}, f_{2,n}, \cdots]$ we have $f_{p,n} \in \mathfrak{M}^{\mathbf{M}}_{f_{m,n}}$. So an $A \in \mathbf{M}$ with $\| f_{p,n} - Af_{m,n} \| < 1$ exists. A fortiori

$$\| P_{[f_{p,1}, f_{p,2}, \cdots]} f_{p,n} - P_{[f_{p,1}, f_{p,2}, \cdots]} Af_{m,n} \| < 1$$

or as

$$P_{[f_{p,1}, f_{p,2}, \cdots]} f_{p,n} = f_{p,n}, \qquad P_{[f_{m,1}, f_{n,2}, \cdots]} f_{m,n} = f_{m,n}$$

we have $\| f_{p,n} - A_0 f_{m,n} \| < 1$, where $A_0 = P_{[f_{p,1}, f_{p,2}, \cdots]}\, AP_{[f_{m,1}, f_{m,2}, \cdots]}$. By what we prove above $A_0 = 0$ or $A_0 = \beta_0 U$, where $\beta_0 \neq 0$ and U is partially isometric with the initial resp. final set $[f_{m,1}, f_{m,2}, \cdots]$ resp. $[f_{p,1}, f_{p,2}, \cdots]$. Now $\| f_{p,n} \| = 1$ implies $A_0 f_{m,n} \neq 0$ and so the latter alternative holds.

As $A_0 \in \mathbf{M}$, $A_0 f_{m,n} \in \mathfrak{M}^{\mathbf{M}}_{f_{m,n}} = \mathfrak{M}^{\mathbf{M}}_{f_{p,n}}$, as $A_0 = P_{[f_{p,1}, f_{p,2}, \cdots]}\, A_0$ therefore $A_0 f_{m,n} \in [f_{p,1}, f_{p,2}, \cdots] = \mathfrak{M}^{\mathbf{M}'}_{f_{p,n}}$. Thus $A_0 f_{m,n} \in \mathfrak{M}^{\mathbf{M}}_{f_{p,n}} \cdot \mathfrak{M}^{\mathbf{M}}_{f_{p,n}} = (\alpha f_{p,n})$, $A_0 f_{m,n} = \alpha_0 f_{p,n}$, $Uf_{m,n} = (\alpha_0/\beta_0) f_{p,n}$. As $\| f_{m,n} \| = \| f_{p,n} \| = 1$ and $f_{m,n}$ is in the initial set of U, we have $|\alpha_0/\beta_0| = 1$. Replacing U by $(\beta_0/\alpha_0)U$ does not affect its other properties, but makes $Uf_{m,n} = f_{p,n}$. So we have:

$$P_{[f_{p,1}, f_{p,2}, \cdots]}\, U = UP_{[f_{m,1}, f_{m,2}, \cdots]} = U, \qquad Uf_{m,n} = f_{p,n}.$$

As U depends on m, n, p write $U = U^{(n)}_{m,p}$. The first equations determine it uniquely up to a constant factor, the last one makes this factor equal to 1. Comparing $U^{(n)}_{m,p}$, $U^{(q)}_{m,p}$ we see that the first equations still make them agree up to a constant factor:

$$U^{(n)}_{m,p} = \theta^{(n,q)}_{m,p}\, U^{(q)}_{m,p}$$

and $\theta^{(n,q)}_{m,p}$ is obviously of absolute value 1. On the other hand the uniqueness property gives $U^{(n)}_{m,p} \cdot U^{(n)}_{p,q} = U^{(n)}_{m,r}$.

Replace every $f_{p,q}$, $p \neq 1$, by $\theta^{(1,q)}_{1,p} f_{p,q}$. This makes every $\theta^{(1,q)}_{1,p} = 1$. Now obviously $\theta^{(1,n)}_{1,p} \cdot \theta^{(n,q)}_{1,p} = \theta^{(1,q)}_{1,p}$, therefore $\theta^{(n,q)}_{1,p} = 1$. Furthermore it is easy

to see, that $\theta_{1,m}^{(n,q)} \cdot \theta_{m,p}^{(n,q)} = \theta_{1,p}^{(n,q)}$ therefore $\theta_{m,p}^{(n,q)} = 1$. Thus $U_{m,p}^{(n)}$ does not depend on n, $U_{m,p}^{(n)} = U_{m,p}$.

$U_{m,p}$ is partially isometric, its initial and final sets are $[f_{m,1}, f_{m,2}, \cdots]$ resp. $[f_{p,1}, f_{p,2}, \cdots]$, and $U_{m,p} f_{m,n} = f_{p,n}$. If $r \neq m$, then $f_{r,n}$ is orthogonal to $[f_{m,1}, f_{m,,2}, \cdots]$, and so $U_{m,p} f_{r,n} = 0$. $U_{m,p}$ is uniquely determined by these properties, because already the $U_{m,p}^{(n)}$ were.

Thus we have proved the first statement of our Lemma. The second results by replacing in our result on A, B at the beginning of this proof A, B by A, $U_{m,p}$.

Lemma 5.3.7. *If a factor* $\mathbf{M}$ *fulfills the conditions of Lemma* 5.3.1, *then we have with the* $U_{m,p}$ *of Lemma* 5.3.6 $\mathbf{M} = \mathbf{R}(U_{m,p}; m, p = 1, 2, \cdots)$.

Proof: As all $U_{m,p} \in \mathbf{M}$, we have $\mathbf{R}(U_{m,p}; m, p = 1, 2, \cdots) \subset \mathbf{M}$. Assume now conversely $A \in \mathbf{M}$. Put $A_{m,p} = P_{[f_{p,1}, f_{p,2}, \cdots]} A P_{[f_{m,1}, f_{m,2}, \cdots]}$. Then $A_{m,p} \in \mathbf{M}$, $P_{[f_{p,1}, f_{p,2}, \cdots]} A_{m,p} = A_{m,p} P_{[f_{m,1}, f_{m,2}, \cdots]} = A_{m,p}$ so by Lemma 5.3.6 $A_{m,p} = \alpha_{m,p} U_{m,p}$. Thus $A_{m,p} \in \mathbf{R}(U_{m',p'}; m', p' = 1, 2, \cdots)$. But

$$\begin{aligned}\sum_{m,p} A_{m,p} &= \sum_{m,p} P_{[f_{p,1}, f_{p,2} \cdots]} A P_{[f_{m,1}, f_{m,2}, \cdots]} \\ &= (\sum_p P_{[f_{p,1}, f_{p,2}, \cdots]}) A (\sum_m P_{[f_{m,1}, f_{m,2}, \cdots]}) \\ &= 1 \cdot A \cdot 1 = A .\end{aligned}$$

If these sums are infinite at all, they are strongly convergent, cf. (16), pp. 77–78), so $A \in \mathbf{R}(U_{m',p'}; m', p' = 1, 2, \cdots)$. This holds for any $A \in \mathbf{M}$, so

$$\mathbf{M} \subset \mathbf{R}(U_{m,p}; m, p, = 1, 2, \cdots) .$$

So we have proved

$$\mathbf{M} = \mathbf{R}(U_{m,p}, m, p = 1, 2, \cdots).$$

Lemma 5.3.8. *If a factor* $\mathbf{M}$ *fulfills the conditions of Lemma* 5.3.1, *then it is a direct factor.*

Proof: Consider the $f_{m,n}$ and the $U_{m,p}$ of Lemma 5.3.7; m, p have the same finite or infinite range, n has another finite of infinite range. Take two spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$ with the same dimensions as there are elements in the resp. ranges of m and n; and let φ_m^0, $m = 1, 2, \cdots$, and ψ_m^0, $n = 1, 2, \cdots$, be complete, normalised orthogonal sets in $\mathfrak{H}_1$ resp. $\mathfrak{H}_2$. Then $\varphi_m^0 \otimes \psi_n^0$, $m = 1, 2, \cdots$, $n = 1, 2, \cdots$, is one in $\mathfrak{H}_1 \otimes \mathfrak{H}_2$. If we let correspond $\varphi_m^0 \otimes \psi_n^0$ to $f_{m,n}$, we obtain an isomorphism of $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ and $\mathfrak{H}$.

Define in $\mathfrak{H}_1$ operators $V_{m,p}$ by

$$V_{m,p} \varphi_m^0 = \varphi_p^0 , \qquad V_{m,p} \varphi_r^0 = 0 \quad \text{if} \quad r \neq m .$$

Then (cf. Lemma 2.3.3)

$$V_{m,p}^{(1)} \varphi_m^0 \otimes \varphi_r^0 = \varphi_p^0 \otimes \varphi_r^0 ,$$

$$V_{m,p}^{(1)} \varphi_r^0 \otimes \varphi_r^0 = 0 \quad \text{if} \quad r \neq m .$$

Thus the above spatial isomorphism, of $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ and $\mathfrak{H}$ transforms $V_{m,p}^{(1)}$ into $U_{m,p}$, and thus $\mathbf{R}(V_{m,p}^{(1)}, m, p = 1, 2, \cdots)$ into $\mathbf{R}(U_{m,p}; m, p = 1, 2, \cdots) = \mathbf{M}$.

Now consider any $A_1 \in \mathbf{B}_1$. Then $P_{[\varphi\ {}^0_p]} A_1 P_{[\varphi\ {}^0_m]} = (A_1\varphi^0_m, \varphi^0_p)\, V_{m,p} = \alpha_{m,p}\, V_{m,p}$. So $P_{[\varphi\ {}^0_p]} A_1 P_{[\varphi\ {}^0_m]} \in \mathbf{R}(V_{m',p'};\ m', p' = 1, 2, \cdots)$. But

$$\sum_{m,p} P_{[\varphi\ {}^0_p]} A_1 P_{[\varphi\ {}^0_m]} = (\sum_p P_{[\varphi\ {}^0_p]}) A_1 (\sum_m P_{[\varphi\ {}^0_m]}) = 1 \cdot A_1 \cdot 1 = A_1,$$

(if the sums are infinite at all, they are strongly convergent), so

$$A_1 \in \mathbf{R}(V_{m,p};\ m,\ p,\ = 1,\ 2, \cdots).$$

Thus $\mathbf{R}(V_{m,p};\ m,\ p = 1, 2, \cdots) = \mathbf{B}_1$. Now Lemmas 2.3.6 and 2.3.7 give $\mathbf{R}(V^{(1)}_{m,p};\ m,\ p = 1, 2, \cdots) = \mathbf{B}^{(1)}_1$. Thus our special isomorphism transforms $\mathbf{B}^{(1)}_1$ into $\mathbf{M}$, proving that $\mathbf{M}$ is a direct factor.

The results of Lemmas 5.3.1, 5.3.2 and 5.3.8 give together:

Theorem IV. *A factor* $\mathbf{M}$ *is direct if and only if it fulfills the (equivalent) conditions of Lemma* 5.3.1.

The condition (i) in Lemma 5.3.1 gives, together with Lemmas 5.2.1 and 5.2.2, the

Theorem V. *Every algebraic ring-isomorphism leaves invariant the notions of a factor and of a direct factor (as applied to the rings* $\mathbf{M}_{\mathrm{I}}$, $\mathbf{M}_{\mathrm{II}}$ *themselves).*

5.4. Theorem V makes it possible to prove the following Lemma, which belongs naturally to §3.2.

Lemma 5.4.1. *If* $\mathbf{M}_1, \cdots, \mathbf{M}_n$ *is a factorisation, and at least* $n - 1$ *of its factors* $\mathbf{M}_i$ *are direct, then the factorisation is a direct one too.*

Proof: For $n = 1$, $\mathbf{R}(\mathbf{M}_1) = \mathbf{B}$ means $\mathbf{M}_1 = \mathbf{B}_1$, thus $\mathbf{M}_1$, is a direct factor: Put $\mathfrak{H}_1$ a one dimensional space, $\mathfrak{H}_2 = \mathfrak{H}$; then the results of §2.4 show that $\mathfrak{H}_1 \otimes \mathfrak{H}_2 = \mathfrak{H}_2 = \mathfrak{H}$, $\mathbf{B}^{(2)}_2 = \mathbf{B}_2 = \mathbf{B} = \mathbf{M}_1$. So the case $n = 1$ is settled, and we must only prove the Lemma for $n = 2, 3, \cdots$, if it is already established for $n - 1$.

There is an i such that all $\mathbf{M}_j, j \neq i$, are direct. As a permutation of all $\mathbf{M}'_j$'s does not matter, we may assume that this i is $\neq 1$, that is, that $\mathbf{M}_1$ is direct.

Then $\mathfrak{H}$ is ismorphic to $\mathfrak{H}_1 \otimes \mathfrak{H}_2$, so that $\mathbf{M}_1$ becomes $\mathbf{B}^{(1)}_1$. Let $\mathbf{M}_2, \cdots, \mathbf{M}_n$ become $\mathbf{N}_2, \cdots, \mathbf{N}_n$. If $j \neq 1$, then $\mathbf{M}_j \subset \mathbf{M}'_1$ so $\mathbf{N}_j \subset (\mathbf{B}^{(1)}_1)' = \mathbf{B}^{(2)}_2$. The correspondence $A^{(2)} \to A$ maps therefore the $\mathbf{N}_j$'s on rings $\mathbf{N}^0_j \subset \mathbf{B}_2$ (cf. Lemma 2.3.7). The $\mathbf{N}_j$'s are algebraically (even fully) ring-isomorphic to the resp. $\mathbf{N}^0_j$ (cf. Lemma 2.3.4, resp. (22), §4). Now all $\mathbf{M}_2, \cdots, \mathbf{M}_n$ are factors, and at least $n - 2$ of them direct, so the same holds for $\mathbf{N}_2, \cdots, \mathbf{N}_n$ and, by Theorem V, for $\mathbf{N}^0_2, \cdots, \mathbf{N}^0_n$.

As the $\mathbf{N}_j$, $\mathbf{N}_k$ commute, the $\mathbf{N}^0_2, \cdots, \mathbf{N}^0_n$ do too; as $\mathbf{R}(\mathbf{N}_2, \cdots, \mathbf{N}_n) = \mathbf{B}^{(2)}_2$, $\mathbf{R}(\mathbf{N}^0_2, \cdots, \mathbf{N}^0_n) = \mathbf{B}_2$ by Lemma 2.3.7. Thus $\mathbf{N}_2, \cdots, \mathbf{N}_n$ is a factorisation in $\mathbf{B}_2$ of $\mathfrak{H}_2$.

So our Lemma for $n - 1$ applies: $\mathbf{N}_2, \cdots, \mathbf{N}_n$ is a direct factorisation in $\mathbf{B}_2$ of $\mathfrak{H}_2$. So $\mathfrak{H}_2$ is isomorphic to $\mathfrak{H}^*_2 \otimes \cdots \otimes \mathfrak{H}^*_n$, and $\mathbf{N}_2, \cdots, \mathbf{N}_n$ become $\mathbf{B}^{*(2)}_2, \cdots, \mathbf{B}^{*(n)}_n$ resp.

Now it is obvious, that $\mathfrak{H}$ is isomorphic to $\mathfrak{H}_1 \otimes \mathfrak{H}_2^* \otimes \cdots \otimes \mathfrak{H}_n^*$ and that $\mathbf{M}_1, \mathbf{M}_2, \cdots, \mathbf{M}_n$ become $\mathbf{B}_1^{(1)}, \mathbf{B}_2^{*(2)}, \cdots, \mathbf{B}_n^{*(n)}$ (the $n-1$ last ones have to be taken now in the product $\mathfrak{H}_1 \otimes \mathfrak{H}_2^* \otimes \cdots \mathfrak{H}_n^*$!) resp. Thus $\mathbf{M}_1, \cdots, \mathbf{M}_n$ is a direct factorisation.

Note that now Lemma 3.2.2 shows, that all $\mathbf{M}_i$'s are direct. Note furthermore, that our Lemma would even then not have been obvious, if all n factors $\mathbf{M}_i$ had been known to be direct. $n-1$ can not be replaced in our Lemma by $n-2$: Because this would give for every factor $\mathbf{M}$, that the factorisation $\mathbf{M}, \mathbf{M}'$ ($n=2$) is direct, and thus $\mathbf{M}$ is direct; and then by our Lemma every factorisation $\mathbf{M}_1, \cdots, \mathbf{M}_n$ would be direct. Thus the answer to both parts of Problem 2 at the end of §3.2 would be affirmative, which is not the case (cf. there and §8.4 and §13.2).

Part II: The General Problem

Chapter VI: Relative Dimensionality

6.1. In what follows $\mathbf{M}$ will always denote a factor in $\mathbf{B}$ of the space $\mathfrak{H}$. Thus $\mathbf{M}'$ is a factor too.

Definition 6.1.1. We write $\mathfrak{M} \sim \mathfrak{N}$ ($\cdots \mathbf{M}$), and for $E = P_{\mathfrak{M}}$, $F = P_{\mathfrak{N}}$, $E \sim F$ ($\cdots \mathbf{M}$), if a partially isometric $U \in \mathbf{M}$ exists, the initial and final sets of which are $\mathfrak{M}$ resp. $\mathfrak{N}$. (If no misunderstanding is possible we will omit the ($\cdots \mathbf{M}$).) We say that $\mathfrak{M}$, $\mathfrak{N}$ have *the same relative dimension* (with respect to $\mathbf{M}$).

In discussing this notion $\sim$, we have always the choice to speak about the linear closed sets $\mathfrak{M}$, or their projections $E = P_{\mathfrak{M}}$. We will mostly use the first mentioned terminology, remembering however the equivalence of the two.

Lemma 6.1.1. $\mathfrak{M} \sim \mathfrak{N}$ *implies* $\mathfrak{M}, \mathfrak{N} \eta \mathbf{M}$. *If* $\mathfrak{M}, \mathfrak{N}, \mathfrak{P} \eta \mathbf{M}$ *then we have:*

$$\mathfrak{M} \sim \mathfrak{M}$$

$$\mathfrak{M} \sim \mathfrak{N} \textit{ implies } \mathfrak{N} \sim \mathfrak{M};$$

$$\mathfrak{M} \sim \mathfrak{N}, \mathfrak{N} \sim \mathfrak{P} \textit{ imply } \mathfrak{M} \sim \mathfrak{P}.$$

Proof: If U is the partially isometric operator in question, we have $P_{\mathfrak{M}} = U^*U \in \mathbf{M}$, $P_{\mathfrak{N}} = UU^* \in \mathbf{M}$, and so $\mathfrak{M}, \mathfrak{N} \eta \mathbf{M}$.

$\mathfrak{M} \sim \mathfrak{M}$ results by putting $U = P_{\mathfrak{M}}$; $\mathfrak{M} \sim \mathfrak{N}$ gives $\mathfrak{N} \sim \mathfrak{M}$ by replacing U by U^*; $\mathfrak{M} \sim \mathfrak{N}$ with U and $\mathfrak{N} \sim \mathfrak{P}$ with V gives $\mathfrak{M} \sim \mathfrak{P}$ by considering VU.

Thus $\sim$ is naturally applied to linear closed sets $\mathfrak{M} \eta \mathbf{M}$, and has there the properties of an equivalence. We may say: Two linear, closed sets $\mathfrak{M}$, $\mathfrak{N}$ which are invariant under all unitary elements of $\mathbf{M}'$, are of the same dimension in the usual sense of the word, if an isometric mapping U of $\mathfrak{M}$ on $\mathfrak{N}$ exists, that is a partially isometric operator U with the initial and final sets $\mathfrak{M}$ resp. $\mathfrak{N}$. But they have the same relative dimension, if even the mapping U can be chosen so as to be invariant under all unitary elements of $\mathbf{M}'$. (Cf. Definition 4.2.1 and Lemma 4.2.1).

We now prove two Lemmas, the first of which expresses an additivity property of our $\sim$, while the second one is the analogon of the Cantor-Bernstein "equivalence theorem" of general set-theory. (In fact, it is a special case of the Kuratowski-Tarski generalisation of that theorem.)

Lemma 6.1.2. *Let* $\mathfrak{M}_1, \mathfrak{M}_2, \cdots$ *and* $\mathfrak{N}_1, \mathfrak{N}_2, \cdots$ *be two (finite or infinite) sequences of the same length; let the* $\mathfrak{M}_i$ *be mutually orthogonal and let the* $\mathfrak{N}_i$ *be mutually orthogonal. If we have* $\mathfrak{M}_i \sim \mathfrak{N}_i$ *for all* i, *then* $[\mathfrak{M}_1, \mathfrak{M}_2, \cdots] \sim [\mathfrak{N}_1, \mathfrak{N}_2, \cdots]$.

Proof: Let $U_i \in \mathbf{M}$ be the partially isometric mapping of $\mathfrak{M}_i$ on $\mathfrak{N}_i$, then $U = \sum_i U_i$ does the desired thing for $[\mathfrak{M}_1, \mathfrak{M}_2, \cdots]$ and $[\mathfrak{N}_1, \mathfrak{N}_2, \cdots]$ by Lemma 4.3.3.

Lemma 6.1.3. *If* $\mathfrak{M} \sim \mathfrak{N}' \subset \mathfrak{N}$ *and* $\mathfrak{N} \sim \mathfrak{M}' \subset \mathfrak{M}$, *then* $\mathfrak{M} \sim \mathfrak{N}$.

Proof: Our relations imply $\mathfrak{M}, \mathfrak{N}, \mathfrak{M}', \mathfrak{N}' \eta \mathbf{M}$. Now let $U \in \mathbf{M}$ be the partially isomorphic mapping of $\mathfrak{N}$ on $\mathfrak{M}'$; it maps $\mathfrak{N}' \subset \mathfrak{N}$ on some $\mathfrak{M}'' \subset \mathfrak{M}'$. Then consideration of $UP_{\mathfrak{N}'}$ proves that $\mathfrak{N}' \sim \mathfrak{M}''$, and so $\mathfrak{M} \sim \mathfrak{M}''$.

Let $V \in \mathbf{M}$ be the partially isomorphic mapping of $\mathfrak{M}$ on $\mathfrak{M}''$. Form for every $n = 0, 1, 2, \cdots$ the V^n-images of $\mathfrak{M}$ and of $\mathfrak{M}'$, and call them $\mathfrak{M}^{(2n)}$ resp. $\mathfrak{M}^{(2n+1)}$ ($\mathfrak{M}, \mathfrak{M}', \mathfrak{M}''$ coincide with $\mathfrak{M}^{(0)}, \mathfrak{M}^{(1)}, \mathfrak{M}^{(2)}$ resp.). We have $\mathfrak{M}^{(0)} \supset \mathfrak{M}^{(1)} \supset \mathfrak{M}^{(2)}$, thus (apply $V^{(n)}$!) $\mathfrak{M}^{(2n)} \supset \mathfrak{M}^{(2n+1)} \supset \mathfrak{M}^{(2n+2)}$, therefore $\mathfrak{M}^{(0)} \supset \mathfrak{M}^{(1)} \supset \mathfrak{M}^{(2)} \supset \cdots$.

V^n is partially isometric with the initial and final sets $\mathfrak{M}^{(0)}$ resp. $\mathfrak{M}^{(2n)}$, $V^n P_{\mathfrak{M}^{(0)}}$ similarly with $\mathfrak{M}^{(1)}$ and $\mathfrak{M}^{(2n+1)}$. Thus all $\mathfrak{M}^{(p)} \eta \mathbf{M}$, $n = 0, 1, 2, \cdots$, and $\mathfrak{M}^{(0)} \sim \mathfrak{M}^{(2n)}$, $\mathfrak{M}^{(1)} \sim \mathfrak{M}^{(2n+1)}$, that is: $\mathfrak{M}^{(p)} \sim \mathfrak{M}$ or $\mathfrak{M}'$ if p is even resp. odd.

V maps $\mathfrak{M}^{(p)}$ on $\mathfrak{M}^{(p+2)}$, $\mathfrak{M}^{(p+1)}$ on $\mathfrak{M}^{(p+3)}$, therefore $\mathfrak{M}^{(p)} - \mathfrak{M}^{(p+1)}$ on $\mathfrak{M}^{(p+2)} - \mathfrak{M}^{(p+3)}$. Using $V(P_{\mathfrak{M}^{(p)}} - P_{\mathfrak{M}^{(p+1)}})$ shows therefore, that $\mathfrak{M}^{(p)} - \mathfrak{M}^{(p+1)} \sim \mathfrak{M}^{(p+2)} - \mathfrak{M}^{(p+3)}$. Now apply Lemma 6.1.2 to the sequences

$$\mathfrak{M}^{(0)}.\mathfrak{M}^{(1)}.\mathfrak{M}^{(2)} \cdots, \mathfrak{M}^{(0)} - \mathfrak{M}^{(1)}, \mathfrak{M}^{(1)} - \mathfrak{M}^{(2)}, \mathfrak{M}^{(2)} - \mathfrak{M}^{(3)},$$

$$\mathfrak{M}^{(3)} - \mathfrak{M}^{(4)}, \cdots,$$

and

$$\mathfrak{M}^{(0)}.\mathfrak{M}^{(1)}.\mathfrak{M}^{(2)} \cdots, \mathfrak{M}^{(2)} - \mathfrak{M}^{(3)}, \mathfrak{M}^{(1)} - \mathfrak{M}^{(2)}, \mathfrak{M}^{(4)} - \mathfrak{M}^{(5)},$$

$$\mathfrak{M}^{(3)} - \mathfrak{M}^{(4)}, \cdots.$$

It gives $\mathfrak{M}^{(0)} \sim \mathfrak{M}^{(1)}$, that is $\mathfrak{M} \sim \mathfrak{M}'$, and thus $\mathfrak{M} \sim \mathfrak{N}$.

6.2. The following Lemma is an important means of establishing equality of relative dimension.

Lemma 6.2.1. *If* X *is a linear, closed operator with an everywhere dense domain, and* $X \eta \mathbf{M}$, *then*

$$[\text{Range } X] \sim [\text{Range } X^*].$$

Proof: Form the canonical decomposition of X in the sense of Definition 4.4.1. $X \eta \mathbf{M}$ implies by Lemma 4.4.1 $W \in \mathbf{M}$. W is partially isometric its initial and final sets being [Range X^*] resp. [Range X]. So

$$[\text{Range } X^*] \sim [\text{Range } X], \qquad [\text{Range } X] \sim [\text{Range } X^*].$$

We now prove in two steps the analogue of the "comparability theorem" of general set-theory.

Lemma 6.2.2. *If* $\mathfrak{M}$, $\mathfrak{N}$ η $\mathbf{M}$ *and* $\neq 0$, *then two* $\mathfrak{P}$, $\mathfrak{Q} \neq 0$, $\mathfrak{P} \subset \mathfrak{M}$, $\mathfrak{Q} \subset \mathfrak{N}$ *with* $\mathfrak{P} \sim \mathfrak{Q}$ *exist.*

Proof: Choose an $f \in \mathfrak{M}$, $f \neq 0$. Then $E_f^{\mathbf{M}} \in \mathbf{M}'$ and $\neq 0$, $P_{\mathfrak{N}} \in \mathbf{M}$ and $\neq 0$, so by the Corollary to Theorem III. $P_{\mathfrak{M}_f^{\mathbf{M}} \cdot \mathfrak{N}} = P_{\mathfrak{N}} \cdot E_f^{\mathbf{M}} \neq 0$, $\mathfrak{M}_f^{\mathbf{M}} \cdot \mathfrak{N} \neq (0)$. Choose $g \in \mathfrak{M}_f^{\mathbf{M}} \cdot \mathfrak{N}$, $g \neq 0$, multiplying with $1/\| g \|$ makes $\| g \| = 1$. As $g \in \mathfrak{M}_f^{\mathbf{M}}$, therefore an $A \in \mathbf{M}$ with $\| g - Af \| < 1$ exists. So $\| P_{\mathfrak{N}} g - P_{\mathfrak{N}} Af \| < 1$. Now $P_{\mathfrak{N}} g = g$, $P_{\mathfrak{M}} f = f$ have the effect that $\| g - P_{\mathfrak{N}} A P_{\mathfrak{M}} f \| < 1$. As $\| g \| = 1$ this implies $P_{\mathfrak{N}} A P_{\mathfrak{M}} f \neq 0$, $P_{\mathfrak{N}} A P_{\mathfrak{M}} \neq 0$.

Now apply Lemma 6.2.1: [Range $P_{\mathfrak{N}} A P_{\mathfrak{M}}$] $\sim$ [Range $(P_{\mathfrak{N}} A P_{\mathfrak{M}})^*$]. As $P_{\mathfrak{N}} A P_{\mathfrak{M}} \neq 0$ and so $(P_{\mathfrak{N}} A P_{\mathfrak{M}})^* \neq 0$, both above sets are $\neq (0)$. Furthermore

$$[\text{Range } P_{\mathfrak{N}} A P_{\mathfrak{M}}] \subset [\text{Range } P_{\mathfrak{N}}] = \mathfrak{N}.$$

$$[\text{Range } (P_{\mathfrak{N}} A P_{\mathfrak{M}})^*] = [\text{Range } P_{\mathfrak{M}} A^* P_{\mathfrak{N}}] \subset [\text{Range } P_{\mathfrak{M}}] = \mathfrak{M}.$$

So $\mathfrak{P} = [\text{Range } (P_{\mathfrak{N}} A P_{\mathfrak{M}})^*]$, $\mathfrak{Q} = [\text{Range } P_{\mathfrak{N}} A P_{\mathfrak{M}}]$ meet our requirements.

Lemma 6.2.3. *If* $\mathfrak{M}$, $\mathfrak{N}$ η $\mathbf{M}$, *then either* $\mathfrak{M} \sim \mathfrak{N}' \subset \mathfrak{N}$ *or* $\mathfrak{N} \sim \mathfrak{M}' \subset \mathfrak{M}$.

Proof: Let Ω be the first uncountable Cantor ordinal number. Define for all $\alpha < \Omega$ a pair $\mathfrak{M}_\alpha$, $\mathfrak{N}_\alpha$ as follows: All $\mathfrak{M}_\alpha$, $\mathfrak{N}_\alpha$ will be linear, closed sets, η $\mathbf{M}$, $\neq (0)$, and $\mathfrak{M}_\alpha \subset \mathfrak{M}$, $\mathfrak{N}_\alpha \subset \mathfrak{N}$, $\mathfrak{M}_\alpha \sim \mathfrak{N}_\alpha$. If $\alpha < \Omega$ and all $\mathfrak{M}_\beta$, $\mathfrak{N}_\beta$, $\beta < \alpha$ are already defined, then form $\mathfrak{M} - [\mathfrak{M}_\beta, \beta < \alpha]$ and $\mathfrak{N} - [\mathfrak{N}_\beta, \beta < \alpha]$. As $\mathfrak{M}$, $\mathfrak{N}$, $\mathfrak{M}_\beta$, $\mathfrak{N}_\beta$ all η $\mathbf{M}$, these two sets are η $\mathbf{M}$ too. If both are $\neq 0$, then there exist two $\mathfrak{P}$, $\mathfrak{Q} \neq (0)$ with $\mathfrak{P} \subset \mathfrak{M} - [\mathfrak{M}_\beta, \beta < \alpha]$, $\mathfrak{Q} \subset \mathfrak{N} - [\mathfrak{N}_\beta, \beta < \alpha]$ $\mathfrak{P} \sim \mathfrak{Q}$. Let them be $\mathfrak{M}_\alpha = \mathfrak{P}$, $\mathfrak{N}_\alpha = \mathfrak{Q}$ for some such pair $\mathfrak{P}$, $\mathfrak{Q}$. As we see $\mathfrak{M}_\alpha$, $\mathfrak{N}_\alpha \neq (0)$, $\mathfrak{M}_\alpha \subset \mathfrak{M}$, $\mathfrak{N}_\alpha \subset \mathfrak{N}$, $\mathfrak{M}_\alpha \sim \mathfrak{N}_\alpha$ are maintained. If the above condition is not fulfilled, let $\mathfrak{M}_\alpha$, $\mathfrak{N}_\alpha$ (and all $\mathfrak{M}_\gamma$, $\mathfrak{N}_\gamma$, $\gamma \geqq \alpha$) be undefined.

Let the first α for which $\mathfrak{M}_\alpha$, $\mathfrak{N}_\alpha$ are undefined, be $\bar{\alpha} \leqq \Omega$. If $\bar{\alpha} < \Omega$, we must have $\mathfrak{M} - [\mathfrak{M}_\alpha; \alpha < \bar{\alpha}] = (0)$ or $\mathfrak{N} - [\mathfrak{N}_\alpha; \alpha < \bar{\alpha}] = (0)$. Consider now all $\mathfrak{M}_\alpha$, $\mathfrak{N}_\alpha$, $\alpha < \bar{\alpha}$. Always $\mathfrak{M}_\alpha \neq (0)$. If α, $\beta < \bar{\alpha}$ assume $\alpha > \beta$. Then $\mathfrak{M}_\alpha \subset \mathfrak{M} - [\mathfrak{M}_{\beta'}, \beta' < \alpha]$, so $\mathfrak{M}_\alpha$ is orthogonal to $\mathfrak{M}_\beta$. This is symmetric in α, β, so it holds for $\alpha < \beta$ too, that is whenever $\alpha \neq \beta$. Thus all $\mathfrak{M}_\alpha$, $\alpha < \bar{\alpha}$, are $\neq (0)$ and mutually orthogonal. The same is of course true for the $\mathfrak{N}_\alpha$, $\alpha < \bar{\alpha}$.

Choose an $f_\alpha \in \mathfrak{M}_\alpha$, $f_\alpha \neq 0$, multiplying with $1/\| f_\alpha \|$ makes $\| f_\alpha \| = 1$. So the f_α, $\alpha < \bar{\alpha}$ form a normalised orthogonal set. It must therefore be finite or countably infinite (cf. (16), p. 66), thus excluding $\bar{\alpha} = \Omega$.

If $\bar{\alpha}$ is finite, we have $\bar{\alpha} = n + 1$, $n = 0, 1, 2, \cdots$, and α runs over $1, \cdots, n$. If $\bar{\alpha}$ is infinite, then $\alpha < \bar{\alpha}$ has the same aleph as $\alpha < \omega$ and may therefore (by a re-indexing of the $\mathfrak{M}_\alpha$) be replaced by it; so we may assume that α runs over $1, 2, \cdots$.

Writing i for α, we see that we have a finite or infinite sequence of pairs $\mathfrak{M}_i$, $\mathfrak{N}_i$, $i = 1, 2, \cdots$; where the $\mathfrak{M}_i$ are mutually orthogonal, the $\mathfrak{N}_i$ are mutually orthogonal; $\mathfrak{M}_i \subset \mathfrak{M}$, $\mathfrak{N}_i \subset \mathfrak{N}$; $\mathfrak{M}_i \sim \mathfrak{N}_i$ and either $[\mathfrak{M}_i, i = 1, 2, \cdots] = \mathfrak{M}$ or $[\mathfrak{N}_i, i = 1, 2, \cdots] = \mathfrak{N}$. Now Lemma 6.1.2 applies: $[\mathfrak{M}_i; i = 1, 2, \cdots] \sim$

$[\mathfrak{N}_i; i = 1, 2, \cdots]$. Thus if we define $\mathfrak{N}' = [\mathfrak{N}_i, i = 1, 2, \cdots]$ resp. $\mathfrak{M}' = [\mathfrak{M}_i, i = 1, 2, \cdots]$ we will have $\mathfrak{M} \sim \mathfrak{N}' \subset \mathfrak{N}$ resp. $\mathfrak{N} \sim \mathfrak{M}' \subset \mathfrak{M}$. So the proof is completed.

6.3. Now we are in the position to define:

Definition 6.3.1. We write $\mathfrak{M} \lesssim \mathfrak{N}$ or $\mathfrak{N} \gtrsim \mathfrak{M}$, if $\mathfrak{M} \sim \mathfrak{N}' \subset \mathfrak{N}$. We write $\mathfrak{M} \prec \mathfrak{N}$ or $\mathfrak{N} \succ \mathfrak{M}$, if $\mathfrak{M} \lesssim \mathfrak{N}$ but not $\mathfrak{M} \sim \mathfrak{N}$. (As to replacing $\mathfrak{M}$, $\mathfrak{N}$ by $E = P_{\mathfrak{M}}$, $F = P_{\mathfrak{N}}$, and the explanatory symbol $(\cdots \mathbf{M})$, cf. Definition 6.1.1).

And we can prove:

Lemma 6.3.1. *If* $\mathfrak{M}$, $\mathfrak{N}$, $\mathfrak{P}$, $\mathfrak{Q}$ η **M**, *then the following statements hold:*

(i) $\mathfrak{M} \lesssim \mathfrak{N}$ *means* $\mathfrak{M} \prec \mathfrak{N}$ *or* $\mathfrak{M} \sim \mathfrak{N}$.

(ii) *If* $\mathfrak{M} \sim \mathfrak{P}$, $\mathfrak{N} \sim \mathfrak{Q}$, *then* $\mathfrak{M} \prec \mathfrak{N}$ *is equivalent to* $\mathfrak{P} \prec \mathfrak{Q}$.

(iii) *One and only one of the three relations* $\mathfrak{M} \begin{matrix} \prec \\ \sim \\ \succ \end{matrix} \mathfrak{N}$ *holds.*

(iv) $\mathfrak{M} \prec \mathfrak{N}$, $\mathfrak{N} \prec \mathfrak{P}$ *imply* $\mathfrak{M} \prec \mathfrak{P}$.

Proof: We prove the statements in a somewhat changed order.

Ad (i): Obvious.

Ad (iv): $\mathfrak{M} \lesssim \mathfrak{N}$, $\mathfrak{N} \lesssim \mathfrak{P}$ imply $\mathfrak{M} \sim \mathfrak{N}' \subset \mathfrak{N}$, $\mathfrak{N} \sim \mathfrak{P}' \subset \mathfrak{P}$. If $U \epsilon \mathbf{M}$ is the isometric mapping of $\mathfrak{N}$ on $\mathfrak{P}'$, and $\mathfrak{P}''$ the U-image of $\mathfrak{N}'$, then we have $\mathfrak{M} \sim \mathfrak{N}' \sim \mathfrak{P}'' \subset \mathfrak{P}' \subset \mathfrak{P}$, so $\mathfrak{M} \lesssim \mathfrak{P}$.

Now if we had $\mathfrak{M} \sim \mathfrak{P}$, then we could write $\mathfrak{P} \sim \mathfrak{M} \subset \mathfrak{M}$, that is $\mathfrak{P} \lesssim \mathfrak{M}$; together with $\mathfrak{M} \lesssim \mathfrak{N}$ this gives $\mathfrak{P} \lesssim \mathfrak{N}$. $\mathfrak{N} \lesssim \mathfrak{P}$ and $\mathfrak{P} \lesssim \mathfrak{N}$ gives by Lemma 6.1.3. $\mathfrak{N} \sim \mathfrak{P}$, contradicting $\mathfrak{N} \prec \mathfrak{P}$. Thus we must have $\mathfrak{M} \prec \mathfrak{P}$.

Ad (ii): Assume $\mathfrak{M} \prec \mathfrak{N}$. As $\mathfrak{M} \sim \mathfrak{P}$, $\mathfrak{N} \prec \mathfrak{Q}$, we can write $\mathfrak{P} \lesssim \mathfrak{N}$, $\mathfrak{M} \lesssim \mathfrak{N}$, $\mathfrak{N} \lesssim \mathfrak{Q}$, implying $\mathfrak{P} \lesssim \mathfrak{Q}$ (cf. the beginning of the proof of (iv)). $\mathfrak{P} \sim \mathfrak{Q}$ would imply $\mathfrak{M} \sim \mathfrak{P}$, $\mathfrak{P} \sim \mathfrak{Q}$, $\mathfrak{Q} \sim \mathfrak{N}$ so $\mathfrak{M} \sim \mathfrak{N}$, contradicting $\mathfrak{M} \prec \mathfrak{N}$. Thus we must have $\mathfrak{P} \prec \mathfrak{Q}$. In the same way $\mathfrak{P} \prec \mathfrak{Q}$ implies $\mathfrak{M} \prec \mathfrak{N}$.

Ad (iii): By Lemma 6.2.3 we have $\mathfrak{M} \lesssim \mathfrak{N}$ or $\mathfrak{N} \lesssim \mathfrak{M}$, by (i) this means $\mathfrak{M} \begin{matrix} \succ \\ \sim \\ \prec \end{matrix} \mathfrak{N}$. $\sim$ and $\succ$ at once imply $\mathfrak{M} \prec \mathfrak{M}$ by (ii); $\prec$ and $\sim$ and similarly; $\prec$ and $\succ$ imply again $\mathfrak{M} \prec \mathfrak{M}$ by (iv). Now $\mathfrak{M} \prec \mathfrak{M}$ is impossible as $\mathfrak{M} \sim \mathfrak{M}$. So precisely one of the three relations holds.

Summing up Lemmas 6.1.1 and 6.3.1 we have:

Theorem VI. *The notions* $\mathfrak{M} \sim \mathfrak{N}$ *and* $\mathfrak{M} \prec \mathfrak{N}$ (*that is* $\mathfrak{N} \succ \mathfrak{M}$) *for* $\mathfrak{M}$, $\mathfrak{N}$ η **M** (*cf. Definition* 6.1.1 *and* 6.3.1) *have the properties of an equivalence and of a complete ordering.*

Further essential properties are given in Lemmas 6.1.2 *and* 6.3.1.

Some immediate consequences:

Lemma 6.3.2. *Assume* $\mathfrak{M}$, $\mathfrak{N}$ η **M**. *Then we have:*

(i) $\mathfrak{M} \subset \mathfrak{N}$ *implies* $\mathfrak{M} \lesssim \mathfrak{N}$.

(ii) *Always* $(0) \lesssim \mathfrak{M} \lesssim \mathfrak{H}$.

(iii) $\mathfrak{M} \sim (0)$ *implies* $\mathfrak{M} = (0)$.

Proof: Ad (i): Write $\mathfrak{M} \sim \mathfrak{M} \subset \mathfrak{N}$.

Ad (ii): By $(0) \subset \mathfrak{M} \subset \mathfrak{H}$ and (i).

Ad (iii): $\mathfrak{M}$ is the image of (0) by some linear U so $\mathfrak{M} = (0)$.

Chapter VII: Finite and Infinite Sets

7.1. We continue along the lines which are familiar from set-theory:

Definition 7.1.1. An $\mathfrak{M}\ \eta\ \mathbf{M}$ is *finite* if $\mathfrak{M} \sim \mathfrak{N} \subset \mathfrak{M}$ implies $\mathfrak{N} = \mathfrak{M}$; it is *infinite*, if this is not the case. (As to replacing $\mathfrak{M}$ by $E = P_{\mathfrak{M}}$, and the explanatory symbol ($\cdots$ **M**), cf. Definition 6.1.1).

Some immediate consequences:

Lemma 7.1.1. *Assume* $\mathfrak{M}, \mathfrak{N}\ \eta\ \mathbf{M}$. *Then we have:*

(i) *If* $\mathfrak{M} \precsim \mathfrak{N}$ *and* $\mathfrak{M}$ *is infinite, then* $\mathfrak{N}$ *is too; if* $\mathfrak{N}$ *is finite,* $\mathfrak{M}$ *is too.*

(ii) (0) *is finite.*

(iii) *If infinite* $\mathfrak{M}$*'s exist at all, then* $\mathfrak{H}$ *is infinite.*

Proof: Ad (i): The second statement follows from the first one, so we consider the first one only. As $\mathfrak{M}$ is infinite, a $\mathfrak{P}$ with $\mathfrak{M} \sim \mathfrak{P} \subsetneqq \mathfrak{M}$ exists. Then $[\mathfrak{M}, \mathfrak{N} - \mathfrak{M}] \sim [\mathfrak{P}, \mathfrak{N} - \mathfrak{M}] \subsetneqq [\mathfrak{M}, \mathfrak{N} - \mathfrak{M}]$, $\mathfrak{N} \sim [\mathfrak{P}, \mathfrak{N} - \mathfrak{M}] \subsetneqq \mathfrak{N}$. So $\mathfrak{N}$ is infinite too.

Ad (ii): $\mathfrak{N} \subset (0)$ alone implies $\mathfrak{N} = (0)$.

Ad (iii): If $\mathfrak{M}$ is infinite, $\mathfrak{M} \precsim \mathfrak{H}$ implies by (i), that $\mathfrak{H}$ is infinite too.

We now begin the preparations for a quantitative comparison of sets $\mathfrak{M}, \mathfrak{N}\ \eta\ \mathbf{M}$.

Lemma 7.1.2. *Assume* $\mathfrak{M}, \mathfrak{N}\ \eta\ \mathbf{M}$, $\mathfrak{M} \neq (0)$. *Then there exists a finite or infinite sequence* $\mathfrak{P}_1, \mathfrak{P}_2, \cdots$ (*its length can be* 0 *too*) *and a* $\mathfrak{Q}$, *such that we have:*

$\mathfrak{P}_1, \mathfrak{P}_2, \cdots, \mathfrak{Q}$ *are mutually orthogonal linear, closed sets, all* $\eta\ \mathbf{M}$,

$$[\mathfrak{P}_1, \mathfrak{P}_2, \cdots, \mathfrak{Q}] = \mathfrak{N},\ \mathfrak{M} \sim \mathfrak{P}_1 \sim \mathfrak{P}_2 \sim \cdots, \mathfrak{Q} \prec \mathfrak{M}$$

and if the sequence is infinite, we may even assume $\mathfrak{Q} = (0)$.

Proof: Let Ω be the first uncountable Cantor ordinal number. Define for all $\alpha < \Omega$ a $\mathfrak{P}_\alpha$ as follows: All $\mathfrak{P}$ will be linear, closed sets,

$$\mathfrak{P}_\alpha\ \eta\ \mathbf{M},\ \mathfrak{P}_\alpha \subset \mathfrak{N},\ \mathfrak{M} \sim \mathfrak{P}_\alpha.$$

If $\alpha < \Omega$ and all $\mathfrak{P}_\beta$, $\beta < \alpha$ are already defined, then form $\mathfrak{N} - [\mathfrak{P}_\beta; \beta < \alpha]$. As $\mathfrak{N}, \mathfrak{P}_\beta$ all $\eta\ \mathbf{M}$, this set is $\eta\ \mathbf{M}$ too. If it is $\succsim \mathfrak{M}$, then there exists a $\mathfrak{P}\ \eta\ \mathbf{M}$ with $\mathfrak{P} \subset \mathfrak{N} - [\mathfrak{P}_\beta; \beta < \alpha]$, $\mathfrak{M} \sim \mathfrak{P}$. Let then $\mathfrak{P}_\alpha$ be some such $\mathfrak{P}$. If the above condition is not fulfilled, let $\mathfrak{P}_\alpha$ (and all $\mathfrak{P}_\gamma$, $\gamma \geqq \alpha$) be undefined.

Let the first α for which $\mathfrak{P}_\alpha$ is undefined, be $\bar{\alpha} \leqq \Omega$. If $\bar{\alpha} < \Omega$, we must have $\mathfrak{N} - [\mathfrak{P}_\alpha; \alpha < \bar{\alpha}] \prec \mathfrak{M}$.

We see in the same way as in the proof of Lemma 6.2.3, that the $\mathfrak{P}_\alpha$, $\alpha < \bar{\alpha}$, are mutually orthogonal; and as $\mathfrak{P}_\alpha \sim \mathfrak{M} \neq (0)$, therefore $\mathfrak{P}_\alpha \neq (0)$. This implies, as above, that their set is finite or countably infinite, thus excluding $\bar{\alpha} = \Omega$.

If $\bar{\alpha}$ is finite, we have $\bar{\alpha} = n + 1$, $n = 0, 1, 2, \cdots$, and α runs over $1, \cdots, n$. If $\bar{\alpha}$ is infinite, then $\alpha < \bar{\alpha}$ has the same aleph as $\alpha < \omega$, and may therefore (by a re-indexing of the $\mathfrak{P}_\alpha$) be replaced by it; so we may assume that α runs over $1, 2, \cdots$.

Writing i for α we have a finite or infinite sequence $\mathfrak{P}_i$, $i = 1, 2, \cdots$; where

the $\mathfrak{P}_i$ are mutually orthogonal; $\mathfrak{M} \sim \mathfrak{P}_1 \sim \mathfrak{P}_2 \sim \cdots$; and $\mathfrak{Q} = \mathfrak{N} - [\mathfrak{P}_1, \mathfrak{P}_2, \cdots] \prec \mathfrak{M}$. Thus $\mathfrak{P}_1, \mathfrak{P}_2, \cdots, \mathfrak{Q}$ are mutually orthogonal and $\mathfrak{N} = [\mathfrak{P}_1, \mathfrak{P}_2, \cdots, \mathfrak{Q}]$.

So the case of a finite sequence is completely settled. In the case of an infinite sequence we must still get rid of $\mathfrak{Q}$. Assume therefore that the $\mathfrak{P}_1, \mathfrak{P}_2, \cdots$ form an infinite sequence.

We have $\mathfrak{Q} \prec \mathfrak{M} \sim \mathfrak{P}_1$, so $\mathfrak{Q} \sim \mathfrak{Q}' \subset \mathfrak{P}_1$. Apply Lemma 6.1.2 to the two sequences $\mathfrak{Q}, \mathfrak{P}_1, \mathfrak{P}_2, \cdots$ and $\mathfrak{Q}', \mathfrak{P}_2, \mathfrak{P}_3, \cdots$. It gives

$$\mathfrak{N} = [\mathfrak{Q}, \mathfrak{P}_1, \mathfrak{P}_2, \cdots] \sim [\mathfrak{Q}', \mathfrak{P}_2, \mathfrak{P}_3, \cdots] \subset [\mathfrak{P}_1, \mathfrak{P}_2, \cdots]$$

$$\subset [\mathfrak{Q}, \mathfrak{P}_1, \mathfrak{P}_2, \cdots] = \mathfrak{N}.$$

So $\mathfrak{N} \precsim [\mathfrak{P}_1, \mathfrak{P}_2, \cdots] \precsim \mathfrak{N}$, $[\mathfrak{P}_1, \mathfrak{P}_2, \cdots] \sim \mathfrak{N}$. Let $U \in \mathbf{M}$ be an isometric mapping of $[\mathfrak{P}_1, \mathfrak{P}_2, \cdots]$ on $\mathfrak{N}$, $\mathfrak{P}_i'$ the U-image of $\mathfrak{P}_i$. Then the $\mathfrak{P}_i'$ are mutually orthogonal and $[\mathfrak{P}_1', \mathfrak{P}_2', \cdots] = \mathfrak{N}$. Using $UP_{\mathfrak{P}_i}$ shows $\mathfrak{P}_i \sim \mathfrak{P}_i'$ so $\mathfrak{M} \sim \mathfrak{P}_1' \sim \mathfrak{P}_2' \sim, \cdots$. Thus we can replace

$$\mathfrak{P}_1, \mathfrak{P}_2, \cdots, \mathfrak{Q} \text{ by } \mathfrak{P}_1', \mathfrak{P}_2', \cdots, (0).$$

This completes the proof.

Lemma 7.1.3. *Assume* $\mathfrak{M}, \mathfrak{N} \eta \mathbf{M}$, $\mathfrak{M} \precneqq (0)$, $\mathfrak{N}$ *finite. Then the sequence* $\mathfrak{P}_1, \mathfrak{P}_2, \cdots$ *from Lemma* 7.1.2 *must be finite, and its length is uniquely determined by* $\mathfrak{M}, \mathfrak{N}$. *Call this number* $\left[\frac{\mathfrak{N}}{\mathfrak{M}}\right] = 0, 1, 2, \cdots$.

Proof: If the sequence $\mathfrak{P}_1, \mathfrak{P}_2, \cdots$ was infinite, we could apply Lemma 6.1.2 to $\mathfrak{P}_1, \mathfrak{P}_2, \cdots$ and $\mathfrak{P}_2, \mathfrak{P}_3, \cdots$, giving $\mathfrak{N} = [\mathfrak{P}_1, \mathfrak{P}_2, \cdots] \sim [\mathfrak{P}_2, \mathfrak{P}_3, \cdots] \subsetneqq [\mathfrak{P}_1, \mathfrak{P}_2, \cdots] = \mathfrak{N}$ ($\subsetneqq$ because $\mathfrak{P}_1 \precneqq (0)$, as $\mathfrak{P}_1 \sim \mathfrak{M} \precneqq (0)$), thus $\mathfrak{N}$ infinite, contradicting the assumption.

Assume now we had two representations

$$\mathfrak{N} = [\mathfrak{P}_1, \cdots, \mathfrak{P}_m, \mathfrak{Q}] = [\mathfrak{P}_1', \cdots, \mathfrak{P}_n', \mathfrak{Q}']$$

in the sense of Lemma 7.1.2, with $m \neq n$. Owing to the symmetry we may assume $m < n$, $m + 1 \leqq n$.

Now $\mathfrak{Q} \prec \mathfrak{M} \sim \mathfrak{P}_{m+1}'$, so $\mathfrak{Q} \sim \mathfrak{Q}^* \subset \mathfrak{P}_{m+1}'$, and $\mathfrak{Q}^* \neq \mathfrak{P}_{m+1}'$. So by Lemma 6.1.2, $\mathfrak{N} = [\mathfrak{P}_1, \cdots, \mathfrak{P}_m, \mathfrak{Q}] \sim [\mathfrak{P}_1', \cdots, \mathfrak{P}_m', \mathfrak{Q}^*] \subsetneqq [\mathfrak{P}_1', \cdots, \mathfrak{P}_m', \mathfrak{P}_{m+1}'] \subset [\mathfrak{P}_1', \cdots, \mathfrak{P}_n', \mathfrak{Q}_0'] = \mathfrak{N}$ and again $\mathfrak{N}$ would be infinite, contradicting the assumption.

7.2. We are now in the position to determine the chief characteristics of infinity.

Lemma 7.2.1. *Assume* $\mathfrak{M}, \mathfrak{N} \eta \mathbf{M}$, $\mathfrak{N}$ *infinite. Then* $\mathfrak{M} \precsim \mathfrak{N}$.

Proof: We have $\mathfrak{N} \sim \mathfrak{N}' \subsetneqq \mathfrak{N}$. Let $U \in \mathfrak{M}$ be the partially isometric mapping of $\mathfrak{N}$ on $\mathfrak{N}'$. Form for every $n = 0, 1, 2, \cdots$ the U^n-image of $\mathfrak{N}$, and call them $\mathfrak{N}^{(n)}$ ($\mathfrak{N}, \mathfrak{N}'$ coincide with $\mathfrak{N}^{(0)}, \mathfrak{N}^{(1)}$ resp.). We have $\mathfrak{N}^{(0)} \supset \mathfrak{N}^{(1)}$, thus (apply U^n !) $\mathfrak{N}^{(n)} \supset \mathfrak{N}^{(n+1)}$; therefore $\mathfrak{N}^{(0)} \supset \mathfrak{N}^{(1)} \supset \mathfrak{N}^{(2)} \supset \cdots$. U^n is partially isometric with the final set $\mathfrak{N}^{(n)}$ therefore $\mathfrak{N}^{(n)} \eta \mathbf{M}$. U maps $\mathfrak{N}^{(n)}$ on

$\mathfrak{N}^{(n+1)}$, $\mathfrak{N}^{(n+1)}$ on $\mathfrak{N}^{(n+2)}$, therefore $\mathfrak{N}^{(n)} - \mathfrak{N}^{(n+1)}$ on $\mathfrak{N}^{(n+1)} - \mathfrak{N}^{(n+2)}$. Using $U(P_{\mathfrak{N}^{(n)}} - P_{\mathfrak{N}^{(n+1)}})$ shows therefore, that $\mathfrak{N}^{(n)} - \mathfrak{N}^{(n+1)} \sim \mathfrak{N}^{(n+1)} - \mathfrak{N}^{(n+2)}$. Put $\mathfrak{P}_n^* = \mathfrak{N}^{(n)} - \mathfrak{N}^{(n+1)}$ then we see: The $\mathfrak{P}_1^*, \mathfrak{P}_2^*, \cdots$ form an infinite sequence, are mutually orthogonal, all $\subset \mathfrak{N}^{(0)} = \mathfrak{N}$ and $\mathfrak{P}_1^* \sim \mathfrak{P}_2^* \sim \cdots$. As

$$\mathfrak{P}_1^* = \mathfrak{N}^{(0)} - \mathfrak{N}^{(1)} = \mathfrak{N} - \mathfrak{N}' \gneq (0)$$

all $\mathfrak{P}_n^* \gneq 0$.

Apply now the construction of Lemma 7.1.2 to $\mathfrak{P}_1^*$, $\mathfrak{N}$ (for $\mathfrak{M}$, $\mathfrak{N}$) and choose $\mathfrak{P}_\alpha$ for $\alpha = 1, 2, \cdots$ equal to $\mathfrak{P}_1^*, \mathfrak{P}_2^*, \cdots$ resp. Then an infinite $\bar{\alpha}$ will result, which will be replaced, as described in the proof of Lemma 7.1.2, by $\bar{\alpha} = \omega$. And then the $\mathfrak{Q}$ will be eliminated. So $\mathfrak{N} = [\mathfrak{P}_1, \mathfrak{P}_2, \cdots]$, where the sequence is infinite, and $\mathfrak{P}_1^* \sim \mathfrak{P}_1 \sim \mathfrak{P}_2 \sim \cdots$.

Apply next the result of Lemma 7.1.2 to $\mathfrak{P}_1^*$, $\mathfrak{M}$ (for $\mathfrak{M}$, $\mathfrak{N}$), then $\mathfrak{M} = [\mathfrak{P}_1', \mathfrak{P}_2', \cdots \mathfrak{Q}']$, where the sequence is finite or infinite, and

$$\mathfrak{P}_1^* \sim \mathfrak{P}_1' \sim \mathfrak{P}_2' \sim \cdots, \mathfrak{Q}' \prec \mathfrak{P}_1^*.$$

If it is infinite, we may put $\mathfrak{Q}' = (0)$, and so, using Lemma 6.1.2,

$$\mathfrak{M} = [\mathfrak{P}_1', \mathfrak{P}_2', \cdots] \sim [\mathfrak{P}_1, \mathfrak{P}_2, \cdots] = \mathfrak{N}.$$

If it is finite, we have

$$\mathfrak{M} = [\mathfrak{P}_1', \cdots, \mathfrak{P}_n', \mathfrak{Q}'] \text{ and } \mathfrak{Q}' \prec \mathfrak{P}_1^* \sim \mathfrak{P}_{n+1}, \mathfrak{Q}' \sim \mathfrak{Q}'' \subset \mathfrak{P}_{n+1}.$$

Thus Lemma 6.1.2 gives

$$\mathfrak{M} = [\mathfrak{P}_1', \cdots, \mathfrak{P}_n', \mathfrak{Q}'] \sim [\mathfrak{P}_1, \cdots, \mathfrak{P}_n, \mathfrak{Q}''] \subset [\mathfrak{P}_1, \cdots, \mathfrak{P}_n, \mathfrak{P}_{n+1}]$$
$$\subset [\mathfrak{P}_1, \mathfrak{P}_2, \cdots] = \mathfrak{N}.$$

So we have $\mathfrak{M} \precsim \mathfrak{N}$ in any event.

Lemma 7.2.2. *There are two possibilities:*

(a) *An* $\mathfrak{M}\ \eta\ \mathbf{M}$ *is infinite if and only if* $\mathfrak{M} \sim \mathfrak{H}$ *and finite if and only if* $\mathfrak{M} \prec \mathfrak{H}$.

(b) *All* $\mathfrak{M}\ \eta\ \mathbf{M}$ *are finite. In this case* $\mathfrak{M} \sim \mathfrak{H}$ *implies* $\mathfrak{M} = \mathfrak{H}$.

Proof: If $\mathfrak{H}$ is finite, every $\mathfrak{M}$ is so, by Lemma 7.1.1, (iii). If $\mathfrak{M} \gneq \mathfrak{H}$, then $\mathfrak{M} \supsetneq \mathfrak{H}$, and $\mathfrak{H} \sim \mathfrak{M}$ would imply that $\mathfrak{H}$ is infinite. Thus if $\mathfrak{H}$ is finite, case (b) holds.

If $\mathfrak{H}$ is infinite, then $\mathfrak{M} \sim \mathfrak{H}$ implies that $\mathfrak{M}$ too is infinite by Lemma 7.1.1, (i). If conversely $\mathfrak{M}$ is infinite, Lemma 7.2.1 gives if applied to $\mathfrak{H}$, $\mathfrak{M}$ (for $\mathfrak{M}$, $\mathfrak{N}$) $\mathfrak{H} \precsim \mathfrak{M}$. By Lemma 6.3.2, (i), $\mathfrak{M} \precsim \mathfrak{H}$, so $\mathfrak{M} \sim \mathfrak{H}$. And as we have always $\mathfrak{M} \precsim \mathfrak{H}$ (cf. as above), therefore $\mathfrak{M} \prec \mathfrak{H}$ is characteristic for finite $\mathfrak{M}$'s. Thus case (a) holds.

A convenient characterisation of infinity is this:

Lemma 7.2.3. *An* $\mathfrak{M}\ \eta\ \mathbf{M}$ *is infinite if and only if* $\mathfrak{M} \gneq (0)$ *and an* $\mathfrak{N} \subset \mathfrak{M}$ *with* $\mathfrak{M} \sim \mathfrak{N} \sim \mathfrak{M} - \mathfrak{N}$ *exists.*

Proof: The condition is sufficient: If $\mathfrak{N}$ is such, then

$$\mathfrak{M} - \mathfrak{N} \sim \mathfrak{M} \neq (0), \mathfrak{M} - \mathfrak{N} \neq (0), \mathfrak{N} \neq \mathfrak{M},$$

so $\mathfrak{M} \sim \mathfrak{N} \subsetneq \mathfrak{M}$.

It is necessary: If $\mathfrak{M}$ is infinite, then the proof of Lemma 7.2.1 shows that an infinite sequence $\mathfrak{P}_1, \mathfrak{P}_2, \cdots$ exists such that all $\mathfrak{P}_i$ are mutually orthogonal, $\neq (0)$, $\mathfrak{M} = [\mathfrak{P}_1, \mathfrak{P}_2, \cdots]$, $\mathfrak{P}_1 \sim \mathfrak{P}_2 \sim \mathfrak{P}_3 \sim \cdots$. Now Lemma 6.1.2 gives $[\mathfrak{P}_1, \mathfrak{P}_2, \cdots] \sim [\mathfrak{P}_1, \mathfrak{P}_3, \cdots] \sim [\mathfrak{P}_2, \mathfrak{P}_4, \cdots]$ so for $\mathfrak{N} = [\mathfrak{P}_1, \mathfrak{P}_3, \cdots]$, $\mathfrak{N} \subset \mathfrak{M}$, $\mathfrak{M} \sim \mathfrak{N} \sim \mathfrak{M} - \mathfrak{N}$. As $\mathfrak{P}_1 \neq (0)$, necessarily $\mathfrak{M} \neq (0)$.

7.3. Continuing our discussion along the lines which are customary in general set-theory, it is natural to establish the additivity of the notion of finiteness. We will prove it in five successive steps, the first Lemma having a certain interest of its own.

Lemma 7.3.1. *Assume* $\mathfrak{M}, \mathfrak{N}, \mathfrak{P} \eta \mathbf{M}$; $\mathfrak{M}, \mathfrak{N}$ *orthogonal;* $\mathfrak{P} \subset [\mathfrak{M}, \mathfrak{N}]$. *Then* $\mathfrak{P}$ *can be represented in the following form (all sets which follow being linear and closed):*

There exist two orthogonal sets $\mathfrak{M}', \mathfrak{M}'' \eta \mathbf{M}$ *and* $\subset \mathfrak{M}$; *two orthogonal sets* $\mathfrak{N}', \mathfrak{N}'' \eta \mathbf{M}$ *and* $\subset \mathfrak{N}$; *and a linear, closed operator* $A \eta \mathbf{M}$, *with the following properties;* Domain A *is part of and dense in* $\mathfrak{M}''$; Range A *is part of and dense in* $\mathfrak{N}''$; $Af = 0$ *implies* $f = 0$. *The set* $\mathfrak{P}^*$ *of all* $f + Af$ ($f \epsilon$ *Domain* A) *is linear, closed, and* $\eta \mathbf{M}$.

$$\mathfrak{P}^* \sim \mathfrak{M}'' \sim \mathfrak{N}''$$

$\mathfrak{M}', \mathfrak{N}', \mathfrak{P}^*$ *are mutually orthogonal and*

$$\mathfrak{P} = [\mathfrak{M}', \mathfrak{N}', \mathfrak{P}^*].$$

Proof: Put $\mathfrak{M}' = \mathfrak{M} \cdot \mathfrak{P}$, $\mathfrak{N}' = \mathfrak{N} \cdot \mathfrak{P}$, then $\mathfrak{M}', \mathfrak{N}' \eta \mathbf{M}$, $\mathfrak{M}' \subset \mathfrak{M}$, $\mathfrak{N}' \subset \mathfrak{N}$, thus $\mathfrak{M}', \mathfrak{N}'$ are orthogonal; and $\mathfrak{M}', \mathfrak{N}' \subset \mathfrak{P}$. Thus we can form $\mathfrak{P}^* = \mathfrak{P} - [\mathfrak{M}', \mathfrak{N}']$, then $\mathfrak{P}^* \eta \mathbf{M}$; $\mathfrak{P}^*$ is orthogonal to both $\mathfrak{M}', \mathfrak{N}'$ and $\mathfrak{P} = [\mathfrak{M}', \mathfrak{N}', \mathfrak{P}^*]$. Thus we must only give now the characterisation of $\mathfrak{P}^*$ given in our Lemma in terms of $\mathfrak{M}'', \mathfrak{N}''$, and A.

An $f \epsilon \mathfrak{P}^*$ is $f \epsilon [\mathfrak{M}, \mathfrak{N}]$, thus $f = g + h, g \epsilon \mathfrak{M}, h \epsilon \mathfrak{N}$ (cf. (16), p. 76), g, h being clearly uniquely determined. If in such a decomposition $g = 0$, then $f = h \epsilon \mathfrak{N}$, $f \epsilon \mathfrak{P}^* \subset \mathfrak{P}$ so $f \epsilon \mathfrak{N} \cdot \mathfrak{P} = \mathfrak{N}' \subset \mathfrak{P} - \mathfrak{P}^*$. As $f \epsilon \mathfrak{P}^*$ this implies $f = 0, h = 0$. Similarly $h = 0$ implies $g = 0$. Owing to the linearity this means, that g determines h uniquely and conversely (as far as they belong to any $f \epsilon \mathfrak{P}^*$ at all). Thus we can define a one-valued operator A with a one-valued inverse A^{-1} by $Ag = h$, whenever g, h belong to an $f \epsilon \mathfrak{P}^*$ in the above way. Thus $\mathfrak{P}^*$ is the set of all $f + Af$. As it is $\eta \mathbf{M}$, linear, and closed, the same is true for A.

Put now $\mathfrak{M}'' =$ [Domain A], $\mathfrak{N}'' =$ [Range A]. All we must prove is, that $\mathfrak{M}''$ is $\subset \mathfrak{M}$ and orthogonal to $\mathfrak{M}'$; that $\mathfrak{N}'' \subset \mathfrak{N}$ and orthogonal to $\mathfrak{N}'$; and that $\mathfrak{P}^* \sim \mathfrak{N}'' \sim \mathfrak{M}''$.

If $g \epsilon$ Domain A, then $f = g + h, f \epsilon \mathfrak{P}^*, g \epsilon \mathfrak{M}, h \epsilon \mathfrak{N}$. So $g \epsilon \mathfrak{M}$. f is orthogonal to $\mathfrak{M}'$, and $h \epsilon \mathfrak{N}$ is orthogonal to $\mathfrak{M}$, so to $\mathfrak{M}'$ too. Thus $g = f - h$ is

orthogonal to $\mathfrak{M}'$, $g \in \mathfrak{M} - \mathfrak{M}'$. Thus Domain $A \subset \mathfrak{M} - \mathfrak{M}'$, $\mathfrak{M}'' = [\text{Domain } A] \subset \mathfrak{M} - \mathfrak{M}'$. Similarly $\mathfrak{N}'' \subset \mathfrak{N} - \mathfrak{N}'$.

Consider now the operator $A_1 = (1 + A)P_{\mathfrak{M}''}$ (the linear operator, which agrees with $1 + A$ in $\mathfrak{M}''$ and with 0 in $\mathfrak{H} - \mathfrak{M}''$). A_1 is clearly η **M**, linear, and closed, and its domain is everywhere dense. Now [Range A_1] = [Range $(1 + A)$] = $\mathfrak{P}^*$, [Range A_1] = $\mathfrak{P}^*$. On the other hand [Range A_1^*] = $\mathfrak{H} - (f, A_1 f = 0) = \mathfrak{H} - (f, (1 + A)\mathfrak{P}_{\mathfrak{M}''} f = 0) = \mathfrak{H} - (f, \mathfrak{P}_{\mathfrak{M}''} f = 0)$. $\mathfrak{H} - (\mathfrak{H} - \mathfrak{M}'') = \mathfrak{M}''$. Now Lemma 6.2.1 gives, applied to A_1, $\mathfrak{P}^* \sim \mathfrak{M}''$. Similarly $\mathfrak{P}^* \sim \mathfrak{N}''$.

Thus all parts of our Lemma are proved.

Lemma 7.3.2. *Assume* $\mathfrak{M}$, $\mathfrak{N}$, $\mathfrak{P}$ η **M**; $\mathfrak{M}$, $\mathfrak{N}$ *orthogonal*; $\mathfrak{P} \subset [\mathfrak{M}, \mathfrak{N}]$. *Then either* $\mathfrak{P} \precsim \mathfrak{M}$, *or* $[\mathfrak{M}, \mathfrak{N}] - \mathfrak{P} \precsim \mathfrak{N}$.

Proof: Apply Lemma 7.3.1 to $\mathfrak{M}$, $\mathfrak{N}$, $\mathfrak{P}$ obtaining the sets $\mathfrak{M}'$, $\mathfrak{M}''$, $\mathfrak{N}'$, $\mathfrak{N}''$, $\mathfrak{P}^*$ and the operator A; and to $\mathfrak{M}$, $\mathfrak{N}$, $[\mathfrak{M}, \mathfrak{N}] - \mathfrak{P}$ obtaining the sets $\overline{\mathfrak{M}}'$, $\overline{\mathfrak{M}}''$, $\overline{\mathfrak{N}}'$, $\overline{\mathfrak{N}}''$, $\overline{\mathfrak{P}}^*$ and the operator B. As $\mathfrak{P}$, $[\mathfrak{M}, \mathfrak{N}] - \mathfrak{P}$ are orthogonal, $\mathfrak{M}' = \mathfrak{M} \cdot \mathfrak{P}$, $\overline{\mathfrak{M}}' = \mathfrak{M} \cdot ([\mathfrak{M}, \mathfrak{N}] - \mathfrak{P})$ are too; similarly $\mathfrak{N}'$, $\overline{\mathfrak{N}}'$. Form $\mathfrak{M}^0 = \mathfrak{M} - [\mathfrak{M}', \overline{\mathfrak{M}}']$, $\mathfrak{N}^0 = \mathfrak{N} - [\mathfrak{N}', \overline{\mathfrak{N}}']$.

If $f \in$ Domain A then $f + Af \in \mathfrak{P}^* \subset \mathfrak{P}$ is orthogonal to $\overline{\mathfrak{M}}' \subset [\mathfrak{M}, \mathfrak{N}] - \mathfrak{P}$. $Af \in \mathfrak{N}$ is orthogonal too to $\overline{\mathfrak{M}}' \subset \mathfrak{M}$. Thus f is orthogonal to $\overline{\mathfrak{M}}'$. Thus all $\mathfrak{M}'' = [\text{Domain } A]$ is orthogonal to $\overline{\mathfrak{M}}'$. Besides we know that it is orthogonal to $\mathfrak{M}'$, and $\subset \mathfrak{M}$; so $\mathfrak{M}'' \subset \mathfrak{M} - [\mathfrak{M}', \overline{\mathfrak{M}}'] = \mathfrak{M}^0$. Similarly $\overline{\mathfrak{M}}'' \subset \mathfrak{M}^0$. Similarly $\mathfrak{N}''$, $\overline{\mathfrak{N}}'' \subset \mathfrak{N}^0$.

Now consider $\mathfrak{N}'$, $\overline{\mathfrak{M}}'$. We have $\mathfrak{N}' \overset{\succsim}{\precsim} \overline{\mathfrak{M}}'$ (by Theorem VI), that is either $\mathfrak{N}' \precsim \overline{\mathfrak{M}}'$, $\mathfrak{N}' \sim \mathfrak{M}^* \subset \overline{\mathfrak{M}}'$ or $\mathfrak{N}' \succsim \overline{\mathfrak{M}}'$, $\overline{\mathfrak{M}}' \sim \overline{\mathfrak{N}}^* \subset \mathfrak{N}'$. In the first case (by Lemma 6.1.2) $\mathfrak{P} = [\mathfrak{M}', \mathfrak{N}', \mathfrak{P}^*] \sim [\mathfrak{M}', \mathfrak{M}^*, \mathfrak{M}''] \subset [\mathfrak{M}', \overline{\mathfrak{M}}', \mathfrak{M}^0] = \mathfrak{M}$; in the second case (as above)

$$[\mathfrak{M}', \mathfrak{N}] - \mathfrak{P} = [\overline{\mathfrak{M}}', \overline{\mathfrak{N}}', \overline{\mathfrak{P}}^*] \sim [\overline{\mathfrak{N}}^*, \overline{\mathfrak{N}}', \mathfrak{N}''] \subset [\mathfrak{N}', \overline{\mathfrak{N}}', \mathfrak{N}^0] = \mathfrak{N}.$$

Thus $\mathfrak{P} \precsim \mathfrak{M}$ resp. $[\mathfrak{M}, \mathfrak{N}] - \mathfrak{P} \precsim \mathfrak{N}$.

Lemma 7.3.3. *Assume* $\mathfrak{M}$, $\mathfrak{N}$ η **M**; $\mathfrak{M}$, $\mathfrak{N}$ *orthogonal. If* $\mathfrak{M}$, $\mathfrak{N}$ *are finite, then* $[\mathfrak{M}, \mathfrak{N}]$ *is finite too.*

Proof: Assume that $[\mathfrak{M}, \mathfrak{N}]$ is infinite. Then there exists by Lemma 7.2.3 a $\mathfrak{P} \subset [\mathfrak{M}, \mathfrak{N}]$ such that $[\mathfrak{M}, \mathfrak{N}] \sim \mathfrak{P} \sim [\mathfrak{M}, \mathfrak{N}] - \mathfrak{P}$. By Lemma 7.3.2 we have $\mathfrak{P} \precsim \mathfrak{M}$ or $[\mathfrak{M}, \mathfrak{N}] - \mathfrak{P} \precsim \mathfrak{N}$, so $\mathfrak{M}$ or $\mathfrak{N} \succsim [\mathfrak{M}, \mathfrak{N}]$. By Lemma 7.1.1, (i), then $\mathfrak{M}$ or $\mathfrak{N}$ must be infinite, contradicting our assumption.

Lemma 7.3.4. *If* $\mathfrak{M}$, $\mathfrak{N}$ η **M**, *then* $[\mathfrak{M}, \mathfrak{N}] - \mathfrak{N} \precsim \mathfrak{M}$.

Proof: $[\mathfrak{M}, \mathfrak{N}] - \mathfrak{N}$ is obviously the set of all $P_{\mathfrak{H}-\mathfrak{N}} f$ where f runs over $[\mathfrak{M}, \mathfrak{N}]$. So it is equal to $[P_{\mathfrak{H}-\mathfrak{N}} f; f \in \{\mathfrak{M}, \mathfrak{N}\}] = [P_{\mathfrak{H}-\mathfrak{N}}(g + h), g \in \mathfrak{M}, h \in \mathfrak{N}] = [P_{\mathfrak{H}-\mathfrak{N}} g; g \in \mathfrak{M}, h \in \mathfrak{N}] = [P_{\mathfrak{H}-\mathfrak{N}} g; g \in \mathfrak{M}] = [P_{\mathfrak{H}-\mathfrak{N}} P_{\mathfrak{M}} g', g' \in \mathfrak{H}] = [\text{Range } P_{\mathfrak{H}-\mathfrak{N}} \cdot P_{\mathfrak{M}}]$. At the same time [Range $(P_{\mathfrak{H}-\mathfrak{N}} P_{\mathfrak{M}})^*$] = [Range $P_{\mathfrak{M}} P_{\mathfrak{H}-\mathfrak{N}}$] $\subset$ [Range $P_{\mathfrak{M}}$] = $\mathfrak{M}$. Now applying Lemma 6.2.1 to $P_{\mathfrak{H}-\mathfrak{N}} P_{\mathfrak{M}}$ gives:

$$[\mathfrak{M}, \mathfrak{N}] - \mathfrak{N} = [\text{Range } P_{\mathfrak{H}-\mathfrak{N}} \cdot P_{\mathfrak{M}}] \sim [\text{Range } (P_{\mathfrak{H}-\mathfrak{N}} \cdot P_{\mathfrak{M}})^*] \subset \mathfrak{M}$$

and therefore $[\mathfrak{M}, \mathfrak{N}] - \mathfrak{N} \precsim \mathfrak{M}$.

Lemma 7.3.5. *If* $\mathfrak{M}_1, \cdots, \mathfrak{M}_n \eta \mathbf{M}$, *and all* $\mathfrak{M}_i$ *are finite, then* $[\mathfrak{M}_1, \cdots, \mathfrak{M}_n]$ *is finite too.*

Proof: The statement is obvious for $n = 1$. Consider now $n = 2$. By Lemma 7.3.4 and Lemma 7.1.1, (i), $[\mathfrak{M}_1, \mathfrak{M}_2] - \mathfrak{M}_2$ is finite. As $[\mathfrak{M}_1, \mathfrak{M}_2] - \mathfrak{M}_2$ and $\mathfrak{M}_2$ are orthogonal, Lemma 7.3.3 applies: $[[\mathfrak{M}_1, \mathfrak{M}_2] - \mathfrak{M}_2, \mathfrak{M}_2]$, that is $[\mathfrak{M}_1, \mathfrak{M}_2]$, is finite. Thus the statement holds for $n = 2$, too.

Assume now $n = 3, 4, \cdots$, and that the statement is already established for $n - 1$. Then $[\mathfrak{M}_1, \cdots, \mathfrak{M}_{n-1}]$ is finite, and as our statement holds for $n = 2$, $[[\mathfrak{M}_1, \cdots, \mathfrak{M}_{n-1}], \mathfrak{M}_n]$, that is $[\mathfrak{M}_1, \cdots, \mathfrak{M}_n]$ is finite too. This completes the proof.

Chapter VIII: Numerical Dimensionality

8.1. We will use the information obtained in the foregoing about the relative dimensionality to define a quantitative measurement for it. We will do this by refining the number $\left[\frac{\mathfrak{N}}{\mathfrak{M}}\right]$ in Lemma 7.1.3, which gave an integral and therefore only approximate measure of the ratio of the sets.

Our task is identical with that one which has to be met if one desires to pass from the purely geometrical calculus with intervals to the analytical one with numerical lengths: the problem which was solved by Euclid's famous algorithm. Our Lemma 7.1.3 is indeed the first step in this algorithm.

It is, however, technically preferable to use a somewhat different procedure. We will have to distinguish for a short time between the two possibilities that there is or is not a minimal $\mathfrak{M} \eta \mathbf{M}$. The first case is not really interesting, because if a minimal $\mathfrak{M} \eta \mathbf{M}$ exists, we possess already the complete characterization of $\mathbf{M}$ (as a direct factor) by Theorem IV. But as it makes no difference for the possibility of a quantitative measurement of the relative dimensionality, we will preserve the completeness of our discussion by including this case.

Definition 8.1.1. Assume $\mathfrak{M}, \mathfrak{N} \eta \mathbf{M}$, finite, and $\gtrdot (0)$. If we have

$$\mathfrak{N} = [\mathfrak{P}_1, \cdots, \mathfrak{P}_n, \mathfrak{Q}],$$

where $n = 0, 1, 2, \cdots$ (finite!); $\mathfrak{P}_1, \cdots, \mathfrak{P}_n, \mathfrak{Q}$ are mutually orthogonal; $\mathfrak{P}_i \sim \mathfrak{M}$; then we use the abbreviated notation $\mathfrak{N} \approx n \odot \mathfrak{M} \ast \mathfrak{Q}$. If $\mathfrak{Q} = (0)$, we omit it.

Lemma 8.1.1. *If* $\mathfrak{M} \eta \mathbf{M}$, *finite,* $\gtrdot (0)$, *and not minimal, then an* $\mathfrak{N} \eta \mathbf{M}$, *finite,* $\gtrdot (0)$, *with* $\left[\frac{\mathfrak{M}}{\mathfrak{N}}\right] \geqq 2$ *exists.*

Proof: As $\mathfrak{M}$ is not minimal, an $\mathfrak{N} \eta \mathbf{M}$, $\mathfrak{N} \subset \mathfrak{M}$, $\mathfrak{N} \gtrdot (0)$, $\mathfrak{M}$ exists. So $\mathfrak{N}, \mathfrak{M} - \mathfrak{N} \gtrdot (0)$. We have $\mathfrak{N} \gtrless \mathfrak{M} - \mathfrak{N}$, that is $\mathfrak{N} \gtrsim \mathfrak{M} - \mathfrak{N}$ or $\mathfrak{N} \lesssim \mathfrak{M} - \mathfrak{N}$. In the second case replace $\mathfrak{N}$ by $\mathfrak{M} - \mathfrak{N}$, so we have at any rate $\mathfrak{N} \lesssim \mathfrak{M} - \mathfrak{N}$.

$\mathfrak{N} \subset \mathfrak{M}$ implies the finiteness of $\mathfrak{N}$. As $\mathfrak{N} \sim \mathfrak{N}' \subset \mathfrak{M} - \mathfrak{N}$, and $\mathfrak{N}, \mathfrak{N}'$ are orthogonal and $\subset \mathfrak{M}$, we have with $\mathfrak{P}_1 = \mathfrak{N}$, $\mathfrak{P}_2 = \mathfrak{N}'$, $\mathfrak{Q} = \mathfrak{M} - [\mathfrak{N}, \mathfrak{N}']$, obviously $\mathfrak{M} \approx 2 \odot \mathfrak{N} \ast \mathfrak{Q}$. Apply now Lemma 7.1.3 to $\mathfrak{N}$, $\mathfrak{Q}$; $\mathfrak{Q} \approx p \odot \mathfrak{N} \ast \mathfrak{Q}'$, $p = 0, 1, 2, \cdots$, $\mathfrak{Q}' \prec \mathfrak{N}$. Then

$$\mathfrak{M} \approx (2 + p) \odot \mathfrak{N} \not\approx \mathfrak{Q}', \mathfrak{Q}' \prec \mathfrak{N}$$

and so $\left[\frac{\mathfrak{M}}{\mathfrak{N}}\right] = 2 + p \geqq 2$.

Lemma 8.1.2. *Every minimal* $\mathfrak{M}\ \eta\ \mathbf{M}$ *is finite and* $\succneqq (0)$.

Proof: $\mathfrak{M} \succneqq (0)$ is part of the definition. $\mathfrak{M}$ infinite would imply $\mathfrak{M} \sim \mathfrak{N} \subsetneqq \mathfrak{M}$, so $\mathfrak{N}\ \eta\ \mathbf{M}$, and as $\mathfrak{M}$ is minimal, $\mathfrak{N} = (0)$. Then $\mathfrak{M} \sim (0)$, $\mathfrak{M} = (0)$ which is contradictory.

Definition 8.1.2. Certain (finite or infinite) sequences $\mathfrak{S} = (\overline{\mathfrak{M}}_1, \overline{\mathfrak{M}}_2, \cdots)$ of elements which are $\eta\mathbf{M}$, finite, $\succneqq (0)$, will be called *fundamental sequences.* They are these:

(α) Any minimal $\overline{\mathfrak{M}}_1\ \eta\ \mathbf{M}$ is a fundamental sequence by itself (length one!).

(β) Any infinite sequence $\overline{\mathfrak{M}}_1, \overline{\mathfrak{M}}_2, \cdots$ of $\eta\ \mathbf{M}$, finite, $\succneqq (0)$ elements is a fundamental sequence if $\left[\frac{\mathfrak{M}_i}{\mathfrak{M}_{i+1}}\right] \geqq 2$ for $i = 1, 2, \cdots$.

Lemma 8.1.3. *If sets* $\mathfrak{M}\ \eta\ \mathbf{M}$ *which are finite and* $\succneqq (0)$ *do exist at all, then there exists at least one fundamental sequence.*

Proof: If a minimal $\mathfrak{M}\ \eta\ \mathbf{M}$ exists, put $\overline{\mathfrak{M}}_1 = \mathfrak{M}$. By Lemma 8.1.2 this meets the requirements of case (α). If no minimal $\mathfrak{M}\ \eta\ \mathbf{M}$ exists, choose a finite $\mathfrak{M}\ \eta\ \mathbf{M}$, $\mathfrak{M} \succneqq (0)$, and put $\overline{\mathfrak{M}}_1 = \mathfrak{M}$; and obtain $\overline{\mathfrak{M}}_{i+1}$ from $\overline{\mathfrak{M}}_i$ by means of Lemma 8.1.1. This meets the requirements of case (β).

We now prove two important evaluations.

Lemma 8.1.4. *Assume* $\mathfrak{M}, \mathfrak{N}, \mathfrak{P}\ \eta\ \mathbf{M}$ *finite, and* $\succneqq (0)$. *Then*

$$\left[\frac{\mathfrak{N}}{\mathfrak{M}}\right]\left[\frac{\mathfrak{P}}{\mathfrak{N}}\right] \leqq \left[\frac{\mathfrak{P}}{\mathfrak{M}}\right] < \left(\left[\frac{\mathfrak{N}}{\mathfrak{M}}\right] + 1\right)\left(\left[\frac{\mathfrak{P}}{\mathfrak{N}}\right] + 1\right)$$

and, if $\mathfrak{N}, \mathfrak{P}$ *are orthogonal,*

$$\left[\frac{\mathfrak{N}}{\mathfrak{M}}\right] + \left[\frac{\mathfrak{P}}{\mathfrak{M}}\right] \leqq \left[\frac{[\mathfrak{N}, \mathfrak{P}]}{\mathfrak{M}}\right] \leqq \left[\frac{\mathfrak{N}}{\mathfrak{M}}\right] + \left[\frac{\mathfrak{P}}{\mathfrak{M}}\right] + 1\,.$$

Proof: First put $\left[\frac{\mathfrak{N}}{\mathfrak{M}}\right] = m$, $\mathfrak{N} \approx m \odot \mathfrak{M} \not\approx \mathfrak{M}'$, $\mathfrak{M}' \prec \mathfrak{M}$ and $\left[\frac{\mathfrak{P}}{\mathfrak{N}}\right] = n$, $\mathfrak{P} \approx n \odot \mathfrak{N} \not\approx \mathfrak{N}'$, $\mathfrak{N}' \prec \mathfrak{N}$. This clearly implies $\mathfrak{P} \approx m\,n \odot \mathfrak{M} \not\approx \mathfrak{M}''$, from which $\left[\frac{\mathfrak{P}}{\mathfrak{M}}\right] \geqq m\,n$ follows as at the end of the proof of Lemma 8.1.1. If we had however $\left[\frac{\mathfrak{P}}{\mathfrak{M}}\right] \geqq (m + 1)\ (n + 1)$ that would mean $\mathfrak{P} \approx \left[\frac{\mathfrak{P}}{\mathfrak{M}}\right] \odot \mathfrak{M} \not\approx \mathfrak{M}''' \approx (m + 1)\ (n + 1) \odot \mathfrak{M} \not\approx \mathfrak{M}^{\mathrm{IV}}$ where

$$\mathfrak{M}^{\mathrm{IV}} \approx \left(\left[\frac{\mathfrak{P}}{\mathfrak{M}}\right] - (m + 1)\ (n + 1)\right) \odot \mathfrak{M} \not\approx \mathfrak{M}'''\,.$$

Or $(n + 1) \odot \mathfrak{N}^\circ \approx \mathfrak{P}^\circ \subset \mathfrak{P}$ where $\mathfrak{N}^\circ \approx (m + 1) \odot \mathfrak{M}$. Now clearly

$$\mathfrak{N} \approx m \odot \mathfrak{M} \not\approx \mathfrak{M}' \precnsim \mathfrak{N}^\circ\,,$$

so we may as well write $(n+1) \odot \mathfrak{N} \approx \mathfrak{P}^{\circ\circ} \subset \mathfrak{P}^{\circ} \subset \mathfrak{P}$, that is

$$\mathfrak{P} \approx (n+1) \odot \mathfrak{N} \ast \mathfrak{N}'' .$$

This gives, as above, $\left[\frac{\mathfrak{P}}{\mathfrak{N}}\right] \geqq n+1$, contradicting $\left[\frac{\mathfrak{P}}{\mathfrak{N}}\right] = n$. So we must have $mn \leqq \left[\frac{\mathfrak{P}}{\mathfrak{M}}\right] < (m+1)(n+1)$ which is the first set of inequalities.

Next assume that $\mathfrak{N}$, $\mathfrak{P}$ are orthogonal, and put

$$\left[\frac{\mathfrak{N}}{\mathfrak{M}}\right] = m,\ \mathfrak{M} \approx m \odot \mathfrak{M} \ast \mathfrak{M}',\ \mathfrak{M}' \prec \mathfrak{M} ;$$

and

$$\left[\frac{\mathfrak{P}}{\mathfrak{M}}\right] = p,\ \mathfrak{P} \approx p \odot \mathfrak{M} \ast \mathfrak{M}^{\mathrm{V}},\ \mathfrak{M}^{\mathrm{V}} \prec \mathfrak{M} .$$

Then $[\mathfrak{N}, \mathfrak{P}] \approx (m+p) \odot \mathfrak{M} \ast [\mathfrak{M}', \mathfrak{M}^{\mathrm{V}}]$ and so, as above, $\left[\frac{[\mathfrak{N}, \mathfrak{P}]}{\mathfrak{M}}\right] \geqq m+p$. If we had, however, $\left[\frac{[\mathfrak{N}, \mathfrak{P}]}{\mathfrak{M}}\right] \geqq m+p+2$, that would mean

$$[\mathfrak{N}, \mathfrak{P}] \approx \left[\frac{[\mathfrak{N}, \mathfrak{P}]}{\mathfrak{M}}\right] \odot \mathfrak{M} \ast \mathfrak{M}^{\mathrm{VI}} \approx (m+p+2) \odot \mathfrak{M} \ast \mathfrak{M}^{\mathrm{VII}}$$

where

$$\mathfrak{M}^{\mathrm{VII}} = \left(\left[\frac{[\mathfrak{N}, \mathfrak{P}]}{\mathfrak{M}}\right] - (m+p+2)\right) \odot \mathfrak{M} \ast \mathfrak{M}^{\mathrm{VI}} .$$

Thus $[\mathfrak{N}, \mathfrak{P}] = [\overline{\mathfrak{N}}, \overline{\mathfrak{P}}, \mathfrak{M}^{\mathrm{VII}}]$ where $\overline{\mathfrak{N}}$, $\overline{\mathfrak{P}}$, $\mathfrak{M}^{\mathrm{VII}}$ are mutually orthogonal, and $\overline{\mathfrak{N}} \approx (m+1) \odot \mathfrak{M}$, $\overline{\mathfrak{P}} \approx (n+1) \odot \mathfrak{N}$. This implies $\mathfrak{N} \sim \overline{\mathfrak{N}}^{\circ} \underset{\precsim}{\subset} \overline{\mathfrak{N}}$, $\mathfrak{P} \sim \overline{\mathfrak{P}}^{\circ} \underset{\precsim}{\subset} \overline{\mathfrak{P}}$, and so $[\mathfrak{N}, \mathfrak{P}] \sim [\overline{\mathfrak{N}}^{\circ}, \overline{\mathfrak{P}}^{\circ}] \underset{\precsim}{\subset} [\overline{\mathfrak{N}}, \overline{\mathfrak{P}}] \subset [\overline{\mathfrak{N}}, \overline{\mathfrak{P}}, \mathfrak{M}^{\mathrm{VII}}] = [\mathfrak{N}, \mathfrak{P}]$ contradicting the finiteness of $[\mathfrak{N}, \mathfrak{P}]$, which follows from the finiteness of $\mathfrak{N}$ and of $\mathfrak{P}$ (by Lemma 7.3.3 or 7.3.5). So $\left[\frac{[\mathfrak{N}, \mathfrak{P}]}{\mathfrak{M}}\right] < m+p+2$ and therefore $m+p \leqq \left[\frac{[\mathfrak{N}, \mathfrak{P}]}{\mathfrak{M}}\right] \leqq m+p+1$, which is the second set of inequalities.

These evaluations put us in position to establish the quantitative ratios of relative dimensionalities as follows.

Lemma 8.1.5. *If* $\mathfrak{S} = (\overline{\mathfrak{M}}_1, \overline{\mathfrak{M}}_2, \cdots)$ *is a fundamental sequence, and* $\mathfrak{M}$, $\mathfrak{N}$, $\eta\,\mathbf{M}$ *are finite and* $\succnsim (0)$, *then*

$$\lim_i \frac{\left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_i}\right]}{\left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_i}\right]} = \left(\frac{\mathfrak{N}}{\mathfrak{M}}\right)_{\mathfrak{S}}$$

exists. (*In case* (I) *we mean by* lim *the value at* $i = 1$, *in case* (II) *we mean by it* $\lim_{i\to\infty}$.) *It is a finite and positive real number.*

Proof: In case (I) we must only prove that $\left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_1}\right]$, $\left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_1}\right]$ are both $\neq 0$. $\left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_1}\right] = 0$ would imply $\mathfrak{N} \approx \mathfrak{Q} \prec \overline{\mathfrak{M}}_1$ so $\mathfrak{N} \sim \mathfrak{N}'' \subset \overline{\mathfrak{M}}_1$ excluding $\mathfrak{N}' = \overline{\mathfrak{M}}_1$. As $\mathfrak{N}' \eta \mathbf{M}$ and $\overline{\mathfrak{M}}_1$ minimal, this necessitates $\mathfrak{N}' = (0)$, $\mathfrak{N} \sim \mathfrak{N}' = (0)$, $\mathfrak{N} = (0)$ thus contradicting $\mathfrak{N} \neq (0)$. Therefore $\left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_1}\right] \neq 0$, similarly $\left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_1}\right] \neq 0$.

Consider now case (II). We have by Lemma 8.1.4 $\left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_{i+1}}\right] \geqq \left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_i}\right] \times \left[\frac{\overline{\mathfrak{M}}_i}{\overline{\mathfrak{M}}_{i+1}}\right] \geqq 2\left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_i}\right]$, so either always $\left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_i}\right] = 0$, or $\lim_{i\to\infty}\left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_i}\right] = \infty$. The former would mean $\mathfrak{N} < \overline{\mathfrak{M}}_i$ for all $i = 1, 2, \cdots$. Now Lemma 8.1.4 gives

$$\left[\frac{\overline{\mathfrak{M}}_i}{\overline{\mathfrak{M}}_{i+j}}\right] \geqq \left[\frac{\overline{\mathfrak{M}}_i}{\overline{\mathfrak{M}}_{i+1}}\right] \cdots \left[\frac{\overline{\mathfrak{M}}_{i+j-1}}{\overline{\mathfrak{M}}_{i+j}}\right] \geqq 2^j,$$

and so in particular $\left[\frac{\overline{\mathfrak{M}}_1}{\overline{\mathfrak{M}}_i}\right] \geqq 2^{i-1}$ (write $1, i-1$ for i, j). So $\overline{\mathfrak{M}}_1 \approx 2^{i-1}\overline{\mathfrak{M}}_i \not\approx \mathfrak{M}_i'$, and *a fortiori* $\overline{\mathfrak{M}}_1 \approx 2^{i-1} \odot \mathfrak{N} \not\approx \mathfrak{M}_i''$. This would imply, as we frequently concluded in the proofs of Lemmas 8.1.1 and 8.1.4, $\left[\frac{\overline{\mathfrak{M}}_1}{\mathfrak{N}}\right] \geqq 2^{i-1}$ for all $i = 1, 2, \cdots$, which is impossible. So we have proved $\lim_{i\to\infty}\left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_i}\right] = \infty$ similarly $\lim_{i\to\infty}\left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_i}\right] = \infty$.

Another application of Lemma 8.1.4 gives

$$\left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_{i+j}}\right] \leqq \left(\left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_i}\right] + 1\right)\left(\left[\frac{\overline{\mathfrak{M}}_i}{\overline{\mathfrak{M}}_{i+j}}\right] + 1\right)$$

$$\left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_{i+j}}\right] \geqq \left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_i}\right] \cdot \left[\frac{\overline{\mathfrak{M}}_i}{\overline{\mathfrak{M}}_{i+j}}\right],$$

so

$$\frac{\left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_{i+j}}\right]}{\left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_{i+j}}\right]} \leqq \frac{\left(\left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_i}\right] + 1\right)\left[\frac{\overline{\mathfrak{M}}_i}{\overline{\mathfrak{M}}_{i+j}}\right] + 1}{\left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_i}\right]} \leqq \frac{\left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_i}\right] + 1}{\left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_i}\right]} \cdot \frac{2^j + 1}{2^j}.$$

Now $j \to \infty$ gives

$$\limsup_{j\to\infty} \frac{\left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_{i+j}}\right]}{\left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_{i+j}}\right]} \leqq \frac{\left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_i}\right] + 1}{\left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_i}\right]},$$

that is

$$\limsup_{i\to\infty} \frac{\left[\frac{\mathfrak{N}}{\mathfrak{M}_i}\right]}{\left[\frac{\mathfrak{M}}{\mathfrak{M}_i}\right]} \leqq \frac{\left[\frac{\mathfrak{N}}{\mathfrak{M}_i}\right]+1}{\left[\frac{\mathfrak{M}}{\mathfrak{M}_i}\right]}.$$

For a sufficiently large i $\left[\frac{\mathfrak{M}}{\mathfrak{M}_i}\right] \neq 0$, so the expression on the right side is finite, and thus the $\limsup_{i\to\infty}$ on the left side is finite too. Now $i \to \infty$ gives, considering $\lim_{i\to\infty} \left[\frac{\mathfrak{M}}{\mathfrak{M}_i}\right] = \infty$,

$$\limsup_{i\to\infty} \frac{\left[\frac{\mathfrak{N}}{\mathfrak{M}_i}\right]}{\left[\frac{\mathfrak{M}}{\mathfrak{M}_i}\right]} \leqq \liminf_{i\to\infty} \frac{\left[\frac{\mathfrak{N}}{\mathfrak{M}_i}\right]+1}{\left[\frac{\mathfrak{M}}{\mathfrak{M}_i}\right]} = \liminf_{i\to\infty} \frac{\left[\frac{\mathfrak{N}}{\mathfrak{M}_i}\right]}{\left[\frac{\mathfrak{M}}{\mathfrak{M}_i}\right]}.$$

Thus $\lim_{i\to\infty} \frac{\left[\frac{\mathfrak{N}}{\mathfrak{M}_i}\right]}{\left[\frac{\mathfrak{M}}{\mathfrak{M}_i}\right]}$ exists and is finite. It is non negative by its nature, and interchanging $\mathfrak{M}$, $\mathfrak{N}$ proves that it cannot be **0**.

This completes the proof.

Lemma 8.1.6. *If* $\mathfrak{S} = (\mathfrak{M}_1, \mathfrak{M}_2, \cdots)$ *is a fundamental sequence, and* $\mathfrak{M}$, $\mathfrak{N}$, $\mathfrak{P}$ η **M** *are finite and* $\neq (0)$, *then the following statements hold*:

(i) *If* $\mathfrak{N} \sim \mathfrak{P}$, *then* $\left(\frac{\mathfrak{N}}{\mathfrak{M}}\right)_{\mathfrak{S}} = \left(\frac{\mathfrak{P}}{\mathfrak{M}}\right)_{\mathfrak{S}}$.

(ii) *If* $\mathfrak{M} \sim \mathfrak{N}$, *then* $\left(\frac{\mathfrak{P}}{\mathfrak{M}}\right)_{\mathfrak{S}} = \left(\frac{\mathfrak{P}}{\mathfrak{N}}\right)_{\mathfrak{S}}$.

(iii) $\left(\frac{\mathfrak{M}}{\mathfrak{M}}\right)_{\mathfrak{S}} = 1$; $\left(\frac{\mathfrak{M}}{\mathfrak{N}}\right)_{\mathfrak{S}} = \frac{1}{\left(\frac{\mathfrak{N}}{\mathfrak{M}}\right)_{\mathfrak{S}}}$; $\left(\frac{\mathfrak{P}}{\mathfrak{M}}\right)_{\mathfrak{S}} = \left(\frac{\mathfrak{N}}{\mathfrak{M}}\right)_{\mathfrak{S}} \cdot \left(\frac{\mathfrak{P}}{\mathfrak{N}}\right)_{\mathfrak{S}}$.

(iv) *If* $\mathfrak{N}$, $\mathfrak{P}$ *are orthogonal, then*

$$\left(\frac{[\mathfrak{N}, \mathfrak{P}]}{\mathfrak{M}}\right)_{\mathfrak{S}} = \left(\frac{\mathfrak{N}}{\mathfrak{M}}\right)_{\mathfrak{S}} + \left(\frac{\mathfrak{P}}{\mathfrak{M}}\right)_{\mathfrak{S}}.$$

Proof: We prove these statements in a somewhat changed order:
Ad (iii): Obvious from the definition.

Ad (i): Clearly $\left[\frac{\mathfrak{N}}{\mathfrak{M}_i}\right] = \left[\frac{\mathfrak{P}}{\mathfrak{M}_i}\right]$, this implies $\left(\frac{\mathfrak{N}}{\mathfrak{M}}\right)_{\mathfrak{S}} = \left(\frac{\mathfrak{P}}{\mathfrak{M}}\right)_{\mathfrak{S}}$.

Ad (ii): Follows from (i), (iii).

Ad (iv): Consider first case (α). Then $\mathfrak{N} \approx \left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_1}\right] \odot \mathfrak{M}_1 \not\approx \mathfrak{M}'$, $\mathfrak{M}' \prec \overline{\mathfrak{M}}_1$. So $\mathfrak{M}' \sim \mathfrak{M}'' \subset \overline{\mathfrak{M}}_1$, excluding $\mathfrak{M}'' = \overline{\mathfrak{M}}_1$. As $\mathfrak{M}'' \eta \mathbf{M}$ and $\overline{\mathfrak{M}}_1$ minimal, this necessitates $\mathfrak{M}'' = (0)$, $\mathfrak{M}' \sim \mathfrak{M}''' = (0)$, $\mathfrak{M}' = (0)$. So $\mathfrak{N} \approx \left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_1}\right] \odot \overline{\mathfrak{M}}_1$, similarly $\mathfrak{P} \approx \left[\frac{\mathfrak{P}}{\overline{\mathfrak{M}}_1}\right] \odot \overline{\mathfrak{M}}_1$, and therefore $[\mathfrak{N}, \mathfrak{P}] \approx \left(\left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_1}\right] + \left[\frac{\mathfrak{P}}{\overline{\mathfrak{M}}_1}\right]\right) \odot \overline{\mathfrak{M}}_1$. Thus $\left[\frac{[\mathfrak{N}, \mathfrak{P}]}{\overline{\mathfrak{M}}_1}\right] = \left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_1}\right] + \left[\frac{\mathfrak{P}}{\overline{\mathfrak{M}}_1}\right]$.

This means $\dfrac{\left[\frac{[\mathfrak{N}, \mathfrak{P}]}{\overline{\mathfrak{M}}_1}\right]}{\left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_1}\right]} = \dfrac{\left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_1}\right]}{\left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_1}\right]} + \dfrac{\left[\frac{\mathfrak{P}}{\overline{\mathfrak{M}}_1}\right]}{\left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_1}\right]}$, and under the conditions of case (α),

$$\left(\frac{[\mathfrak{N}, \mathfrak{P}]}{\mathfrak{M}}\right)_{\mathfrak{S}} = \left(\frac{\mathfrak{N}}{\mathfrak{M}}\right)_{\mathfrak{S}} + \left(\frac{\mathfrak{P}}{\mathfrak{M}}\right)_{\mathfrak{S}}.$$

Consider next case (β). Then Lemma 8.1.4 gives

$$\frac{\left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_i}\right] + \left[\frac{\mathfrak{P}}{\overline{\mathfrak{M}}_i}\right]}{\left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_i}\right]} \leqq \frac{\left[\frac{[\mathfrak{N}, \mathfrak{P}]}{\overline{\mathfrak{M}}_i}\right]}{\left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_i}\right]} \leqq \frac{\left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_i}\right] + \left[\frac{\mathfrak{P}}{\overline{\mathfrak{M}}_i}\right] + 1}{\left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_i}\right]}.$$

Now if $i \to \infty$ this becomes, considering $\lim_{i \to \infty} \left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_i}\right] = \infty$ (cf. the proof of Lemma 8.1.5),

$$\left(\frac{\mathfrak{N}}{\mathfrak{M}}\right)_{\mathfrak{S}} + \left(\frac{\mathfrak{P}}{\mathfrak{M}}\right)_{\mathfrak{S}} \leqq \left(\frac{[\mathfrak{N}, \mathfrak{P}]}{\mathfrak{M}}\right)_{\mathfrak{S}} \leqq \left(\frac{\mathfrak{N}}{\mathfrak{M}}\right)_{\mathfrak{S}} + \left(\frac{\mathfrak{P}}{\mathfrak{M}}\right)_{\mathfrak{S}},$$

that is

$$\left(\frac{[\mathfrak{N}, \mathfrak{P}]}{\mathfrak{M}}\right)_{\mathfrak{S}} = \left(\frac{\mathfrak{N}}{\mathfrak{M}}\right)_{\mathfrak{S}} + \left(\frac{\mathfrak{P}}{\mathfrak{M}}\right)_{\mathfrak{S}}.$$

8.2. We introduce the numerical relative dimensions by an implicit definition.

Definition 8.2.1. A real-valued function $D(\mathfrak{M})$, which is defined for all linear, closed sets $\mathfrak{M} \eta \mathbf{M}$, is a *relative dimension function,* if it has the following properties:

(i) $D(\mathfrak{M})$ is 0 if $\mathfrak{M} = (0)$; it is finite and positive if $\mathfrak{M}$ is finite and $\neq (0)$; it is ∞ if $\mathfrak{M}$ is infinite.

(ii) If $\mathfrak{M} \sim \mathfrak{N}$, then $D(\mathfrak{M}) = D(\mathfrak{N})$.

(iii) If $\mathfrak{M}$, $\mathfrak{N}$ are orthogonal, then $D([\mathfrak{M}, \mathfrak{N}]) = D(\mathfrak{M}) + D(\mathfrak{N})$. (As to replacing $\mathfrak{M}$ by $E = P_{\mathfrak{M}}$, and the use of the explanatory symbol $(\cdots \mathbf{M})$, cf. Definition 6.1.1. If we want to make the relationship between $\mathbf{M}$ and the function $D(\mathfrak{M})$ explicit, we will write $D_{\mathbf{M}}(\mathfrak{M})$.)

The results of §8.1 make it possible to deal exhaustively with the existence question.

Lemma 8.2.1. *There always exists at least one relative dimension function* $D(\mathfrak{M})$.

Proof: If no finite $\mathfrak{M} \eta \mathbf{M}$, $\mathfrak{M} \precneqq (0)$, exists, put $D(\mathfrak{M}) \begin{cases} = 0 \text{ for } \mathfrak{M} = (0) \\ = \infty \text{ otherwise} \end{cases}$. This meets all requirements. If finite $\mathfrak{M} \eta \mathbf{M}$, $\precneqq (0)$ do exist, choose one, say $\mathfrak{M}_0$. Furthermore choose a fundamental sequence $\mathfrak{S}$ by Lemma 8.1.3. Now define

$$D(\mathfrak{M}) \begin{cases} = 0 \text{ for } \mathfrak{M} = (0), \\ = \left(\frac{\mathfrak{M}}{\mathfrak{M}_0}\right)_{\mathfrak{S}} \text{ for } \mathfrak{M} \text{ finite and } \precneqq (0), \\ = \infty \text{ for } \mathfrak{M} \text{ infinite}. \end{cases}$$

Remember that $[\mathfrak{M}, \mathfrak{N}]$ is infinite if and only if $\mathfrak{M}$ or $\mathfrak{N}$ is infinite, by Lemmas 7.1.1 (i), and 7.3.3 or 7.3.5. Now (i) is obviously fulfilled, and (ii), (iii) follow from Lemma 8.1.6, (i), (iv), resp.

Lemma 8.2.2. *If* $D(\mathfrak{M})$ *is a relative dimension function and* $\mathfrak{S}$ *a fundamental sequence, then we have, whenever* $\mathfrak{M}$, $\mathfrak{N} \eta \mathbf{M}$, *finite and* $\precneqq (0)$ *the relation*

$$\frac{D(\mathfrak{N})}{D(\mathfrak{M})} = \left(\frac{\mathfrak{N}}{\mathfrak{M}}\right)_{\mathfrak{S}}.$$

Proof: Distinguish the two cases (α) and (β) of Definition 8.1.2.

In the case (α) we have (cf. the proof of Lemma 8.1.6, (iv),

$$\mathfrak{N} \approx \left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_1}\right] \odot \overline{\mathfrak{M}}_1, \qquad \mathfrak{M} \approx \left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_1}\right] \odot \overline{\mathfrak{M}}_1,$$

therefore by Definition 8.2.1 $D(\mathfrak{N}) = \left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_1}\right] D(\overline{\mathfrak{M}}_1)$, $D(\mathfrak{M}) = \left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_1}\right] D(\overline{\mathfrak{M}}_1)$ and $D(\mathfrak{M}_1)$ finite and positive, so

$$\frac{D(\mathfrak{N})}{D(\mathfrak{M})} = \frac{\left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_1}\right]}{\left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_1}\right]} = \left(\frac{\mathfrak{N}}{\mathfrak{M}}\right)_{\mathfrak{S}}.$$

In the case (β) we have

$$\mathfrak{N} \approx \left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_1}\right] \odot \overline{\mathfrak{M}}_i + \mathfrak{M}_i', \mathfrak{M}_i' \prec \overline{\mathfrak{M}}_i; \quad \mathfrak{M} \approx \left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_i}\right] \odot \overline{\mathfrak{M}}_i + \mathfrak{M}_i'', \mathfrak{M}_i'' \prec \overline{\mathfrak{M}}_i.$$

So $\mathfrak{M}_i' \sim \mathfrak{M}_i''' \subset \overline{\mathfrak{M}}_i$, implying (using Definition 8.2.1 here and below as we did above)

$$D(\overline{\mathfrak{M}}_i) = D(\mathfrak{M}_i''') + D(\overline{\mathfrak{M}}_i - \mathfrak{M}_i''') \geqq D(\mathfrak{M}_i''') = D(\mathfrak{M}_i').$$

Therefore

$$D(\mathfrak{N}) = \left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_i}\right] D(\overline{\mathfrak{M}}_i) + D(\mathfrak{M}_i') \leqq \left(\left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_i}\right] + 1\right) \cdot D(\overline{\mathfrak{M}}_i).$$

$$D(\mathfrak{M}) = \left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_i}\right] D(\overline{\mathfrak{M}}_i) + D(\mathfrak{M}_i'') \geqq \left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_i}\right] \cdot D(\overline{\mathfrak{M}}_i),$$

and so

$$\frac{D(\mathfrak{N})}{D(\mathfrak{M})} \leqq \frac{\left[\frac{\mathfrak{N}}{\overline{\overline{\mathfrak{M}_i}}}\right] + 1}{\left[\frac{\mathfrak{M}}{\overline{\overline{\mathfrak{M}_i}}}\right]}.$$

Now $i \to \infty$ gives, remembering $\lim_{i\to\infty}\left[\frac{\mathfrak{N}}{\overline{\overline{\mathfrak{M}_i}}}\right] = \infty$ from the proof of Lemma 8.1.5,

$$\frac{D(\mathfrak{N})}{D(\mathfrak{M})} \leqq \lim_{i\to\infty} \frac{\left[\frac{\mathfrak{N}}{\overline{\overline{\mathfrak{M}_i}}}\right]}{\left[\frac{\mathfrak{M}}{\overline{\overline{\mathfrak{M}_i}}}\right]} = \left(\frac{\mathfrak{N}}{\mathfrak{M}}\right)_{\mathfrak{S}}.$$

Interchanging $\mathfrak{M}$, $\mathfrak{N}$ gives, considering Lemma 8.1.6, (iii), that $\geqq$ holds too; so we have equality again.

Lemma 8.2.3. *All relative dimension functions $D(\mathfrak{M})$ are obtained by taking one of them, say $D_0(\mathfrak{M})$, and forming all $aD_0(\mathfrak{M})$, for all finite and positive constants a.*

Proof: It is clear, that every $aD_0(\mathfrak{M})$ is a relative dimension function, along with $D_0(\mathfrak{M})$.

Consider now two relative dimension functions $D_0(\mathfrak{M})$ and $D(\mathfrak{M})$. If no finite $\mathfrak{M} \eta \mathbf{M}$, $\mathfrak{M} \gtrneqq (0)$ exist, we have $D_0(\mathfrak{M}) = D(\mathfrak{M}) = 0$ resp. ∞ for $\mathfrak{M} = (0)$ resp. $\gtrneqq (0)$, so we have $D(\mathfrak{M}) = aD_0(\mathfrak{M})$ with $a = 1$. Let us therefore assume, that finite $\mathfrak{M} \eta \mathbf{M}$, $\mathfrak{M} \gtrneqq (0)$ do exist, and choose a fundamental sequence $\mathfrak{S}$ by Lemma 8.1.3. Then Lemma 8.2.2 gives for $\mathfrak{M}$, $\mathfrak{N} \eta \mathbf{M}$, finite and $\gtrneqq 0$,

$$\frac{D(\mathfrak{N})}{D(\mathfrak{M})} = \frac{D_0(\mathfrak{N})}{D_0(\mathfrak{M})} = \left(\frac{\mathfrak{N}}{\mathfrak{M}}\right)_{\mathfrak{S}},$$

that is $\frac{D(\mathfrak{M})}{D_0(\mathfrak{M})} = \frac{D(\mathfrak{N})}{D_0(\mathfrak{N})}$. Thus $\frac{D(\mathfrak{M})}{D_0(\mathfrak{M})}$ is, for these $\mathfrak{M}$, independent of $\mathfrak{M}$; therefore it is a finite, positive constant a. So we have $D(\mathfrak{M}) = a\, D_0(\mathfrak{M})$ for every $\mathfrak{M} \eta \mathbf{M}$ which is finite and $\gtrneqq (0)$. But this holds too, if $\mathfrak{M}$ is $= (0)$ or infinite, as then both sides become $= 0$ resp. $= \infty$. So the desired relation holds without exceptions.

This completes the proof.

We may remark, on the other hand, that Lemma 8.2.2 shows too, that $\left(\frac{\mathfrak{M}}{\mathfrak{N}}\right)_{\mathfrak{S}}$ is independent of the choice of the fundamental sequence $\mathfrak{S}$. It is nevertheless dependent on its existence, whereas $D(\mathfrak{M})$ is not.

8.3. The connection between the ordering $\gtrsim$ and the relative dimension functions is given by this Lemma.

Lemma 8.3.1. *If $D(\mathfrak{M})$ is a relative dimension function and $\mathfrak{M}$, $\mathfrak{N} \eta \mathbf{M}$, then $\mathfrak{M} \gtrsim \mathfrak{N}$ are equivalent to $D(\mathfrak{M}) \geqq D(\mathfrak{N})$ resp.*

Proof: $\mathfrak{M} \sim \mathfrak{N}$ implies $D(\mathfrak{M}) = D(\mathfrak{N})$ by Definition 8.2.1, (ii). Consider now $\mathfrak{M} \prec \mathfrak{N}$. Then $\mathfrak{M}$ is finite, because $\mathfrak{M}$ infinite would imply $\mathfrak{M} \succsim \mathfrak{N}$; so $D(\mathfrak{M})$ is finite too. Now $\mathfrak{M} \sim \mathfrak{N}' \subset \mathfrak{N}$, and $\mathfrak{N}' = \mathfrak{N}$ is excluded. So $\mathfrak{N} - \mathfrak{N}' \neq (0)$, $D(\mathfrak{N} - \mathfrak{N}') > 0$. (Use both times Definition 8.2.1, (i).) Now Definition 8.2.1, (ii) and (iii) give $D(\mathfrak{N}) = D(\mathfrak{N}') + D(\mathfrak{N} - \mathfrak{N}') > D(\mathfrak{N}') = D(\mathfrak{M})$, $D(\mathfrak{M}) < D(\mathfrak{N})$.

Interchanging $\mathfrak{M}$, $\mathfrak{N}$ shows that $\mathfrak{M} \succ \mathfrak{N}$ implies $D(\mathfrak{M}) > D(\mathfrak{N})$. So $\mathfrak{M} \gtrless \mathfrak{N}$ imply $D(\mathfrak{M}) \gtrless D(\mathfrak{N})$ resp. As we have complete disjunctions on both sides, the converse is true too.

Summing up Lemmas 8.2.1, 8.2.3, and 8.3.1, we have:

Theorem VII. *A relative dimension function $D(\mathfrak{M})$ as characterized by Definition* 8.2.1, *does always exist, and it is unique, except for an arbitrary constant (real, finite, and positive) factor a. A particular choice of $D(\mathfrak{M})$ that is of a, is a normalization of the relative dimension.*

*If $\mathfrak{M}$, $\mathfrak{N}$ η **M** then $\mathfrak{M} \gtrless \mathfrak{N}$ is equivalent to $D(\mathfrak{M}) \gtrless D(\mathfrak{N})$ resp.*

We will use $D(\mathfrak{M})$ to obtain a detailed characterization of **M** itself. But first we establish some continuity properties of $D(\mathfrak{M})$.

Lemma 8.3.2. *Let $\mathfrak{M}_1, \mathfrak{M}_2, \cdots$ be a (finite or infinite) sequence, all $\mathfrak{M}_i$ η **M** and mutually orthogonal. Then*

$$D([\mathfrak{M}_1, \mathfrak{M}_2, \cdots]) = \sum_i D(\mathfrak{M}_i) .$$

Proof: Those $\mathfrak{M}_i$ which are $= (0)$ have no effect on either side of this equation, and we can omit them. So we may assume that all $\mathfrak{M}_i \neq (0)$, $D(\mathfrak{M}_i) > 0$.

Assume first that the sequence is finite, say $\mathfrak{M}_1, \cdots, \mathfrak{M}_n$. For $n = 0, 1$ the statement is obvious, for $n = 2$ it coincides with Definition 8.2.1, (iii). So we need only to consider $n = 3, 4, \cdots$, assuming that the statement is already proved for $n - 1$. Now as $[\mathfrak{M}_1, \cdots, \mathfrak{M}_n] = [[\mathfrak{M}_1, \cdots, \mathfrak{M}_{n-1}], \mathfrak{M}_n]$ and as the statement holds for $n - 1$ and for two addends, it holds for n too.

Assume next that the sequence is infinite. Then we have

$$[\mathfrak{M}_1, \mathfrak{M}_2, \cdots] = [\mathfrak{M}_1, \cdots, \mathfrak{M}_n, [\mathfrak{M}_{n+1}, \mathfrak{M}_{n+2}, \cdots]]$$

so

$$D([\mathfrak{M}_1, \mathfrak{M}_2, \cdots]) = \sum_1^n{}_i D(\mathfrak{M}_i) + D([\mathfrak{M}_{n+1}, \mathfrak{M}_{n+2}, \cdots]) \geqq \sum_1^n{}_i D(\mathfrak{M}_i) .$$

Thus $\sum_{i=1}^n D(\mathfrak{M}_i) \leqq D([\mathfrak{M}_1, \mathfrak{M}_2, \cdots]$ for every $n = 1, 2, \cdots$, and therefore $\sum_{i=1}^\infty D(\mathfrak{M}_i) \leqq D([\mathfrak{M}_1, \mathfrak{M}_2, \cdots])$.

Assume now, that $<$ does hold. Then $\sum_{i=1}^\infty D(\mathfrak{M}_i)$ is finite. Therefore $\lim_{i\to\infty} D(\mathfrak{M}_i) = 0$. So there exists for every $\epsilon > 0$ a finite $\mathfrak{N}$ η **M** (in fact an $\mathfrak{M}_i$) with $0 < D(\mathfrak{N}) < \epsilon$. Choose such an $\mathfrak{N}$ for

$$\epsilon = D([\mathfrak{M}_1, \mathfrak{M}_2, \cdots]) - \sum_1^\infty D(\mathfrak{M}_i) > 0 .$$

Consider the expression $D([\mathfrak{M}_1, \mathfrak{M}_2, \cdots]) - \sum_{i=1}^{\infty} D(\mathfrak{M}_i)$. It is equal to $(D(\mathfrak{M}_1) + D([\mathfrak{M}_2, \mathfrak{M}_3, \cdots])) - \sum_1^{\infty} D(\mathfrak{M}_i) = D([\mathfrak{M}_2, \mathfrak{M}_3, \cdots]) - \sum_{i=2}^{\infty} D(\mathfrak{M}_i)$. Thus it is unchanged if we replace $\mathfrak{M}_1, \mathfrak{M}_2, \cdots$ by $\mathfrak{M}_2, \mathfrak{M}_3, \cdots$. Applying this i_0 times, we see, that it is unchanged even if we pass to $\mathfrak{M}_{i_0+1}, \mathfrak{M}_{i_0+2}, \cdots$. Now $\sum_{i=1}^{\infty} D(\mathfrak{M}_{i_0+i}) = \sum_{i_0+1}^{\infty} D(\mathfrak{M}_i)$ is arbitrarily small if i_0 is sufficiently great. So we can make it $\leqq D(\mathfrak{N})$. Now write again $\mathfrak{M}_1, \mathfrak{M}_2, \cdots$ for $\mathfrak{M}_{i_0+1}, \mathfrak{M}_{i_0+2}, \cdots$. Then we see: $\sum_{i=1}^{\infty} D(\mathfrak{M}_i) \leqq D(\mathfrak{N})$ and still $D(\mathfrak{N}) < D([\mathfrak{M}_1, \mathfrak{M}_2, \cdots] - \sum_{i=1}^{\infty} D(\mathfrak{M}_i)$ and *a fortiori* $D(\mathfrak{N}) < D([\mathfrak{M}_1, \mathfrak{M}_2, \cdots])$.

We will now construct a sequence $\mathfrak{N}_1, \mathfrak{N}_2, \cdots$ of mutually orthogonal sets $\mathfrak{N}_i \eta \mathbf{M}$, $\mathfrak{N}_i \subset \mathfrak{N}$ with $\mathfrak{M}_i \sim \mathfrak{N}_i$. We proceed by induction. Assume that for an $i = 1, 2, \cdots$ all $\mathfrak{N}_1, \cdots, \mathfrak{N}_{i-1}$ have already been constructed fulfilling these conditions, then we construct $\mathfrak{N}_i$ as follows. $D(\mathfrak{N}_j) = D(\mathfrak{M}_j)$, $j = 1, \cdots, i-1$, is finite, so $D(\mathfrak{N} - [\mathfrak{N}_1, \cdots, \mathfrak{N}_{i-1}]) = D(\mathfrak{N}) - \sum_{j=1}^{i-1} D(\mathfrak{N}_j) \geqq \sum_1^{\infty} D(\mathfrak{M}_j) - \sum_{j=1}^{i-1} D(\mathfrak{M}_j) = \sum_{j=i}^{\infty} D(\mathfrak{M}_j) \geqq D(\mathfrak{M}_i)$, $\mathfrak{M}_i \lesssim \mathfrak{N} - [\mathfrak{N}_1, \cdots, \mathfrak{N}_{i-1}]$, and thus $\mathfrak{M}_i \sim \mathfrak{N}_i \subset \mathfrak{N} - [\mathfrak{N}_1, \cdots, \mathfrak{N}_{i-1}]$. This definition of $\mathfrak{N}_i$ meets all requirements.

Now $\mathfrak{M}_i \sim \mathfrak{N}_i$, all $\mathfrak{M}_i$ are mutually orthogonal, and all $\mathfrak{N}_i$ are too, so Lemma 6.1.2 applies: $[\mathfrak{M}_1, \mathfrak{M}_2, \cdots] \sim [\mathfrak{N}_1, \mathfrak{N}_2, \cdots] \subset \mathfrak{N}$, $[\mathfrak{M}_1, \mathfrak{M}_2, \cdots] \lesssim \mathfrak{N}$, $D([\mathfrak{M}_1, \mathfrak{M}_2, \cdots]) \leqq D(\mathfrak{N})$. But on the other hand we had $D(\mathfrak{N}) < D([\mathfrak{M}_1, \mathfrak{M}_2, \cdots])$ which is a contradiction. So our original assumption concerning the $<$ must have been wrong, and there must be for the original sequence

$$\sum_1^{\infty i} D(\mathfrak{M}_i) = D([\mathfrak{M}_1, \mathfrak{M}_2, \cdots]) .$$

This completes the proof.

Lemma 8.3.3. *Let* $\mathfrak{M}_1, \mathfrak{M}_2, \cdots$ *be an infinite sequence, all* $\mathfrak{M}_i \eta \mathbf{M}$ *and* $\mathfrak{M}_1 \subset \mathfrak{M}_2 \subset \cdots$. *Then*

$$\lim_{i \to \infty} D(\mathfrak{M}_i) = D([\mathfrak{M}_1, \mathfrak{M}_2, \cdots]) .$$

Proof: Put $\mathfrak{N}_1 = \mathfrak{M}_1$, $\mathfrak{N}_i = \mathfrak{M}_i - \mathfrak{M}_{i-1}$ for $i = 2, 3, \cdots$. Then all $\mathfrak{N}_i \eta \mathbf{M}$ and they are mutually orthogonal. So Lemma 8.3.2 applies to them and gives

$$\sum_1^{\infty i} D(\mathfrak{N}_i) = D([\mathfrak{N}_1, \mathfrak{N}_2, \cdots]) .$$

Now $[\mathfrak{N}_1, \cdots, \mathfrak{N}_i] = \mathfrak{M}_i$ and so $[\mathfrak{N}_1, \mathfrak{N}_2, \cdots] = [\mathfrak{M}_1, \mathfrak{M}_2, \cdots]$ and furthermore

$$\sum_1^{\infty i} D(\mathfrak{N}_i) = \lim_{n \to \infty} \sum_1^{n i} D(\mathfrak{N}_i) = \lim_{n \to \infty} D([\mathfrak{N}_1, \cdots, \mathfrak{N}_n]) = \lim_{n \to \infty} D(\mathfrak{M}_n) .$$

Therefore we have $\lim_{i \to \infty} D(\mathfrak{M}_i) = D([\mathfrak{M}_1, \mathfrak{M}_2, \cdots])$.

Lemma 8.3.4. *Let* $\mathfrak{M}_1, \mathfrak{M}_2, \cdots$ *be an infinite sequence, all* $\mathfrak{M}_i \eta \mathbf{M}$ *and* $\mathfrak{M}_1 \supset \mathfrak{M}_2 \supset \cdots$. *If not all* $\mathfrak{M}_i$ *are infinite, then*

$$\lim_{i \to \infty} D(\mathfrak{M}_i) = D(\mathfrak{M}_1 \cdot \mathfrak{M}_2 \cdot \mathfrak{M}_3 \cdots) .$$

Proof: Assume that $\mathfrak{M}_{i_0}$ is finite. Put $\mathfrak{M}_i' = \mathfrak{M}_{i_0} - \mathfrak{M}_{i_0+i}$. Then all $\mathfrak{M}_i' \eta \mathbf{M}$ and $\mathfrak{M}_1' \subset \mathfrak{M}_2' \subset \cdots$ so Lemma 8.3.3 applies to them and gives $\lim_{i\to\infty} D(\mathfrak{M}_i') = D([\mathfrak{M}_1', \mathfrak{M}_2', \cdots])$. But $D(\mathfrak{M}_i') = D(\mathfrak{M}_{i_0} - \mathfrak{M}_{i_0+i}) = D(\mathfrak{M}_{i_0}) - D(\mathfrak{M}_{i_0+i})$ and $[\mathfrak{M}_1', \mathfrak{M}_2', \cdots] = [\mathfrak{M}_{i_0} - \mathfrak{M}_{i_0+1}, \mathfrak{M}_{i_0} - \mathfrak{M}_{i_0+2}, \cdots] = \mathfrak{M}_{i_0} - (\mathfrak{M}_{i_0+i} \cdot \mathfrak{M}_{i_0+2} \cdots) = \mathfrak{M}_{i_0} - (\mathfrak{M}_1 \cdot \mathfrak{M}_2 \cdots)$. Therefore we have

$$D(\mathfrak{M}_{i_0}) - \lim_{i\to\infty} D(\mathfrak{M}_{i_0+i}) = D(\mathfrak{M}_{i_0}) - D(\mathfrak{M}_1 \cdot \mathfrak{M}_2 \cdots), \text{ so that}$$

$$\lim_{i\to\infty} D(\mathfrak{M}_{i_0+i}) = D(\mathfrak{M}_1 \cdot \mathfrak{M}_2 \cdots),$$

or, which is the same thing $\lim_{i\to\infty} D(\mathfrak{M}_i) = D(\mathfrak{M}_1 \cdot \mathfrak{M}_2 \cdots)$.

The following criterion is occasionally useful, because it contains a sufficient condition for relative dimension functions which makes no explicit use of the notions of finiteness and infiniteness for sets $\mathfrak{M} \eta \mathbf{M}$.

Lemma 8.3.5. *If a real-valued function $D'(\mathfrak{M})$ which is defined for all linear, closed sets $\mathfrak{M} \eta \mathbf{M}$ assumes only values $\geqq 0$, $\leqq \infty$ and if some of its values are actually $\neq 0$, ∞ then the conditions* (ii), (iii) *in Definition* 8.2.1 *are sufficient in order that $D'(\mathfrak{M})$ be a relative dimension function.*

Proof: We must derive (i) in Definition 8.2.1 from these conditions. Choose an $\overline{\mathfrak{M}} \eta \mathbf{M}$ with $D'(\overline{\mathfrak{M}}) > 0$, $< \infty$.

As (0) is orthogonal to $\overline{\mathfrak{M}}$ and $[\overline{\mathfrak{M}}, (0)] = \overline{\mathfrak{M}}$; (iii) gives $D'(\overline{\mathfrak{M}}) + D'(0) = D'(\overline{\mathfrak{M}})$, as $D'(\overline{\mathfrak{M}})$ is finite, this means $D'(0) = 0$.

If any $\mathfrak{M}$ is infinite, then $\mathfrak{H}$ is it, too, and $\mathfrak{M} \sim \mathfrak{H}$. Then by (ii) we must only prove $D'(\mathfrak{H}) = \infty$. By Lemma 7.2.3 applied to $\mathfrak{H}$ we have $2D'(\mathfrak{H}) = D'(\mathfrak{H})$, $D'(\mathfrak{H}) = 0$ or ∞. But $D'(\mathfrak{H}) = D'(\overline{\mathfrak{M}}) + D'(\mathfrak{H} - \overline{\mathfrak{M}}) \geqq D'(\overline{\mathfrak{M}}) > 0$ so $D'(\mathfrak{H}) = \infty$. This disposes of all infinite $\mathfrak{M}$'s.

If $\mathfrak{M}$ is finite and $\neq (0)$ then apply Lemma 7.1.3 to $\mathfrak{M}$, $\overline{\mathfrak{M}}$ and to $\overline{\mathfrak{M}}$, $\mathfrak{M}$ (for its $\mathfrak{M}$, $\mathfrak{N}$). Then (ii), (iii) give $D'(\mathfrak{M}) \leqq \left(\left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}}\right] + 1\right) D'(\overline{\mathfrak{M}})$. Thus $D'(\overline{\mathfrak{M}}) < \infty$ necessitates $D'(\mathfrak{M}) < \infty$. Next $D'(\overline{\mathfrak{M}}) \leqq \left(\left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}}\right] + 1\right) D'(\mathfrak{M})$. Thus $D'(\overline{\mathfrak{M}}) > 0$ necessitates $D'(\mathfrak{M}) > 0$. So $0 < D'(\mathfrak{M}) < \infty$, completing the proof.

8.4. We now undertake to find invariant characteristics of $\mathbf{M}$ in terms of its relative dimension functions $D(\mathfrak{M})$.

Lemma 8.4.1. *The range Δ of $D(\mathfrak{M})$ (that is the set of all values of a given $D(\mathfrak{M})$ for all $\mathfrak{M} \eta \mathbf{M}$) has these properties*:

(i) *The elements of Δ are real numbers, $\geqq 0$, $\leqq \infty$.*

(ii) $0 \in \Delta$, *Δ contains a greatest element $\bar{\alpha} > 0$.*

(iii) *$\alpha, \beta \in \Delta$ and $\alpha > \beta$ imply $\alpha - \beta \in \Delta$.*

(iv) *$\alpha_1, \alpha_2, \cdots \in \Delta$ and $\sum_{i=1}^{\infty} \alpha_i \leqq \bar{\alpha}$ imply $\sum_1^{\infty} \alpha_i \in \Delta$.*

Proof: Ad (i): Obvious.

Ad (ii): $D((0)) = 0$; $D(\mathfrak{H}) = \bar{\alpha}$, as $\mathfrak{H} \neq (0)$ therefore $D(\mathfrak{H}) > 0$.

Ad (iii): β must be finite. Choose $\mathfrak{M}$, $\mathfrak{N}\ \eta\ \mathbf{M}$ with $D(\mathfrak{M}) = \alpha$, $D(\mathfrak{N}) = \beta$. Then $\mathfrak{N} < \mathfrak{M}$, $\mathfrak{N} \sim \mathfrak{N}' \subset \mathfrak{M}$, $D(\mathfrak{N}') = D(\mathfrak{N}) = \beta$ finite and so $D(\mathfrak{M} - \mathfrak{N}') = D(\mathfrak{M}) - D(\mathfrak{N}') = \alpha - \beta$.

Ad (iv): If any $\alpha_j = \infty$ then $\sum_1^\infty \alpha_i = \infty$ so $\sum_{i=1}^\infty \alpha_i = \alpha_j\ \epsilon\ \Delta$ so we may assume that all α_j are finite. Define a sequence $\mathfrak{M}_1, \mathfrak{M}_2, \cdots$ of mutually orthogonal sets $\mathfrak{M}_i\ \eta\ \mathbf{M}$, $D(\mathfrak{M}_i) = \alpha_i$ by induction. Assume that for an $i = 1, 2, \cdots$ $\mathfrak{M}_1, \cdots, \mathfrak{M}_{i-1}$ have already been constructed fulfilling these conditions, then construct $\mathfrak{M}_i$ as follows. $D(\mathfrak{H} - [\mathfrak{M}_1, \cdots \mathfrak{M}_{i-1}]) = D(\mathfrak{H}) - \sum_1^{i-1} D(\mathfrak{M}_j) = \bar{\alpha} - \sum_{j=1}^{i-1} \alpha_j \geqq \sum_1^\infty \alpha_j - \sum_1^{i-1} \alpha_j = \sum_i^\infty \alpha_j \geqq \alpha_j$. Choose an $\mathfrak{M}_i'\ \eta\ \mathbf{M}$ with $D(\mathfrak{M}_i') = \alpha_i$ then $\mathfrak{M}_i' \lesssim \mathfrak{H} - [\mathfrak{M}_1, \cdots \mathfrak{M}_{i-1}]$, $\mathfrak{M}_i' \sim \mathfrak{M}_i \subset \mathfrak{H} - [\mathfrak{M}_1, \cdots, \mathfrak{M}_{i-1}]$. This definition of $\mathfrak{M}_i$ meets all requirements. Now by Lemma 8.3.2 $D([\mathfrak{M}_1, \mathfrak{M}_2, \cdots]) = \sum_{i=1}^\infty D(\mathfrak{M}_i) = \sum_{i=1}^\infty \alpha_i$.

Lemma 8.4.2. *The only sets* Δ *which fulfill the requirements* (i)–(iv) *of Lemma* 8.4.1 *are the following ones*: ($\bar{\epsilon}$, $\bar{\alpha}$ *are finite*).

(I_n) $\bar{\epsilon} > 0$, $n = 1, 2, \cdots$, Δ *consists of all* $i\,\bar{\epsilon}$, $i = 0, 1, \cdots, n$.

(I_∞) $\bar{\epsilon} > 0$, Δ *consists of all* $i\bar{\epsilon}$, $i = 0, 1, \cdots, \infty$.

(II_1) $\bar{\alpha} > 0$, Δ *consists of all* α, $0 \leqq \alpha \leqq \bar{\alpha}$.

(II_∞) , Δ *consists of all* α, $0 \leqq \alpha \leqq \infty$.

(III_∞) , Δ *consists of* 0, ∞.

Proof: Δ contains 0 by (ii) and some element >0 by (i), (ii). If it contains no element >0, $<\infty$ then it consists of 0, ∞ which is precisely case (III_∞).

Assume now that it does contain elements >0, $<\infty$.

Assume first, that this set contains a smallest element $\bar{\epsilon}$. If $\alpha\ \epsilon\ \Delta$ and finite, then there exists an $i = 0, 1, 2, \cdots$ with $i\,\bar{\epsilon} \leqq \alpha < (i+1)\,\bar{\epsilon}$. If $i\bar{\epsilon} \neq \alpha$ then by (iii) all α, $\alpha - \bar{\epsilon}, \cdots, \alpha - i\,\bar{\epsilon}$ belong to Δ. But $0 < \alpha - i\bar{\epsilon} < \bar{\epsilon}$, which is impossible. So $\alpha = i\bar{\epsilon}$. In particular $\bar{\alpha} = n\bar{\epsilon}$, $n = 0, 1, 2, \cdots$ if it is finite, and as $\bar{\alpha} > 0$, $n \neq 0$. If $\bar{\alpha}$ is infinite we may write, $\bar{\alpha} = \infty\,\bar{\epsilon}$ so at any rate $\bar{\alpha} = n\bar{\epsilon}$, $n = 1, 2, \cdots \infty$.

For all other $\alpha\ \epsilon\ \Delta$, $\alpha < \bar{\alpha}$ so α finite, $\alpha = i\epsilon$ and as $\alpha < \bar{\alpha}$, $i < n$, $i = 0, 1, 2, \cdots, n-1$. Conversely, if $i = 0, 1, \cdots, n-1$, then $i\bar{\epsilon} < n\bar{\epsilon}, \bar{\alpha}$ and (iv) with $\alpha_1 = \cdots = \alpha_i = \bar{\epsilon}$, $\alpha_{i+1} = \alpha_{i+2} = \cdots = 0$ gives $i\bar{\epsilon}\ \epsilon\ \Delta$. Thus Δ consists of all $i\bar{\epsilon}$, $i = 0, 1, \cdots n$ which is case (I_n) if $n = 1, 2, \cdots$, and case (I_∞) if $n = \infty$.

Assume next that there is no smallest element >0, $<\infty$ in Δ. Let ϵ' be the gr.l.b. of the elements $\epsilon \geqq 0$; if we had $\epsilon' > 0$ then there would exist an $\alpha\ \epsilon\ \Delta$, $\alpha > 0$ with $\alpha < 2\epsilon'$ and as $\alpha = \epsilon'$ would imply that α is the smallest element >0, $<\infty$ in Δ, therefore $\alpha > \epsilon'$. Now there exists a $\beta\ \epsilon\ \Delta$, $\beta > 0$ with $\beta < \alpha$ and $\beta > \epsilon'$. So $\alpha, \beta\ \epsilon\ \Delta$, $\alpha > \beta$ and by (iii) $\alpha - \beta\ \epsilon\ \Delta$, but $\alpha - \beta > 0$, $\alpha - \beta < 2\epsilon' - \bar{\epsilon}' = \epsilon'$ which is impossible. Thus $\epsilon' = 0$ and so for every $\epsilon > 0$ an $\alpha\ \epsilon\ \Delta$, $0 < \alpha < \epsilon$ with $0 < \alpha < \epsilon$ exists.

Now assume $0 \leqq \beta_1 < \beta_2 \leqq \bar{\alpha}$. Choose an $\alpha \in \Delta$ with $0 < \alpha < \beta_2 - \beta_1$. Then an $i = 0, 1, 2, \cdots$ with $i\alpha \leqq \beta_1 < (i+1)\alpha$ exists, so $(i+1)\alpha = i\alpha + \alpha < \beta_1 + \beta_2 - \beta_1 = \beta_2$. As $(i+1)\alpha < \bar{\alpha}$, therefore (iv) with $\alpha_1 = \cdots = \alpha_{i+1} = \alpha$, $\alpha_{i+2} = \alpha_{i+3} = \cdots = 0$ gives $(i+1)\alpha \in \Delta$. So we see: There exists a $\beta \in \Delta$ with $\beta_1 < \beta < \beta_2$.

Now consider any α with $0 < \alpha < \bar{\alpha}$ and form a sequence $\beta_1, \beta_2, \cdots$ with $0 < \beta_1 < \beta_2 < \cdots < \alpha$; $\lim_{i\to\infty} \beta_i = \alpha$. For each i choose a $\beta_i' \in \Delta$ with $\beta_i < \beta_i' < \beta_{i+1}$. So $\beta_i' > \beta_{i-1}'$, $\beta_i' - \beta_{i-1}' \in \Delta$, $i = 2, 3, \cdots$. Put $\alpha_1 = \beta_1'$, $\alpha_i = \beta_i' - \beta_{i-1}'$ for $i = 2, 3, \cdots$, and apply (iv): $\sum_1^\infty \alpha_i = \lim_{i\to\infty} \beta_i' = \alpha < \bar{\alpha}$ and so $\alpha \in \Delta$.

Thus Δ consists of all α with $0 \leqq \alpha \leqq \bar{\alpha}$ which is case (II_1) if $\bar{\alpha}$ is finite, and case (II_∞) if $\bar{\alpha} = \infty$.

These considerations exhaust all possibilities.

We now apply Lemmas 8.4.1 and 8.4.2 simultaneously, remembering that we can normalise $D(\mathfrak{M})$ so as to make $\bar{\epsilon} = 1$ (cases (I_n) and (I_∞)) resp. $\bar{\alpha} = 1$ (case (II_1)). Then we have:

Theorem VIII. The range of $D(\mathfrak{M})$ (that is the set of all values of a given $D(\mathfrak{M})$ for all $\mathfrak{M}$ η $\mathbf{M}$) is one of the following sets:

(I_n) $n = 1, 2, \cdots$	*The set* $0, 1, \cdots, n$.
(I_∞)	*The set* $0, 1, \cdots \infty$.
(II_1)	*The set of all* α, $0 \leqq \alpha \leqq 1$.
(II_∞)	*The set of all* α, $0 \leqq \alpha \leqq \infty$.
(III_∞)	*The set* $0, \infty$.

In the cases (I_n), (I_∞), (II_1) we have included a normalisation of $D(\mathfrak{M})$, the *standard normalisation*. Case (II_∞) is not normalised; in case (III_∞) no normalisation is needed, because $0, \infty$ are unaltered by multiplication with a finite positive number.

We call the cases (I_n), (II_1) the *finite* cases, the cases I_∞, II_∞, III_∞, the *infinite* cases. On the other hand we call the cases (I) (that is (I_n), (I_∞)) the *discrete* cases, the cases (II), (that is (II_1), (II_∞)) the *continuous* cases, and the case (III), (that is (III_∞)) the *purely infinite* case.

The following problems arise immediately:

Problem 3. *Which ones of the classes* (I_n)–(III_∞) *do really exist?*

Problem 4. *Which combinations of these classes do occur for coupled factors* $\mathbf{M}$, $\mathbf{M}'$? *For general factorisations* $\mathbf{M}_1, \cdots, \mathbf{M}_n$?

Problem 5. *To which classes do the direct factors* $\mathbf{M}$ *belong?*

Problem 6. *Are all factors* $\mathbf{M}$ *belonging to the same class ring-isomorphic, or do there exist further invariant characteristics?*

Problem 7. *Are all factorisations* $\mathbf{M}_1, \cdots, \mathbf{M}_n$ *in which each* $\mathbf{M}_i$ *belongs to a given class ring-isomorphic or even spatially isomorphic, or do there exist further invariant characteristics?*

We will be able to contribute considerable material to clarify these questions, but the only one to which we can give a complete answer is Problem 5.

8.5. We establish the invariance of the subjects of the last §§ under ring-isomorphisms.

Lemma 8.5.1. *Replace the* $\mathfrak{M}$, $\mathfrak{N} \eta \mathbf{M}$ *by their* $E = \mathfrak{P}_{\mathfrak{M}}$, $F = \mathfrak{P}_{\mathfrak{N}}$ *which are* $\epsilon \mathbf{M}$ (*Cf. Definitions* 6.1.1, 7.1.1, *and* 8.2.1). *The notions* $E \sim F(\cdots \mathbf{M})$, *finiteness and infiniteness of an* E *with respect to* $\mathbf{M}$, $D_{\mathbf{M}}(E)$ (*apart from its normalisation*) *are invariant under algebraic ring-isomorphisms.*

Proof: $E \sim F$ means that a $U \epsilon \mathbf{M}$ with $UU^*U = U$, $U^*U = E$, $UU^* = F$ exists. E is infinite, if an $F \epsilon \mathbf{M}$ with $F^2 = F^* = F$, $EF = F$, $F \precneqq E$, $F \sim E$ exists. $D(E)$ is characterised in Definition 8.2.1 by the sole use of the notions 0, finiteness, infiniteness, $E \sim F$ and $E + F$ with $EF = 0$. Thus all these constructions are invariant under algebraic ring-isomorphisms.

Now we can state, combining Lemma 8.5.1 and Theorem VIII:

Theorem IX. The cases (I_n)–(III_∞) *of Theorem VIII are invariant under algebraic ring-isomorphisms. So is* $D_{\mathbf{M}}(E)$ (*we write* $E = P_{\mathfrak{M}} \epsilon \mathbf{M}$ *for* $\mathfrak{M} \eta \mathbf{M}$), *apart from its normalisation, and even the standard normalisation* (*in the cases* (I_n)–(II_1)).

8.6. We now locate the direct factors in this classification.

Lemma 8.6.1. $\mathfrak{M}$ *is a direct factor if and only if it belongs to the discrete cases:* (I_n) *or* (I_∞). *If we then use the spatial isomorphism of* $\mathfrak{H}$ *with* $\mathfrak{H}_1 \otimes \cdots \otimes \mathfrak{H}_m$ *where* $\mathbf{M}$ *becomes* $\mathbf{B}_i^{(i)}$ *then the corresponding* $\mathfrak{H}_i$ *will be an* n*-dimensional Euclidean space resp. a Hilbert space. If* $E_i = P_{\mathfrak{M}_i}$ *is a projection in* $\mathfrak{H}_i$ *then* $D_{\mathbf{M}}(E_i^{(i)})$ *is, in the standard normalisation, simply the common notion of dimension of* $\mathfrak{M}_i$.

Proof: In the cases (I) an $\mathfrak{M} \eta \mathbf{M}$ with $D(\mathfrak{M}) = 1$ exists. If $\mathfrak{N} \eta \mathbf{M}$, $\mathfrak{N} \subset \mathfrak{M}$ then $D(\mathfrak{N}) \leqq D(\mathfrak{M})$, $D(\mathfrak{N}) = 0, 1$. The former implies $\mathfrak{N} = (0)$, the latter $\mathfrak{N} \sim \mathfrak{M}$ and as $\mathfrak{M}$ is finite, it excludes $\mathfrak{N} \subsetneqq \mathfrak{M}$, so it implies $\mathfrak{N} = \mathfrak{M}$. Obviously $\mathfrak{M} \precneqq (0)$. Thus $\mathfrak{M}$ is minimal, and therefore $\mathbf{M}$ is a direct factor (cf. Theorem IV). So our condition is sufficient.

All other statements assume that $\mathbf{M}$ is a direct factor, and thus algebraically (even fully) ring-isomorphic to $\mathbf{B}_i$. Then we may replace (by Theorem IX) $\mathfrak{H}$, $\mathbf{M}$, by $\mathfrak{H}_i$, $\mathbf{B}_i$ that is, we may assume $\mathbf{M} = \mathbf{B}$.

Every one-dimensional $\mathfrak{M} = [\varphi]$, $\varphi \precneqq 0$ (now all $\mathfrak{M} \eta \mathbf{M}$) is $\precneqq (0)$ and finite ($\mathfrak{N} \subsetneqq \mathfrak{M}$ implies $\mathfrak{N} = (0)$ excluding $\mathfrak{N} \sim \mathfrak{M}$) so $D_{\mathbf{B}}([\varphi]) > 0, < \infty$. Besides $[\varphi] \sim [\psi]$ so $D_{\mathbf{B}}([\varphi]) = D_{\mathbf{B}}([\psi])$, $D_{\mathbf{B}}([\varphi])$ is a constant. Normalise $D_{\mathbf{B}}(\mathfrak{M})$ so as to have $D_{\mathbf{B}}([\varphi]) = 1$. Then Lemma 8.3.2 gives $D_{\mathbf{B}}(\mathfrak{M})$ = common notion of dimension of $\mathfrak{M}$.

Thus if $\mathfrak{H}$ is an n-dimensional Euclidean space, this $D_{\mathbf{B}}(\mathfrak{M})$ has the range $0, 1, \cdots, n$; and if it is a Hilbert space, it has the range $0, 1, \cdots, \infty$. This proves that $\mathbf{M}$ is in the cases (I_n) resp. (I_∞), and that this normalisation of $D_{\mathbf{B}}(\mathfrak{M})$ is the standard one.

This completes the proof.

Lemma 8.6.1 makes it possible to solve Problems 3–7 in the discrete cases (that is for direct factors). The details are these:

The discrete cases (I_n), for every $n = 1, 2, \cdots$ and (I_∞) do occur; and for factorisations $\mathbf{M}_1, \cdots, \mathbf{M}_m$ we can prescribe any combination of the discrete cases (I_n) and (I_∞) for the $\mathbf{M}_i$'s. We need only form $\mathfrak{H} = \mathfrak{H}_1 \otimes \cdots \otimes \mathfrak{H}_m$ where the $\mathfrak{H}_i$ are the corresponding n-dimensional Euclidean or Hilbert spaces, and put $\mathbf{M}_i = \mathbf{B}_i^{(i)}$. Furthermore if $m - 1$ of the factors $\mathbf{M}_i$ are in discrete cases, then all are (Lemma 5.4.1); and so in particular two factors $\mathbf{M}_1$, $\mathbf{M}_2$ (and *a fortiori* two coupled factors) are either both in discrete cases, or none of them is. The direct factors are identical with those in discrete cases. If in a factorisation $\mathbf{M}_1, \cdots, \mathbf{M}_m$ all $\mathbf{M}_i$ are in discrete cases, then they are all direct factors, so $\mathbf{M}_1, \cdots, \mathbf{M}_m$ is a direct factorisation (Lemma 5.4.1). Thus a spatial isomorphism carries $\mathfrak{H}$ into $\mathfrak{H}_1 \otimes \cdots \otimes \mathfrak{H}_m$ and each $\mathbf{M}_i$ into $\mathbf{B}_i^{(i)}$, therefore $\mathfrak{H}_i$ is determined by the class of $\mathbf{M}_i$ (cf. above). Thus if two factors $\mathbf{M}$, $\mathbf{N}$ have the same discrete class, they are fully ring isomorphic (Lemma 2.3.4 and (22) §4, compare the factorisations $\mathbf{M}$, $\mathbf{M}'$ and $\mathbf{N}$, $\mathbf{N}'$); and if in two factorisations $\mathbf{M}_1, \cdots, \mathbf{M}_m$ and $\mathbf{N}_1, \cdots, \mathbf{N}_m$ the corresponding $\mathbf{M}_i$, $\mathbf{N}_i$ have the same discrete classes for $i = 1, \cdots, m$, then we have even spatial isomorphism.

Part III: Pairs of factors

Chapter IX: Qualitative Comparison of $\mathfrak{M}_f^{\mathbf{M}'}$ and $\mathfrak{M}_f^{\mathbf{M}}$

9.1. We need first a discussion of infinite sums of definite operators. The considerations which follow are based on a construction of K. Friedrichs, (7), pp. 472, 476. The situation which we will investigate is described as follows:

Definition 9.1.1. Let $A_1, A_2, \cdots$ be a sequence of everywhere defined, bounded, Hermitian, (semi-)definite operators. In other words; For each $i = 1, 2, \cdots$ an α_i exists, so that we have identically $0 \leqq (A_i f, f) \leqq \alpha_i \| f \|^2$ for all $f \in \mathfrak{H}$ (Cf. (16) pp. 73–74).

Put $A_0 = 1$. Define $\mathfrak{A}$ as the set of $f \in \mathfrak{H}$ for which $\sum_{i=0}^{\infty} (A_i f, f)$ is finite. For any two $f, g \in \mathfrak{H}$ for which $\sum_{i=0}^{\infty} (A_i f, g)$ is absolutely convergent, call this $Q(f, g)$.

Lemma 9.1.1. *$\mathfrak{A}$ is a linear set. If $f, g \in \mathfrak{A}$ then $Q(f, g)$ is defined. With af, $f + g$ as linear operations and $Q(f, g)$ as inner product $\mathfrak{A}$ fulfills the conditions* A, B, E *of* (16) *pp.* 64–66. (*We may however have* $\mathfrak{A} = (0)$.)

Proof: As $(A_i \alpha f, \alpha f) = |\alpha|^2 (A_i f, f)$ therefore $f \in \mathfrak{A}$ implies $\alpha f \in \mathfrak{A}$. As

$$(A_i(f + g), f + g) \leqq (A_i(f + g), f + g) + (A_i(f - g), f - g)$$
$$= 2(A_i f, f) + 2(A_i g, g)$$

therefore $f, g \in \mathfrak{A}$ imply $f + g \in \mathfrak{A}$. Thus $\mathfrak{A}$ is a linear set. We have $|(A_i f, g)| \leqq \frac{1}{2}(A_i f, f) + \frac{1}{2}(A_i g, g)$, (this is proved literally as Schwarz's inequality cf. (16), p. 64), thus if $f, g \in \mathfrak{A}$ then $\sum_{i=0}^{\infty} (A_i f, g)$ is absolutely convergent, and so $Q(f, g)$ is defined.

We now verify A, B, E, loc. cit. All parts of A, B are obvious, except B, 4. This holds, as $f \neq 0$ implies $Q(f, f) = \sum_{i=0}^{\infty} (A_i f, f) \geqq (A_0 f, f) = \| f \|^2 > 0$. Consider now E. We must prove: If $f_1, f_2, \cdots \in \mathfrak{A}$ and $\lim_{m, n \to \infty} Q(f_m - f_n, f_m - f_n) = 0$ then there exists an $f \in \mathfrak{A}$ with $\lim_{n \to \infty} Q(f_n - f, f_n - f) = 0$.

As $Q((f_m - f_n), (f_m - f_n)) \geqq \| f_m - f_n \|^2$ we have $\lim_{m,n\to\infty} \| f_m - f_n \| = 0$. So an $f \in \mathfrak{H}$ with $\lim_{n\to\infty} \| f_n - f \| = 0$ exists. As

$$\lim_{m,n\to\infty} Q(f_m - f_n, f_m - f_n) = 0$$

there exists for every $\epsilon > 0$ an $n_0 = n_0(\epsilon)$ so that $m, n \geqq n_0$ imply

$$Q(f_m - f_n, f_m - f_n) \leqq \epsilon$$

and *a fortiori* $\sum_{i=0}^{j} (A_i(f_m - f_n), f_m - f_n) \leqq \epsilon$. Let now $n \to \infty$ as all A_i are continuous, this gives $\sum_{i=0}^{j} (A_i(f_m - f), f_m - f) \leqq \epsilon$. This holds for every $j = 0, 1, 2, \cdots$, therefore $\sum_{0}^{\infty} (A_i(f_m - f), (f_m - f)) \leqq \epsilon$. This proves $f_m - f \in \mathfrak{A}$, $f = f_m - (f_m - f) \in \mathfrak{A}$, and then $Q(f_m - f, f_m - f) \leqq \epsilon$. So we have $f \in \mathfrak{A}$ and $\lim_{m\to\infty} Q(f_m - f, f_m - f) = 0$. This completes the proof.

Lemma 9.1.2. *For every $f \in \mathfrak{H}$ there exists one and only one $f^* \in \mathfrak{A}$ so that $(f, g) = Q(f^*, g)$ for every $g \in \mathfrak{A}$. We have $\| f^* \| \leqq \| f \|$.*

Proof: Consider $L(g) = (f, g)$ for $g \in \mathfrak{A}$ as a functional in $\mathfrak{A}$. We have obviously 1) $L(ag) = \bar{a}L(g)$, 2) $L(g_1 + g_2) = L(g_1) + L(g_2)$. Furthermore we have $| L(g) | = | (f, g) | \leqq \| f \| \cdot \| g \| \leqq \| f \| \sqrt{Q(g, g)}$, that is 3) $| L(g) | \leqq a_0 \sqrt{Q(g, g)}$, where $a_0 = \| f \|$. Under these conditions the existence of an $f^* \in \mathfrak{A}$ is certain, for which $L(g) = Q(f^*, g)$ for all $g \in \mathfrak{A}$ and this f^* is unique. (Cf. (11), p. 11, and (13a), p. 34.) In other words: $(f, g) = Q(f^*, g)$, $g = f^*$ gives in particular: $Q(f^*, f^*) = (f, f^*) \leqq \| f \| \cdot \| f^* \|$; but as $Q(f^*, f^*) \geqq \| f^* \|^2$ this implies $\| f^* \| \leqq \| f \|$.

Definition 9.1.2. Define an operator B by $Bf = f^*$, in the sense of Lemma 9.1.2.

Lemma 9.1.3. *B is everywhere defined, bounded, Hermitian, (semi-) definite (in $\mathfrak{H}$); and Range $B \subset \mathfrak{A}$.*

Proof: B is obviously everywhere defined. $(Bf, f) = (f^*, f) = Q(f^*, f^*)$ and $\geqq \| f^* \| \cdot \| f \| \geqq \| f \|^2$, thus B is bounded, Hermitian, and (semi-) definite. As $Bf = f^* \in \mathfrak{A}$ we have Range $B \subset \mathfrak{A}$.

Lemma 9.1.4. *Form $B^{\frac{1}{2}}$ in the sense of F. Riesz (cf. in the proof). $B^{\frac{1}{2}}$ is everywhere defined, bounded, Hermitean, (semi-) definite. Range $B^{\frac{1}{2}} = \mathfrak{A}$. For $f \in \mathfrak{H} - [\mathfrak{A}]$, $B^{\frac{1}{2}}f = 0$, while for $f, g \in [\mathfrak{A}]$, $(f, g) = Q(B^{\frac{1}{2}}f, B^{\frac{1}{2}}f)$.*

Proof: As to the definition and character of $B^{\frac{1}{2}}$ cf. for instance (16), p. 113.

$B^{\frac{1}{2}}f = 0$ implies $Bf = B^{\frac{1}{2}}(B^{\frac{1}{2}}f) = 0$; $Bf = 0$ implies $\| B^{\frac{1}{2}}f \|^2 = (B^{\frac{1}{2}}f, B^{\frac{1}{2}}f) = (Bf, f) = 0$, $B^{\frac{1}{2}}f = 0$; $Bf = 0$ means that the f^* of Lemma 9.1.2 is 0, that is: $(f, g) = 0$ for $g \in \mathfrak{A}$. This means $f \in \mathfrak{H} - [\mathfrak{A}]$. So we have:

$$(f; B^{\frac{1}{2}}f = 0) = (f; Bf = 0) = \mathfrak{H} - [\mathfrak{A}].$$

$$[\text{Range } B^{\frac{1}{2}}] = [\text{Range } B] = [\mathfrak{A}].$$

Consider an $f \in [\mathfrak{A}]$. Then f is a condensation point of Range $B^{\frac{1}{2}}$ so a sequence $f_1, f_2, \cdots$ with $\lim_{n\to\infty} \| B^{\frac{1}{2}}f_n - f \| = 0$ exists. Now all $Bf_n \in \mathfrak{A}$ and

$$Q(Bf_m - Bf_n, Bf_m - Bf_n) = (f_m - f_n, Bf_m - Bf_n)$$
$$= (B^{\frac{1}{2}}f_m - B^{\frac{1}{2}}f_n, B^{\frac{1}{2}}f_m - B^{\frac{1}{2}}f_n) = \| B^{\frac{1}{2}}f_m - B^{\frac{1}{2}}f_n \|^2,$$

so $\lim_{m,n\to\infty} Q(Bf_m - Bf_n, Bf_m - Bf_n) = 0$. But $\mathfrak{A}$ fulfills the completeness postulate E (for $Q(\cdots, \cdots)$ cf. Lemma 9.1.1), so an $f^\circ \in \mathfrak{A}$ with

$$\lim_{n\to\infty} Q(Bf_n - f^\circ, Bf_n - f^\circ) = 0$$

exists. Now $Q(g, g) \geqq \| g \|^2$ so $\lim_{n\to\infty} \| Bf_m - f^\circ \| = 0$. On the other hand the continuity of $B^{\frac{1}{2}}$ implies $\lim_{n\to\infty} \| Bf_n - B^{\frac{1}{2}}f \| = \lim_{n\to\infty} \| B^{\frac{1}{2}}(B^{\frac{1}{2}}f_n - f) \| = 0$. So $f^\circ = B^{\frac{1}{2}}f$, proving $B^{\frac{1}{2}}f \in \mathfrak{A}$. Besides we found

$$\lim_{n\to\infty} Q(Bf_n - B^{\frac{1}{2}}f, Bf_n - B^{\frac{1}{2}}f) = 0.$$

Consider two $f, g \in [\mathfrak{A}]$. Then $B^{\frac{1}{2}}f, B^{\frac{1}{2}}g \in \mathfrak{A}$ and forming the above sequence $f_1, f_2, \cdots$ for f and its analogue $g_1, g_2, \cdots$ for g, the continuity of the inner product of $\mathfrak{A}$ in the metric of $\mathfrak{A}$ (all $Q(\cdots, \cdots)$ cf. (16), p. 65) necessitates

$$\begin{aligned} Q(B^{\frac{1}{2}}f, B^{\frac{1}{2}}g) &= \lim_{n\to\infty} Q(Bf_n, Bg_n) = \lim_{n\to\infty} (f_n, Bg_n) \\ &= \lim_{n\to\infty} (B^{\frac{1}{2}}f_n, B^{\frac{1}{2}}g_n) = (f, g). \end{aligned}$$

Thus $B^{\frac{1}{2}}$ maps $[\mathfrak{A}]$ on some subset $\mathfrak{A}'$ of $\mathfrak{A}$ and this mapping is isomorphic, if we use the inner product (f, g) in $[\mathfrak{A}]$ but $Q(f, g)$ in $\mathfrak{A}'$.

Therefore it is one-to-one. Now $[\mathfrak{A}]$ is a closed subset in the topologically complete set $\mathfrak{H}$ (topology of the metric $\| f - g \| = (f - g, f - g)^{\frac{1}{2}}$, therefore $\mathfrak{A}'$ is complete too, and so it must be a closed subset of $\mathfrak{A}$ (topology of the metric $[Q(f - g, f - g)]^{\frac{1}{2}}$). In this sense $\mathfrak{A}'$ is a closed linear set in $\mathfrak{A}$. Consider therefore $\mathfrak{A} - \mathfrak{A}'$ (for the inner product $Q(f, g)$). $f \in \mathfrak{A} - \mathfrak{A}'$ means $f \in \mathfrak{A}$ and $Q(f, g) = 0$ for every $g \in \mathfrak{A}'$ that is $Q(f, B^{\frac{1}{2}}g') = 0$ for every $g' \in [\mathfrak{A}]$. For every $g'', g' = B^{\frac{1}{2}}g'' \in [\mathfrak{A}]$ so we have *a fortiori* $Q(f, Bg'') = 0$, $(f, g'') = 0$ for every g'', and so $f = 0$. This proves $\mathfrak{A} - \mathfrak{A}' = (0)$. Now every $f \in \mathfrak{A}$ is the sum of an element of $\mathfrak{A}'$ and one of $\mathfrak{A} - \mathfrak{A}'$ (this is the fundamental theorem on projections, which applies to spaces which fulfill, like $\mathfrak{A}$, only A., B., E., by (11), p. 14, or (13a), p. 34); as $\mathfrak{A} - \mathfrak{A}' = (0)$ this means $f \in \mathfrak{A}'$. Thus we have proved $\mathfrak{A}' = \mathfrak{A}$. We see that $B^{\frac{1}{2}}$ maps $[\mathfrak{A}]$ on $\mathfrak{A}$. In $\mathfrak{H} - [\mathfrak{A}]$ (we use again the inner product (f, g) of $\mathfrak{H}$) however it is $= 0$. Therefore Range $B^{\frac{1}{2}} = \mathfrak{A}$.

Thus we have proved all statements of our Lemma.

Another essential property of $B^{\frac{1}{2}}$ is this:

Lemma 9.1.5. *Let* $\mathbf{M}$ *be a ring which contains* 1. *If the above* $A_1, A_2, \cdots$ *are elements of* $\mathbf{M}$ *then* $B^{\frac{1}{2}} \in \mathbf{M}$.

Proof: $A_n \in \mathbf{M}$, $A_n \eta \mathbf{M}$ so A_n is invariant under every unitary $U' \in \mathbf{M}'$. Therefore $\mathfrak{A}$ and $Q(f, g)$ are invariant under these and with them B, which was uniquely characterised with their help. So $B \eta \mathbf{M}$ and by Lemma 4.2.1 $B \in \mathbf{M}$. This implies $B^{\frac{1}{2}} \in \mathbf{M}$ for instance by (19), p. 213.

9.2. Consider now the sets $\mathfrak{M}_f^{\mathbf{M}'}$ of Definition 5.1.1.

Lemma 9.2.1. *Let* $\mathbf{M}$ *be a ring which contains* 1. *If* $g_0 \in \mathfrak{M}_{f_0}^{\mathbf{M}'}$ *then two linear, closed operators* $X', Y' \eta \mathbf{M}'$ *with everywhere dense domains can be found such that* $g_0 = X'Y'f_0$. (*Of course* $f_0 \in$ *Domain* Y', $Y'f_0 \in$ *Domain* X'. X' *is even bounded.*)

Proof: g_0 is a condensation point of the set of all $A'f_0$, $A' \in \mathbf{M}$. So we can find for every $n = 1, 2, \cdots$ an $A_n' \in \mathbf{M}'$ with $\| A_n'f_0 - g_0 \| \leqq 1/2^n$.

Thus $\| A_{n+1}'f_0 - A_n'f_0 \| \leqq (1/2^{n+1}) + (1/2^n) = 3/2^{n+1}$, and so

$$(2^n(A_{n+1}' - A_n')^*(A_{n+1}' - A_n')f_0, f_0) = 2^n((A_{n+1}' - A_n')f_0, (A_{n+1}' - A_n')f_0)$$
$$= 2^n \| (A_{n+1}' - A_n')f_0 \|^2 \leqq 2^n(3/2^{n+1})^2 = 9/2^{n+2}.$$

We now perform the construction of Definition 9.1.1 with $2^n(A_{n+1}' - A_n')^*(A_{n+1} - A_n')$ in place of A_n, $n = 1, 2, \cdots$. We thus obtain $\mathfrak{A}$ and $Q(f, g)$, and the B, $B^{\frac{1}{2}}$ of Definition 9.1.2 and Lemma 9.1.4, which we will denote by B', $B'^{\frac{1}{2}}$. Lemma 9.1.5 gives $B'^{\frac{1}{2}} \in \mathbf{M}'$. Note that our above evaluation secures $f_0 \in \mathfrak{A}$.

$B'^{\frac{1}{2}}$ is a one-to-one mapping of $[\mathfrak{A}]$ on $\mathfrak{A}$ (even isomorphic!, cf. Lemma 9.1.4). Denote its inverse (in $\mathfrak{A}$ only!) by Y_0'. As $B'^{\frac{1}{2}} \in \mathbf{M}'$ we have $\mathfrak{A} = \text{Range } B'^{\frac{1}{2}} \eta \mathbf{M}$ and so $Y_0' \eta \mathbf{M}$; by its very nature Y_0' is linear and closed. But Domain $Y_0' = \mathfrak{A}$ is only dense in $\mathfrak{A}$. If we put $Y' = Y_0'P_{[\mathfrak{A}]}$ then Y' is clearly $\eta \mathbf{M}$, linear, closed, and has an everywhere dense domain.

Consider an $f \in \mathfrak{H}$. Then $B'^{\frac{1}{2}}f = B'^{\frac{1}{2}}P_{[\mathfrak{A}]}f \in \mathfrak{A}$ and $Q(B'^{\frac{1}{2}}f, B'^{\frac{1}{2}}f) = \| P_{[\mathfrak{A}]}f \|^2 \leqq \| f \|^2$. But

$$Q(B'^{\frac{1}{2}}f, B'^{\frac{1}{2}}f) = (B'^{\frac{1}{2}}f, B'^{\frac{1}{2}}f) + \sum_1^\infty{}_n (2^n(A_{n+1}' - A_n')^*(A_{n+1}' - A_n')B'^{\frac{1}{2}}f, B'^{\frac{1}{2}}f)$$
$$= \| B'^{\frac{1}{2}}f \|^2 + \sum_1^\infty{}_n 2^n \| (A_{n+1}' - A_n')B'^{\frac{1}{2}}f \|^2$$
$$\geqq 2^n \| (A_{n+1}' - A_n')B'^{\frac{1}{2}}f \|^2$$

for every $n = 1, 2, \cdots$; so

$$\| (A_{n+1}' - A_n')\, B'^{\frac{1}{2}}f \| \leqq \frac{1}{\sqrt{2^n}} \| f \|$$

$$\| (A_{n+p}' - A_n')\, B'^{\frac{1}{2}}f \| \leqq \left(\frac{1}{\sqrt{2^n}} + \frac{1}{\sqrt{2^{n+1}}} + \cdots + \frac{1}{\sqrt{2^{n+p-1}}} \right) \| f \|$$
$$\leqq \frac{1}{\sqrt{2^n}} \left(\frac{1}{1 - \frac{1}{\sqrt{2}}} \right) \| f \| = \frac{2 + \sqrt{2}}{\sqrt{2^n}} \| f \|,$$

or, which is the same thing:

$$\| (A_m' - A_n')\, B'^{\frac{1}{2}}f \| \leqq (2 + \sqrt{2})/\sqrt{2^{M \text{ in } (m, n)}} \| f \|.$$

Thus $\lim_{m, n \to \infty} \| (A_m' - A_n')\, B'^{\frac{1}{2}}f \| = 0$ and therefore a unique f^* with $\lim_{n \to \infty} \| A_n'B'^{\frac{1}{2}}f - f^* \| = 0$ exists. We define an operator X' by $X'f = f^*$; then X' is clearly the uniform limit of the sequence $A_1'B'^{\frac{1}{2}}$, $A_2B'^{\frac{1}{2}}$, $\cdots$ (cf. (18), p. 384), thus bounded and $\in \mathbf{M}'$ too.

Now $f_0 \in \mathfrak{A}$, therefore $Y'f_0 = Y_0'f_0 \in [\mathfrak{A}]$ and $B'^{\frac{1}{2}}Y'f_0 = B'^{\frac{1}{2}}Y_0'f_0 = f_0$, g_0 is the

$n \to \infty$ limit of the sequence $A_n' f_0 = (A_n' B^{\frac{1}{2}}) Y' f_0$, so $g_0 = X'Y' f_0$. This completes the proof.

Note that we cannot make much use of the operator-product $X'Y'$ itself. It is not closed in general, and it is uncertain whether it possesses a single valued closure (cf. (21), p. 300). We will see in §16.3–16.4 how the notion of finiteness helps to bridge this gap; for the moment this is unimportant, because we are able to deal with the X', Y' separately.

9.3. We will compare the $\mathfrak{M}_f^{\mathbf{M}'}$ and the $\mathfrak{M}_f^{\mathbf{M}}$. From now on we assume that $\mathbf{M}$ is a factor.

Lemma 9.3.1. *If $g_0 = X' f_0$ where $X' \eta \mathbf{M}'$ is a linear, closed operator with an everywhere dense domain, then*

$$\mathfrak{M}_{g_0}^{\mathbf{M}} \precsim \mathfrak{M}_{f_0}^{\mathbf{M}} (\cdots \mathbf{M}') .$$

Proof: $X' \eta \mathbf{M}'$ means that X' commutes with all unitary $U \epsilon \mathbf{M}'' = \mathbf{M}$ that is with all elements of $\mathbf{M}^{(U)}$ (cf. (18), p. 392), as it is linear and closed however, it will even commute with those of $\mathbf{R}(\mathbf{M}^U) = \mathbf{M}$ (cf. (18), p. 392 and 405). So we have

$$\begin{aligned} \mathfrak{M}_{g_0}^{\mathbf{M}} &= [Ag_0, A \epsilon \mathbf{M}] = [AX'f_0; A \epsilon \mathbf{M}] \\ &= [X'Af_0; A \epsilon \mathbf{M}] = [X'f, f \epsilon (Af_0, A \epsilon \mathbf{M})] \\ &\subset [X'f, f \epsilon \text{ Domain } X' \cdot \mathfrak{M}_{f_0}^{\mathbf{M}}] . \end{aligned}$$

Put [Domain $X' \cdot \mathfrak{M}_{f_0}^{\mathbf{M}}$] = $\mathfrak{N}'$; clearly $\mathfrak{N}' \subset \mathfrak{M}_{f_0}^{\mathbf{M}}$ and $X' \eta \mathbf{M}'$, $\mathfrak{M}_{f_0}^{\mathbf{M}} \eta \mathbf{M}'$ imply $\mathfrak{N}' \eta \mathbf{M}'$. It is clear that $X' \cdot P_{\mathfrak{N}'}$ is a linear, closed operator with an everywhere dense domain, and $\eta \mathbf{M}$. Our above relation proves $\mathfrak{M}_{g_0}^{\mathbf{M}} \subset$ [Range $(X' \cdot P_{\mathfrak{N}})$]. On the other hand

$$\begin{aligned} [\text{Range } (X'P_{\mathfrak{N}'})^*] &= \mathfrak{H} - (f; X'P_{\mathfrak{N}'}f = 0) \subset \mathfrak{H} - (f; P_{\mathfrak{N}'}f = 0) \\ &= \mathfrak{H} - (\mathfrak{H} - \mathfrak{N}') = \mathfrak{N}' \subset \mathfrak{M}_{f_0}^{\mathbf{M}} . \end{aligned}$$

Using Lemma 6.2.1 we obtain:

$\mathfrak{M}_{g_0}^{\mathbf{M}} \subset$ [Range $(X'P_{\mathfrak{N}'})$] $\sim$ [Range $(X'P_{\mathfrak{N}})^*$] $\subset \mathfrak{M}_{f_0}^{\mathbf{M}}$ (the $\sim$ is $(\cdots \mathbf{M}')$, and so $\mathfrak{M}_{g_0}^{\mathbf{M}} \precsim \mathfrak{M}_f^{\mathbf{M}} (\cdots , \mathbf{M}')$.

Lemma 9.3.2. $\mathfrak{M}_{g_0}^{\mathbf{M}'} \precsim \mathfrak{M}_{f_0}^{\mathbf{M}'} (\cdots \mathbf{M})$ *implies* $\mathfrak{M}_{g_0}^{\mathbf{M}} \precsim \mathfrak{M}_{f_0}^{\mathbf{M}} (\mathbf{M}')$.

Proof: $\mathfrak{M}_{g_0}^{\mathbf{M}'} \precsim \mathfrak{M}_{f_0}^{\mathbf{M}'} (\cdots \mathbf{M})$ means $\mathfrak{M}_{g_0}^{\mathbf{M}'} \sim \mathfrak{N}$, $(\cdots \mathbf{M})$, $\mathfrak{N} \subset \mathfrak{M}_{g_0}^{\mathbf{M}'}$. So a partially isometric $U \epsilon \mathbf{M}$ maps $\mathfrak{M}_{g_0}^{\mathbf{M}'}$ on $\mathfrak{N}$ while U^* maps $\mathfrak{N}$ on $\mathfrak{M}_{g_0}^{\mathbf{M}'}$.

Now $Ug_0 \epsilon \mathfrak{N} \subset \mathfrak{M}_{f_0}^{\mathbf{M}'}$ so by Lemma 9.2.1 $Ug_0 = X'Y'f_0$, X', Y' linear, closed, and with everywhere dense domains. By Lemma 9.3.1 $\mathfrak{M}_{f_0}^{\mathbf{M}} \succsim \mathfrak{M}_{Y'f_0}^{\mathbf{M}} \succsim \mathfrak{M}_{X'Y'f_0}^{\mathbf{M}} = \mathfrak{M}_{Ug_0}^{\mathbf{M}}$ all $\succsim (\cdots \mathbf{M}')$. But $U^*Ug_0 = g_0$ and so

$$\begin{aligned} \mathfrak{M}_{g_0}^{\mathbf{M}} &= [Ag_0; A \epsilon \mathbf{M}] = [AU^*Ug_0; A \epsilon \mathbf{M}] \\ &\subset [BUg_0, B \epsilon \mathbf{M}] = \mathfrak{M}_{Ug_0}^{\mathbf{M}} . \end{aligned}$$

So $\mathfrak{M}_{g_0}^{\mathbf{M}} \subset \mathfrak{M}_{U g_0}^{\mathbf{M}} \lesssim \mathfrak{M}_{f_0}^{\mathbf{M}} (\cdots \mathbf{M}')$, $\mathfrak{M}_{g_0}^{\mathbf{M}} \lesssim \mathfrak{M}_{f_0}^{\mathbf{M}} (\cdots \mathbf{M}')$.

Lemma 9.3.3. $\mathfrak{M}_{f_0}^{\mathbf{M}'} \gtrsim \mathfrak{M}_{g}^{\mathbf{M}'}(\cdots \mathbf{M})$ *is equivalent to* $\mathfrak{M}_{f_0}^{\mathbf{M}} \gtrsim \mathfrak{M}_{g_0}^{\mathbf{M}} (\cdots \mathbf{M}')$ *resp.*

Proof: $\mathfrak{M}_{f_0}^{\mathbf{M}'} \gtrsim \mathfrak{M}_{g_0}^{\mathbf{M}'}(\cdots \mathbf{M})$ implies $\mathfrak{M}_{f_0}^{\mathbf{M}} \gtrsim \mathfrak{M}_{g_0}^{\mathbf{M}}$ by Lemma 9.3.2. The converse follows by replacing $\mathbf{M}$ by $\mathbf{M}'$ (and so $\mathbf{M}'$ by $\mathbf{M}'' = \mathbf{M}$); thus $\mathfrak{M}_{f_0}^{\mathbf{M}'} \gtrsim \mathfrak{M}_{g_0}^{\mathbf{M}'}(\cdots \mathbf{M})$ and $\mathfrak{M}_{f_0}^{\mathbf{M}} \gtrsim \mathfrak{M}_{g_0}^{\mathbf{M}}(\cdots \mathbf{M}')$ are equivalent. The same follows for $\lesssim$ by interchanging f_0 and g_0.

The same follows for $\sim$ because it means that $\lesssim$ and $\gtrsim$ both hold; it follows for $\prec$ because this means that $\lesssim$ holds, but $\sim$ not; finally it follows for $\succ$ by interchanging again f_0 and g_0. Thus all statements of our Lemma are proved.

Chapter X: Ratio of $D_{\mathbf{M}}(\mathfrak{M}_f^{\mathbf{M}'})$ *and* $D_{\mathbf{M}'}(\mathfrak{M}_f^{\mathbf{M}})$

10.1. The last §§ give the basis for a quantitative comparison of $D_{\mathbf{M}}(\mathfrak{M}_f^{\mathbf{M}'})$ with $D_{\mathbf{M}'}(\mathfrak{M}_f^{\mathbf{M}})$. $\mathbf{M}, \mathbf{M}'$ will again be factors; $D_{\mathbf{M}}(\mathfrak{M})$, $D_{\mathbf{M}'}(\mathfrak{M}')$ arbitrarily normalised relative dimension functions of $\mathbf{M}, \mathbf{M}'$ resp.; Δ, Δ' their resp. ranges. We define:

Definition 10.1.1. Denote the range of $D_{\mathbf{M}}(\mathfrak{M}_f^{\mathbf{M}'})$ (for all $f \in \mathfrak{H}$) by Δ_0 and the range of $D_{\mathbf{M}'}(\mathfrak{M}_f^{\mathbf{M}})$ (for all $f \in \mathfrak{H}$) by Δ_0'.

Lemma 10.1.1. *If we let correspond* $D_{\mathbf{M}}(\mathfrak{M}_f^{\mathbf{M}'})$ *to* $D_{\mathbf{M}'}(\mathfrak{M}_f^{\mathbf{M}})$ *a one-to-one mapping of* Δ_0 *on* Δ_0' *results. Describe this mapping by* $\alpha' = \phi(\alpha)$, $\alpha \in \Delta_0$, $\alpha' \in \Delta_0'$. $\phi(\alpha)$ *is a monotone increasing function of* α.

Proof: Lemma 9.3.3 can be formulated like this: $D_{\mathbf{M}}(\mathfrak{M}_f^{\mathbf{M}'}) \gtreqless D_{\mathbf{M}}(\mathfrak{M}_g^{\mathbf{M}'})$ is equivalent to $D_{\mathbf{M}'}(\mathfrak{M}_f^{\mathbf{M}}) \gtreqless D_{\mathbf{M}'}(\mathfrak{M}_g^{\mathbf{M}})$ resp. This implies all parts of our Lemma.

We will now determine Δ_0, Δ_0', $\phi(\alpha)$ and the set of all pairs $\mathfrak{M}_f^{\mathbf{M}'}$, $\mathfrak{M}_f^{\mathbf{M}}$.

Lemma 10.1.2. *If* $A \in \mathbf{M}$, *then* $[Ag; g \in \mathfrak{M}_f^{\mathbf{M}'}] = \mathfrak{M}_{Af}^{\mathbf{M}'}$.

Proof: We have

$[Ag; g \in \mathfrak{M}_f^{\mathbf{M}'}] = [Ag; g \in [B'f, B' \in \mathbf{M}']] = [Ag; g \in (B'f; B' \in \mathbf{M}')]$
$= [AB'f; B' \in \mathbf{M}'] = [B'Af; B' \in \mathbf{M}'] = \mathfrak{M}_{Af}^{\mathbf{M}'}$.

Lemma 10.1.3. *The necessary and sufficient conditions for the existence of an* $f \in \mathfrak{H}$ *with* $\mathfrak{M}_f^{\mathbf{M}'} = \mathfrak{M}$, $\mathfrak{M}_f^{\mathbf{M}} = \mathfrak{N}$ *are these:* $\mathfrak{M} \eta \mathbf{M}$, $\mathfrak{N} \eta \mathbf{M}'$, $D_{\mathbf{M}}(\mathfrak{M}) \in \Delta_0$, $D_{\mathbf{M}'}(\mathfrak{N}) = \phi(D_{\mathbf{M}}(\mathfrak{M}))$.

Proof: The necessity being obvious, let us consider the sufficiency. Our assumptions imply the existence of a g with $D_{\mathbf{M}}(\mathfrak{M}_g^{\mathbf{M}'}) = D_{\mathbf{M}}(\mathfrak{M})$, $D_{\mathbf{M}'}(\mathfrak{M}_g^{\mathbf{M}}) = D_{\mathbf{M}'}(\mathfrak{N})$. So $\mathfrak{M}_g^{\mathbf{M}'} \sim \mathfrak{M}(\cdots \mathbf{M})$, $\mathfrak{M}_g^{\mathbf{M}} \sim \mathfrak{N}(\cdots \mathbf{M}')$. Let $U \in \mathbf{M}$ be partially isometric with the initial and final sets $\mathfrak{M}_g^{\mathbf{M}'}$, $\mathfrak{M}$ and $U' \in \mathbf{M}'$ similarly with $\mathfrak{M}_g^{\mathbf{M}}$, $\mathfrak{N}$.

Consider now $\mathfrak{M}_{U'g}^{\mathbf{M}'}$. It is equal to $[A'U'g; A' \in \mathbf{M}']$, thus certainly a subset of $[B'g, B' \in \mathbf{M}'] = \mathfrak{M}_g^{\mathbf{M}'}$. But on the other hand for every $B' \in \mathbf{M}'$ we have for $A' = B'U'^* \in \mathbf{M}'$, $A'U'g = BU'^*U'g = B'P_{\mathfrak{M}_g^{\mathbf{M}}}g = B'g$. So our set is equal to $\mathfrak{M}_g^{\mathbf{M}'}$. Now Lemma 10.1.2 gives: $\mathfrak{M}_{UU'g}^{\mathbf{M}'} = [Uh, h \in \mathfrak{M}_{U'g}^{\mathbf{M}'}] = [Uh; h \in \mathfrak{M}_g^{\mathbf{M}'}] = [\mathfrak{M}] = \mathfrak{M}$. Similarly $\mathfrak{M}_{U'Ug}^{\mathbf{M}} = \mathfrak{N}$. Thus $f = UU'g = U'Ug$ meets all our requirements.

Lemma 10.1.4. $\alpha_1, \alpha_2, \cdots \in \Delta_0$, $\sum_{i=1}^{\infty} \alpha_i \in \Delta$, $\sum_{i=1}^{\infty} \phi(\alpha_i) \in \Delta'$ *imply* $\sum_{i=1}^{\infty} \alpha_i \in \Delta_0$, $\sum_{i=1}^{\infty} \phi(\alpha_i) = \phi(\sum_{i=1}^{\infty} \alpha_i)$.

Proof: We wish to construct a sequence of mutually orthogonal

$$\mathfrak{M}_1, \mathfrak{M}_2, \cdots \eta \mathbf{M}$$

so that $D_{\mathbf{M}}(\mathfrak{M}_i) = \alpha_i$. We proceed by induction. Assume that

$$\mathfrak{M}_1, \mathfrak{M}_2, \cdots, \mathfrak{M}_{i-1}$$

have already been constructed, fulfilling these requirements as well as $D_{\mathbf{M}}(\mathfrak{H} - [\mathfrak{M}_1, \cdots, \mathfrak{M}_{i-1}]) \geqq \sum_i^\infty \alpha_j$, where $i = 1, 2, \cdots$. (For $i = 1$, this is the case, as $D_{\mathbf{M}}(\mathfrak{H}) \geqq \sum_{j=1}^\infty \alpha_j$ because $\sum_{j=1}^\infty \alpha_j \epsilon \Delta$.) We then construct $\mathfrak{M}_i$ as follows: If $\alpha_i = \infty$ then $\sum_i^\infty \alpha_j = \infty$ and $\mathfrak{H} - [\mathfrak{M}_1, \cdots, \mathfrak{M}_{i-1}]$ is infinite. Apply Lemma 7.2.3 to it, and put $\mathfrak{M}_i = \mathfrak{N}$ for the resulting $\mathfrak{N}$ which obviously meets all requirements. If α_i is finite, choose an $\mathfrak{M}_i' \epsilon \mathbf{M}$ with $D(\mathfrak{M}_i') = \alpha_i$. Then $\mathfrak{M}_i' \lesssim \mathfrak{H} - [\mathfrak{M}_1, \cdots, \mathfrak{M}_{i-1}]$, $\mathfrak{M}_i' \sim \mathfrak{M}_i \subset \mathfrak{H} - [\mathfrak{M}_1, \cdots, \mathfrak{M}_{i-1}]$ and use this $\mathfrak{M}_i$. Then $D_{\mathbf{M}}(\mathfrak{M}_i) = \alpha_i$, $D_{\mathbf{M}}(\mathfrak{H} - [\mathfrak{M}_1, \cdots, \mathfrak{M}_i]) = D_{\mathbf{M}}(\mathfrak{H} - [\mathfrak{M}_1, \cdots, \mathfrak{M}_{i-1}]) - D_{\mathbf{M}}(\mathfrak{M}_i) \geqq \sum_{j=i}^\infty \alpha_j - \alpha_i = \sum_{j=i+1}^\infty \alpha_j$ and again all conditions are fulfilled.

Choose similarly a sequence $\mathfrak{N}_1, \mathfrak{N}_2, \cdots \epsilon \mathbf{M}'$ so that $D_{\mathbf{M}'}(\mathfrak{N}_i) = \phi(\alpha_i)$. (Now $\sum_{i=1}^\infty \phi(\alpha_i) \epsilon \Delta'$ must be used.) By Lemma 10.1.3 an $f_i \epsilon \mathfrak{H}$ with $\mathfrak{M}_{f_i}^{\mathbf{M}'} = \mathfrak{M}_i$, $\mathfrak{M}_{f_i}^{\mathbf{M}} = \mathfrak{N}_i$ exists. As we can replace f_i by any $\alpha_i f_i$ we may assume $\| f_i \| \leqq 1/i$.

As we see, the $\mathfrak{M}_{f_i}^{\mathbf{M}'}$, $i = 1, 2, \cdots$ are mutually orthogonal and so are the $\mathfrak{M}_{f_i}^{\mathbf{M}}$, $i = 1, 2, \cdots$. So the f_i are mutually orthogonal, and as $\sum_{i=1}^\infty \| f_i \|^2 \leqq \sum_{i=1}^\infty 1/i^2$ is finite, we can form $f = \sum_{i=1}^\infty f_i$ (cf. (16), p. 67).

As $f_i \epsilon \mathfrak{M}_{f_i}^{\mathbf{M}'}$ therefore $f \epsilon [\mathfrak{M}_{f_i}^{\mathbf{M}'}, i = 1, 2, \cdots]$ and considering the mutual orthogonality of the $\mathfrak{M}_{f_i}^{\mathbf{M}'}$, $i = 1, 2, \cdots$ we have $f_i = E_{f_i}^{\mathfrak{M}'} f$. Similarly $f \epsilon [\mathfrak{M}_{f_i}^{\mathbf{M}}, i = 1, 2, \cdots]$ and $f_i = E_{f_i}^{\mathbf{M}} f$.

Now

$$\mathfrak{M}_f^{\mathbf{M}'} = [A'f;\ A' \epsilon \mathbf{M}'] = \left[\sum_1^\infty {}_i A'f_i;\ A' \epsilon \mathbf{M}'\right] \subset [[A'f_i;\ A' \epsilon \mathbf{M}'];\ i = 1, 2, \cdots]$$
$$= [\mathfrak{M}_{f_i}^{\mathbf{M}'},\ i = 1, 2, \cdots]$$

and

$$\mathfrak{M}_f^{\mathbf{M}'} = [A'f;\ A' \epsilon \mathbf{M}'] \supset [B'E_{f_i}^{\mathbf{M}} f;\ B' \epsilon \mathbf{M}'] = [B'f_i;\ B' \epsilon \mathbf{M}'] = \mathfrak{M}_{f_i}^{\mathbf{M}'}.$$
$$\mathfrak{M}_f^{\mathbf{M}'} \supset [\mathfrak{M}_{f_i}^{\mathbf{M}'},\ i = 1, 2, \cdots].$$

Thus $\mathfrak{M}_f^{\mathbf{M}'} = [\mathfrak{M}_{f_i}^{\mathbf{M}'},\ i = 1, 2, \cdots]$ similarly $\mathfrak{M}_f^{\mathbf{M}} = [\mathfrak{M}_{f_i}^{\mathbf{M}},\ i = 1, 2, \cdots]$. So we have $D_{\mathbf{M}}(\mathfrak{M}_f^{\mathbf{M}'}) = \sum_{i=1}^\infty D_{\mathbf{M}}(\mathfrak{M}_{f_i}^{\mathbf{M}'}) = \sum_{i=1}^\infty \alpha_i$, $D_{\mathbf{M}'}(\mathfrak{M}_f^{\mathbf{M}}) = \sum_{i=1}^\infty D_{\mathbf{M}'}(\mathfrak{M}_{f_i}^{\mathbf{M}}) = \sum_{i=1}^\infty \phi(\alpha_i)$. Therefore $\sum_1^\infty \alpha_i \epsilon \Delta_0$, $\sum_{i=1}^\infty \phi(\alpha_i) = \phi(\sum_{i=1}^\infty \alpha_i)$.

Lemma 10.1.5. $0 \epsilon \Delta_0$; *an* $\alpha \epsilon \Delta_0$ *with* $\alpha > 0$ *exists; if* $\alpha \epsilon \Delta$, $\alpha \leqq \beta \epsilon \Delta_0$ *then* $\alpha \epsilon \Delta_0$.

Proof: For $f = 0$, $D_{\mathbf{M}}(\mathfrak{M}_f^{\mathbf{M}'}) = D_{\mathbf{M}}((0)) = 0$, for $f \neq 0$, $\mathfrak{M}_f^{\mathbf{M}'} \neq (0)$, $D_{\mathbf{M}}(\mathfrak{M}_f^{\mathbf{M}'}) > 0$. $\alpha \leqq \beta \epsilon \Delta_0$, $\alpha \epsilon \Delta$ implies that an f with $D_{\mathbf{M}}(\mathfrak{M}_f^{\mathbf{M}'}) = \beta$ and an $\mathfrak{N} \eta \mathbf{M}$ with $D_{\mathbf{M}}(\mathfrak{N}) = \alpha$ exist. So $\mathfrak{N} \lesssim \mathfrak{M}_f^{\mathbf{M}'}$, $\mathfrak{N} \sim \mathfrak{N}' \subset \mathfrak{M}_f^{\mathbf{M}}$ and by Lemma 10.1.2, $\mathfrak{M}_{P_{\mathfrak{N}'} f}^{\mathbf{M}'} = [P_{\mathfrak{N}'} g,\ g \epsilon \mathfrak{M}_f^{\mathbf{M}'}] = \mathfrak{N}' \sim \mathfrak{N}$ so $D_{\mathbf{M}}(\mathfrak{M}_{P_{\mathfrak{N}'} f}^{\mathbf{M}'}) = D_{\mathbf{M}}(\mathfrak{N}) = \alpha$.

10.2. With the help of Lemmas 10.1.4, 10.1.5 the discussion of Δ_0, Δ_0' and

$\phi(\alpha)$ can now be completed. We will use Theorem VIII, and discuss the three cases (I), (II), (III) for $\mathbf{M}$ separately.

Assume first case (I) for $\mathbf{M}$. Then we have case (I) for $\mathbf{M}'$ too, by Lemmas 3.2.3, 3.2.4, and 8.6.1. Use $D_{\mathbf{M}}(\mathfrak{M})$, $D_{\mathbf{M}'}(\mathfrak{M})$ in their standard normalisations. By Lemma 10.1.5 $0 \in \Delta_0$; and an $\alpha \in \Delta_0$ with $\alpha > 0$, so $\alpha \geqq 1$ exists, which implies, as $1 \in \Delta$ that $1 \in \Delta_0$. Similarly $0, 1 \in \Delta_0'$. As $\phi(\alpha)$ is monotone increasing (by Lemma 10.1.1) and as 0 is the smallest element of both Δ_0 and Δ_0' so $\phi(0) = 0$, and as 1 is the smallest element $\neq 0$ of both Δ_0 and Δ_0' so $\phi(1) = 1$.

Denote the case of $\mathbf{M}$ more precisely by $\mathrm{I}_{m'}$ where $m' = 1, 2, \cdots$ or ∞; similarly $\mathbf{M}'$s by $\mathrm{I}_{n'}$ where $n' = 1, 2, \cdots$ or ∞. If $p = 0, 1, 2, \cdots$, Min (m', n'), then apply Lemma 10.1.4 with $\alpha_1 = \cdots = \alpha_p = 1$, $\alpha_{p+1} = \alpha_{p+2} = \cdots = 0$ if p is finite, and $\alpha_1 = \alpha_2 = \cdots = 1$ if p is infinite. Then $p \in \Delta_0$, $\phi(p) = p$ obtains.

If Δ_0 had further elements, we would have:

$$q \in \Delta_0,\ q \neq 0, 1, 2, \cdots, \text{Min } (m', n').$$

But $q \in \Delta$, $q = 0, 1, 2, \cdots, m'$, therefore Min $(m', n') < q \leqq m'$. Thus

$$\phi(\text{Min } (m', n')) < \phi(q)$$

(by Lemma 10.1.1), but $\phi(\text{Min } (m', n')) = \text{Min } (m', n')$ and $\phi(q) \in \Delta_0' \subset \Delta'$, so Min $(m', n') < \phi(q) \leqq n'$. So Min $(m', n') < m'$ and $< n'$ which is impossible. Thus Δ_0 is precisely the set $0, 1, 2, \cdots$, Min (m', n') and in it $\phi(\alpha) = \alpha$; therefore Δ_0' is the same set. We see that we have either $\Delta_0 = \Delta$, or $\Delta_0' = \Delta'$.

Assume next case (III) for $\mathbf{M}$. Δ has precisely two elements: $0, \infty$, so that $\Delta_0 = \Delta$ by Lemma 10.1.5. So $\Delta_0 = (0, \infty)$, $\Delta_0' = (\phi(0), \phi(\infty)) = (0, \phi(\infty))$. Lemma 10.1.5 applied to $\mathbf{M}'$ excludes the existence of an $\alpha' \in \Delta'$ with $0 < \alpha' < \phi(\infty)$. This excludes case (II) for $\mathbf{M}'$. Case (I) for $\mathbf{M}'$ is ruled out as above (interchange $\mathbf{M}$, $\mathbf{M}'$), so we have case (III) for $\mathbf{M}'$ too. Now we have for the same reasons as in $\mathbf{M}$, $\Delta_0' = \Delta' = (0, \infty)$, $\phi(\infty) = \infty$. So we have again $\phi(\alpha) = \alpha$, but now $\Delta_0 = \Delta$ and $\Delta_0' = \Delta'$.

Assume finally case (II) for $\mathbf{M}$. The above arguments exclude cases (I) and (III) for $\mathbf{M}'$ (interchange $\mathbf{M}$, $\mathbf{M}'$,!), so we have case (II) for $\mathbf{M}'$ too. We have four possibilities: $\mathbf{M}$ in a case $(\mathrm{II}_{m'})$ and $\mathbf{M}'$ in a case $(\mathrm{II}_{n'})$ with $m', n', = 1, \infty$.

Consider a pair $\alpha \in \Delta_0$, $\alpha' \in \Delta_0'$ with $\alpha' = \phi(\alpha)$ and a $p = 1, 2, \cdots$. Then $(\alpha/p) \in \Delta_0$, $(\alpha'/p) \in \Delta_0'$. Assume $(\alpha'/p) > \phi(\alpha/p)$. Obviously $p \cdot (\alpha/p) = \alpha \in \Delta_0$. Further $p\phi(\alpha/p) < \alpha' \in \Delta_0' \subset \Delta'$ as we have case (II) for $\mathbf{M}$, this implies $p\phi(\alpha/p) \in \Delta'$. Thus Lemma 10.1.4 applies for $\alpha_1 = \cdots = \alpha_p = (\alpha/p)$, $\alpha_{p+1} = \alpha_{p+2} = \cdots = 0$, giving $p\phi(\alpha/p) = \phi(\alpha) = \alpha'$, $(\alpha'/p) = \phi(\alpha/p)$. This contradicts our assumption $(\alpha'/p) > \phi(\alpha/p)$. Interchanging of $\mathbf{M}$, $\mathbf{M}'$ excludes $(\alpha'/p) < \phi(\alpha/p)$ too. So we have proved $\alpha'/p = \phi(\alpha/p)$. In other words:

$$\phi(\alpha/p) = (1/p)\phi(\alpha).$$

Take now a $q = 0, 1, \cdots, p$. We have $(q/p)\alpha \leqq \alpha \in \Delta_0 \subset \Delta$, so (owing to case (II) for $\mathbf{M}$ and for $\mathbf{M}'$) $(q/p)\alpha \in \Delta$, $q\phi(\alpha/p) \in \Delta'$. Therefore Lemma 10.1.4

applies for $\alpha_1 = \cdots = \alpha_q = \alpha/p$, $\alpha_{q+1} = \alpha_{q+2} = \cdots 0$, giving $(q/p)\alpha \in \Delta_0$ and $\phi((q/p)\alpha) = q\,\varphi(\alpha/p) = (q/p)\phi(\alpha)$. In other words: If $0 \leqq \beta \leqq \alpha$ then $\phi(\beta) = (\beta/\alpha)\phi(\alpha)$ if (β/α) is rational.

By Lemma 10.1.5 $0 \leqq \beta \leqq \alpha$ implies $\beta \in \Delta_0$ so $\phi(\beta)$ is defined in all this interval. As it is monotone increasing (Lemma 10.1.1), we have $\phi(\beta) = (\beta/\alpha)\phi(\alpha)$ for all these β that is $\phi(\alpha)/\alpha = \phi(\beta)/\beta$ (assuming $\alpha, \beta \neq 0$).

Assume $\alpha, \beta \in \Delta_0$ and $\neq 0$. If $\alpha \geqq \beta$ then $0 \leqq \beta \leqq \alpha$ and we know that $\phi(\alpha)/\alpha = \phi(\beta)/\beta$. By symmetry this equation holds for $\alpha \leqq \beta$ too, that is, it holds always. So $\phi(\alpha)/\alpha$ is constant, say C. C is finite and positive by its nature. We have $\phi(\alpha) = C\alpha$ if $\alpha \in \Delta_0$, $\alpha \neq 0$ and of course for $\alpha = 0$ too.

Let the l.u.b. of Δ_0 be m'', and that of Δ_0', n''. We have $0 < m'' \leqq m'$, $0 < n'' \leqq n'$, and of course $n'' = Cm''$. m'' belongs to Δ_0: Choose $\alpha_1, \alpha_2, \cdots$ with $\alpha_i \geqq 0$, $\sum_1^i \alpha_j < m''$, $\sum_1^\infty \alpha_j = m''$ (for instance $\alpha_i = m''/2^i$ if m'' is finite, and $\alpha_i = 1$ if m'' is infinite), and apply Lemma 10.1.4. Similarly n'' belongs to Δ_0'.

Assume now $m'' < m'$, $n'' < n'$. Then m'', n'' are both finite, and a $\vartheta > 1$, < 2 with $\vartheta m'' < m'$, $\vartheta n'' < n'$ exists. Then $(\vartheta/2)m'' < m'$ and so $\in \Delta_0$ while $2\cdot(\vartheta/2)m'' = \vartheta m'' \leqq m'$ is $\in \Delta$, $2\phi((\vartheta/2)m'') = 2\cdot C(\vartheta/2)m'' = \vartheta n'' \leqq n'$ is $\in \Delta'$. So Lemma 10.1.4 applies for $\alpha_1 = \alpha_2 = (\vartheta/2)m''$, $\alpha_3 = \alpha_4 = \cdots = 0$ giving $\vartheta m'' \in \Delta_0$. This is impossible, as $\vartheta m'' > m''$.

So we have $m'' = m'$ and $n'' \leqq n'$ or $m'' \leqq m'$ and $n'' = n'$ and besides $n'' = Cm''$. Let us now consider the four possible combinations of $m', n' = 1, \infty$. If $m' = n' = 1$ then $C \geqq 1$ necessitates $n'' = n' = 1$, $m' = (1/C)n' = (1/C) \leqq 1$ so $\Delta_0' = \Delta'$, while $C \leqq 1$ implies $m'' = m' = 1$, $n'' = Cm'' = C \leqq 1$ so $\Delta_0 = \Delta$. As both Δ, Δ' (that is $D_{\mathbf{M}}(\mathfrak{M})$, $D_{\mathbf{M}'}(\mathfrak{M})$) are normalised, C has an invariant meaning. If $m' = 1$, $n' = \infty$ then Δ' is not normalised. So we can make $C = 1$. Now obviously $m'' = m' = 1$, $n'' = m'' = 1 < \infty$. So $\Delta_0 = \Delta$. If $m' = \infty$, $n' = 1$ then Δ is not normalised. So we can make $C = 1$ again. Now obviously $n'' = n' = 1$, $m'' = n' = 1 < \infty$. So $\Delta_0' = \Delta$. If finally $m' = n' = \infty$ then both Δ and Δ' are not normalised. So we can make $C = 1$ again, and then $m'' = n'' = \infty$ and thus $\Delta_0 = \Delta$, $\Delta_0' = \Delta'$.

All these results together with those of Lemmas 10.1.1 and 10.1.3 give the

Theorem X. *In a factorisation* $\mathbf{M}$, $\mathbf{M}'$ *the factors* $\mathbf{M}$, $\mathbf{M}'$ *must both belong to class* (I), *or both to class* (II), *or both to class* (III).

The function $\phi(\alpha)$ *defined in Lemma* 10.1.1 *always has the form* $\phi(\alpha) = C\alpha$ *where* C *is a finite, positive constant. In case* (I), *using the standard normalisations of* Δ, Δ' *necessarily* $C = 1$; *in case* (II), *if both* $\mathbf{M}$, $\mathbf{M}'$ *are in case* (II_1) *and using the standard normalisations of* Δ, Δ' *the value of* C *is uniquely defined; in case* (II), *if at least one of* $\mathbf{M}$, $\mathbf{M}'$ *is in case* (II_∞) *and if we normalise the corresponding* Δ *or* Δ' *suitably, then we can make* $C = 1$; *in case* (III) *the value of* C *is arbitrary* (*as* $\alpha = 0, \infty$) *so we can take* $C = 1$.

Always either $\Delta_0 = \Delta$ *or* $\Delta_0' = \Delta'$; *details are given in the discussion above.*

An f *with* $\mathfrak{M}_f^{\mathbf{M}'} = \mathfrak{M}$, $\mathfrak{M}_f^{\mathbf{M}} = \mathfrak{N}$ *exists if and only if* $\mathfrak{M}\,\eta\,\mathbf{M}$, $\mathfrak{N}\,\eta\,\mathbf{M}'$ *and if* $D_{\mathbf{M}}(\mathfrak{M}) \in \Delta_0$, $D_{\mathbf{M}'}(\mathfrak{N}) \in \Delta_0'$, $D_{\mathbf{M}'}(\mathfrak{N}) = CD_{\mathbf{M}}(\mathfrak{M})$. (*Owing to the last condition each one of the two preceding conditions implies the other*).

This Theorem brings us nearer to the answers of Problems 4 and 7: It excludes certain combinations of cases for $\mathbf{M}$, $\mathbf{M}'$ and it shows that $\mathbf{M}$, $\mathbf{M}'$ possesses a further invariant, if $\mathbf{M}$, $\mathbf{M}'$ are both of class (II_1), the number C. This latter statement is of course only then of value, if such $\mathbf{M}$, $\mathbf{M}'$ both of class (II_1), and with various values of C, do really exist. We will see that this is the case, and besides get further information on all these points in §§13.2 and 13.3.

Chapter XI: Composition and decomposition of factors

11.1. A natural question to ask is this:

Problem 8. *Under what conditions can two given* $\mathbf{M}$, $\mathbf{N}$ *belong both to one factorisation* $\mathbf{M}_1, \cdots, \mathbf{M}_n$?

The following Lemma gives a necessary and sufficient condition but thereby raises a new problem.

Lemma 11.1.1. $\mathbf{M}$, $\mathbf{N}$ *belong both to one (suitably chosen) factorisation if and only if* (i) $\mathbf{M}, \mathbf{N}$ *are factors,* (ii) $\mathbf{M}, \mathbf{N}$ *commute (that is:* $\mathbf{M} \subset \mathbf{N}'$), (iii) $\mathbf{R}(\mathbf{M}, \mathbf{N})$ *is a factor.*

Proof: If $\mathbf{M}$, $\mathbf{N}$ belong to a factorisation $\mathbf{M}_1, \cdots, \mathbf{M}_n$ then $\mathbf{M} = \mathbf{M}_i$, $\mathbf{N} = \mathbf{M}_j$, $i \gtrless j$. Thus (i) is fulfilled by Lemma 3.1.1; (ii) by definition; and (iii) by Lemma 3.1.1 again, as we may make $i = 1, j = 2$ by a permutation of $\mathbf{M}_1, \cdots, \mathbf{M}_n$ and then remember (cf. the remark at the end of §3.1) that $\mathbf{R}(\mathbf{M}_1, \mathbf{M}_2), \mathbf{R}(\mathbf{M}_3, \cdots, \mathbf{M}_n)$ too is a factorisation if $n > 3$ (for $n = 2, \mathbf{R}(\mathbf{M}_1, \mathbf{M}_2) = \mathbf{B}$ is trivially a factor). Thus our condition is necessary.

Assume now that (i)–(iii) are fulfilled. $(\mathbf{R}(\mathbf{M}, \mathbf{N}))' = \mathbf{M}'\,\mathbf{N}'$ commutes with $\mathbf{M}$ and $\mathbf{N}$; $\mathbf{M}$, $\mathbf{N}$ commute with each other; and

$$\mathbf{R}(\mathbf{M}, \mathbf{N}, (\mathbf{R}(\mathbf{M}, \mathbf{N}))') = \mathbf{R}(\mathbf{R}(\mathbf{M}, \mathbf{N}), (\mathbf{R}(\mathbf{M}, \mathbf{N}))') = \mathbf{B}$$

as $\mathbf{R}(\mathbf{M}, \mathbf{N})$ is a factor. So $\mathbf{M}, \mathbf{N}, (\mathbf{R}(\mathbf{M}, \mathbf{N}))'$ is a factorisation and it contains $\mathbf{M}, \mathbf{N}$. Thus our condition is sufficient.

The new problem therefore is:

Problem 9. *If* $\mathbf{M}, \mathbf{N}$ *are two factors which commute, under what conditions will then* $\mathbf{R}(\mathbf{M}, \mathbf{N})$ *be a factor? How does the class of* $\mathbf{R}(\mathbf{M}, \mathbf{N})$ *(in the sense of Theorem VIII) connect with the classes of* $\mathbf{M}$ *and* $\mathbf{N}$?

We will only succeed in giving a partial answer. This answer could be obtained by using somewhat less formal machinery then we are going to use. But it seems reasonable to make use of it, because it makes the algebraic side of these questions clearer.

We define:

Definition 11.1.1. Let $\mathbf{M}$, $\mathbf{P}$ be rings which contain 1. If $\mathbf{M} \subset \mathbf{P}$ we define $\mathbf{M}'_{\mathbf{P}} = \mathbf{P} \cdot \mathbf{M}'$ ($\mathbf{M}'_{\mathbf{P}}$ too is a ring which contains 1 and $\subset \mathbf{P}$).

This notion shares many formal properties with $\mathbf{M}'$ with which it coincides for $\mathbf{P} = \mathbf{B}$. As the decisive property of $\mathbf{M}'$ is $\mathbf{M}'' = \mathbf{M}$ (cf. (18), p. 397), it is reasonable to define:

Definition 11.1.2. $\mathbf{P}$ is *normal*, if $\mathbf{M} \subset \mathbf{P}$ implies $\mathbf{M}''_{\mathbf{PP}} = \mathbf{M}$.

Lemma 11.1.2. *If* $\mathbf{P}$ *is normal, then it is a factor; and every factorisation* $\mathbf{Q}, \mathbf{P}'$ *is a coupled one.*

Proof: Put $\mathbf{M} = (\alpha 1)$. Then $\mathbf{M} \subset \mathbf{P}$ so $\mathbf{M}''_{\mathbf{PP}} = \mathbf{M}$; now $(\alpha 1)'_{\mathbf{P}} = \mathbf{P}(\alpha 1)' = \mathbf{PB} = \mathbf{P}$, $\mathbf{P}'_{\mathbf{P}} = \mathbf{P}\cdot\mathbf{P}'$. So we have $\mathbf{P}\cdot\mathbf{P}' = (\alpha 1)$. So $\mathbf{P}$ is a factor. That $\mathbf{Q}$, $\mathbf{P}'$ is a factorisation, means $\mathbf{Q} \subset \mathbf{P}'' = \mathbf{P}$ and $\mathbf{R}(\mathbf{Q}, \mathbf{P}') = \mathbf{B}$, that is $(\mathbf{R}(\mathbf{Q}, \mathbf{P}'))' = \mathbf{B}'$, $\mathbf{Q}'_{\mathbf{P}} = \mathbf{Q}'\mathbf{P} = (\mathbf{R}(\mathbf{Q}, \mathbf{P}'))' = \mathbf{B}' = (\alpha 1)$.
So $\mathbf{Q} = \mathbf{Q}''_{\mathbf{PP}} = (\alpha 1)'_{\mathbf{P}} = \mathbf{P}$.

Our notion of normalcy could easily be expressed in terms of the conditions which characterise the "B-lattices" of G. Birkhoff (cf. (1), p. 445) but we will not enter on this aspect of the subject at present.

The connection with Problem 9 is established by the following Lemma:

Lemma 11.1.3. *The factors* $\mathbf{M}$, $\mathbf{N}$ *are commutative and* $\mathbf{R}(\mathbf{M}, \mathbf{N})$ *is a factor too (that is: the conditions of Lemma* 11.1.1 *are fulfilled), if a* $\mathbf{P}$ *with* $\mathbf{M} \subset \mathbf{P}$, $\mathbf{N} \subset \mathbf{P}'$ *exists, so that* $\mathbf{P}$, $\mathbf{P}'$ *are both normal.*

This is in particular the case, if $\mathbf{M}$, $\mathbf{N}$ *commute, and besides* $\mathbf{M}$, $\mathbf{M}'$ *or* $\mathbf{N}$, $\mathbf{N}'$ *are both normal.*

Proof: $\mathbf{M} \subset \mathbf{P} \subset \mathbf{N}'$ shows that $\mathbf{M}$, $\mathbf{N}$ commute. We have

$$(\mathbf{R}(\mathbf{M}, \mathbf{M}'_{\mathbf{P}}))'_{\mathbf{P}} = \mathbf{P}\cdot(\mathbf{R}(\mathbf{M}, \mathbf{M}'_{\mathbf{P}}))' = \mathbf{PM}'(\mathbf{M}'_{\mathbf{P}})' = \mathbf{M}'\mathbf{M}''_{\mathbf{PP}}$$

and as $\mathbf{M} \subset \mathbf{P}$, $\mathbf{P}$ normal, this is $\mathbf{M}'\mathbf{M} = (\alpha 1)$. Therefore, considering $\mathbf{R}(\mathbf{M}, \mathbf{M}'_{\mathbf{P}}) \subset \mathbf{P}$, $\mathbf{P}$ normal, we have $\mathbf{R}(\mathbf{M}, \mathbf{M}'_{\mathbf{P}}) = (\mathbf{R}(\mathbf{M}, \mathbf{M}'_{\mathbf{P}})''_{\mathbf{PP}} = (\alpha 1)'_{\mathbf{P}} = \mathbf{P}$. Similarly $\mathbf{R}(\mathbf{N}, \mathbf{N}'_{\mathbf{P}'}) = \mathbf{P}'$. Therefore

$$\mathbf{R}(\mathbf{R}(\mathbf{M}, \mathbf{N}), \mathbf{R}(\mathbf{M}'_{\mathbf{P}}, \mathbf{N}'_{\mathbf{P}'})) = \mathbf{R}(\mathbf{M}, \mathbf{N}, \mathbf{M}'_{\mathbf{P}}, \mathbf{N}'_{\mathbf{P}'}) = \mathbf{R}(\mathbf{R}(\mathbf{M}, \mathbf{M}'_{\mathbf{P}}), \mathbf{R}(\mathbf{N}, \mathbf{N}'_{\mathbf{P}'}))$$

$$= \mathbf{R}(\mathbf{P}, \mathbf{P}') = \mathbf{B}. \quad (\mathbf{P} \text{ is a factor!}).$$

Now $\mathbf{M}$ commutes with $\mathbf{M}'_{\mathbf{P}}$ because $\mathbf{M}'_{\mathbf{P}} \subset \mathbf{M}'$, and with $\mathbf{N}'_{\mathbf{P}'}$ because $\mathbf{M} \subset \mathbf{P}$, $\mathbf{N}'_{\mathbf{P}'} \subset \mathbf{P}'$. So $\mathbf{M}$ commutes with $\mathbf{R}(\mathbf{M}'_{\mathbf{P}}, \mathbf{N}'_{\mathbf{P}'})$. Similarly $\mathbf{N}$ commutes with $\mathbf{R}(\mathbf{M}'_{\mathbf{P}}, \mathbf{N}'_{\mathbf{P}'})$. Therefore $\mathbf{R}(\mathbf{M}'_{\mathbf{P}}, \mathbf{N}'_{\mathbf{P}'})$ commutes with $\mathbf{R}(\mathbf{M}, \mathbf{N})$. Thus $\mathbf{R}(\mathbf{M}, \mathbf{N})$, $\mathbf{R}(\mathbf{M}'_{\mathbf{P}}, \mathbf{N}'_{\mathbf{P}'})$ is a factorisation, and so (by Lemma 3.1.1) $\mathbf{R}(\mathbf{M}, \mathbf{N})$ is a factor.

The second half of our Lemma obtains by putting $\mathbf{P} = \mathbf{M}$ resp. $\mathbf{P} = \mathbf{N}'$.

11.2. We derive some properties of normalcy.

Lemma 11.2.1. *The operation* $\mathbf{M}'_{\mathbf{P}}$ *(for* $\mathbf{M} \subset \mathbf{P}$*) and the normalcy of* $\mathbf{P}$ *are invariants under* (A^*, AB)*-ring-isomorphisms of* $\mathbf{P}$.

Proof: The first statement is obvious, the second follows immediately from the first one.

Lemma 11.2.2. *If* $\mathbf{P}$ *is a factor of class* (I), *then* $\mathbf{P}$, $\mathbf{P}'$ *are both normal.*

Proof: As $\mathbf{P}'$ is of class (I) too (by Lemmas 3.2.4 and 8.6.1, or by Theorem X), it suffices to consider $\mathbf{P}$ alone. Now Lemma 8.6.1 and Theorem II imply that $\mathbf{P}$ is algebraically (even fully) ring-isomorphic to the $\mathbf{B}$ of some space $\mathfrak{H}$. Therefore Lemma 11.2.1 permits us to assume $\mathbf{P} = \mathbf{B}$.

But then $\mathbf{M}'_{\mathbf{P}}$ becomes $\mathbf{M}'_{\mathbf{B}} = \mathbf{M}'$ and as $\mathbf{M}'' = \mathbf{M}$ (cf. (18), p. 397) therefore $\mathbf{P} = \mathbf{B}$ is normal.

Note that Lemmas 11.1.2 and 11.2.2 give together the statement of Lemma 3.2.4 again: Every direct factorisation $\mathbf{M}$, $\mathbf{N}$ is coupled (let $\mathbf{M}$, $\mathbf{N}$ correspond to $\mathbf{Q}$, $\mathbf{P}'$ resp.).

As we will find non-coupled factorisations of class (II) (Cf. 13.4), Lemma 11.1.2 implies that non-normal factors of class (II) do exist. We did not succeed however in showing that no factor of class (II) is normal. Nevertheless the only non-normal **P**'s we know are of class (II).

Thus the following question is only partially answered:

Problem 10. *Which factors are normal?*

The second half of Problem 9 remains to be discussed: Determining the class of $\mathbf{R}(\mathbf{M}, \mathbf{N})$ in terms of the classes of **M** and **N**. If **M** or **N** belongs to case (I) (by symmetry we may assume that **N** does), then the results of the next two §§ permit us to solve this problem. In all other cases our information is incomplete.

11.3. We will now consider an operation which is in a certain sense inverse to the "composition" $\mathbf{R}(\mathbf{M}, \mathbf{N})$. It is based on the following Lemma:

Lemma 11.3.1. *Let* **M** *be a ring which contains* 1, E *a projection* ϵ **M**. *If* $A' \epsilon \mathbf{M}'$ *then form* $EA' = A'E = A'_E$. *Then these* A'_E *are identical with those (bounded) operators* $\bar{A}$ *for which* $E\bar{A} = \bar{A}E = \bar{A}$ *and which commute with all* EBE, $B \epsilon \mathbf{M}$.

Proof: The necessity is clear: If $A' \epsilon \mathbf{M}'$ then A' commutes with $E \epsilon \mathbf{M}$ so we can define

$$EA' = A'E = A'_E\,;\; EA'_E = EEA' = EA' = A'_E,\; A'_E E = A'EE = A'E = A'_E$$

and if $B \epsilon \mathbf{M}$ then

$$EBEA'_E = EBA'_E = EBEA' = A'EBE = A'_E BE = A'_E EBE\,.$$

Let us now consider the sufficiency. Assume therefore $E\bar{A} = \bar{A}E = \bar{A}$ and that $\bar{A}$ commutes with all EBE, $B \epsilon \mathbf{M}$.

As $\bar{A}$ is bounded, there exists an $\alpha > 0$ so that $\alpha^2 \cdot 1 - \bar{A}^*\bar{A}$ is (semi-) definite, and then there exists a unique (semi-) definite $\bar{C}$ with $\bar{C}^2 = \alpha^2 \cdot 1 - \bar{A}^*\bar{A}$. (That is: $\bar{C} = (\alpha^2 \cdot 1 - \bar{A}^*A)^{\frac{1}{2}}$, cf. (21), p. 303.) The EBE, $B \epsilon \mathbf{M}$ commute with $\bar{A}$, therefore with $\bar{A}^*$, $\bar{A}^*\bar{A}$ and $\bar{C}$ too. In particular $E = EEE$ does so.

Assume now $B_1, \cdots, B_i \epsilon \mathbf{M}$, $f_1, \cdots, f_i$ arbitrary. We then form:

$$\left\| \sum_1^i{}_j B_j \bar{C} E f_j \right\|^2 + \left\| \sum_1^i{}_j B_j \bar{A} E f_j \right\|^2$$

$$= \left(\sum_1^i{}_j B_j \bar{C} E f_j, \sum_1^i{}_k B_k \bar{C} E f_k \right) + \left(\sum_1^i{}_j B_j \bar{A} E f_j, \sum_1^i{}_k B_k \bar{A} E f_k \right)$$

$$= \left(\sum_{1\,j,k}^i \{ E\bar{C}B_k^* B_j \bar{C}E + E\bar{A}^* B_k^* B_j \bar{A}E \} f_j, f_k \right)$$

$$= \left(\sum_{1\,j,k}^i \{ \bar{C}EB_k^* B_j E\bar{C} + \bar{A}^* EB_k^* B_j E\bar{A} \} f_j, f_k \right)$$

$$= \left(\sum_{1\,j,k}^i \{ EB_k^* B_j E(\bar{C}^2 + \bar{A}^*\bar{A}) \} f_j, f_k \right)$$

$$= \left(\sum_{1}^{i}{}_{j,k} \{EB_k^* B_j E(\alpha^2 1)\} f_j, f_k \right) = \alpha^2 \left(\sum_{1}^{i}{}_j B_j E f_j, \sum_{1}^{i}{}_k B_k E f_k \right)$$

$$= \alpha^2 \left\| \sum_{1}^{i}{}_j B_j E f_j \right\|^2,$$

and therefore $\| \sum_{j=1}^{i} B_j \bar{A} E f_j \| \leqq \alpha \| \sum_1^i B_j E f_j \|$.

Thus $\sum_1^i B_j E f_j = 0$ implies $\sum_{j=1}^{i} B_j \bar{A} E f_j = 0$ and so, owing to the linearity, $\overline{A}$ $(\sum_{j=1}^{i} B_j E f_j) = \sum_{j=1}^{i} B_j \bar{A} E f_j (i = 0, 1, 2, \cdots ; B_j \epsilon \mathbf{M}, f_j$ arbitrary for $j = 1, \cdots, i)$ defines a one-valued linear operator $\overline{A}$. We can now write $\| \overline{A} f \| \leqq \alpha \| f \|$ so that $\overline{A}$ is continuous. Therefore we can form the closure of $\overline{A}$, $\widetilde{\overline{A}}$. This will be one-valued, again fulfilling $\| \widetilde{\overline{A}} f \| \leqq \alpha \| f \|$ and Domain $\widetilde{\overline{A}}'$ = [Domain $\overline{A}$]. (All this results by direct application of the criterium of (21), p. 300).

If $B \epsilon \mathbf{M}$ then $\overline{A}(B(\sum_{j=1}^{i} B_j E f_j) = \overline{A}(\sum_1^i BB_j E f_j) = \sum_1^i (BB_j)\bar{A} E f_j = B(\sum_{j=1}^{i} B_j \bar{A} E f_j) = B(\overline{A}(\sum_{j=1}^{i} B_j E f_j)$ so $\overline{A} \eta \mathbf{M}$ (cf. Lemma 4.2.2). Therefore $\widetilde{\overline{A}} \eta \mathbf{M}$ and $\mathfrak{M}_0 =$ Domain $\widetilde{\overline{A}} \eta \mathbf{M}'$. Thus $F_0 = P_{\mathfrak{M}_0} \epsilon \mathbf{M}'$. Note that Domain $\widetilde{\overline{A}} \supset$ Domain $\overline{A} \supset \mathfrak{M}$, if $E = P_{\mathfrak{M}}$ (put $i = 1, B_1 = 1$), so $F_0 \geqq E$.

Our definition of F_0 shows, that $\widetilde{\overline{A}} F_0$ is everywhere defined. It is bounded too, and as $\widetilde{\overline{A}} \eta \mathbf{M}', F_0 \epsilon \mathbf{M}'$, therefore $\widetilde{\overline{A}} F_0 \eta \mathbf{M}'$. These facts imply together $\overline{A} F_0 \epsilon \mathbf{M}'$. Put $A' = \overline{A} F_0$.

Now $\widetilde{\overline{A}} E f = \bar{A} E f$ (put $i = 1$; $B_1 = 1$ in the definition), and so $\widetilde{\overline{A}} E = \bar{A} E = \bar{A}$. Therefore $A' E = \widetilde{\overline{A}} F_0 \cdot E = \widetilde{\overline{A}} E = \bar{A}$. In other words: $EA' = A'E = A'_E = \bar{A}$, completing the proof.

We can now carry out the constructions which are necessary for our "decomposition."

Definition 11.3.1. Let $\mathbf{M}$ be a ring which contains 1 and $\mathfrak{M}$ a closed linear set $\neq (0)$. Consider those operators $A \epsilon \mathbf{M}$ which are reduced by $\mathfrak{M}$ and form their parts in $\mathfrak{M}$, $A_{(\mathfrak{M})}$ (cf. (16), p. 78). These are bounded operators in $\mathfrak{M}$. Denote the set of all these $A_{(\mathfrak{M})}$ ($A \epsilon \mathbf{M}$ and reduced by $\mathfrak{M}$) by $\mathbf{M}_{(\mathfrak{M})}$.

Lemma 11.3.2. *Let* $\mathbf{M}$ *be a ring which contains* 1 *and* $\mathfrak{M}$ *a closed linear set* $\neq (0)$, $\mathfrak{M} \eta \mathbf{M}, E = P_{\mathfrak{M}}$. *Then* $(\mathbf{M}_{(\mathfrak{M})})' = \mathbf{M}'_{(\mathfrak{M})}$. (These are rings of operators in the space $\mathfrak{M}$.)

Proof: An operator A_0 in $\mathfrak{M}$ is at the same time an operator in $\mathfrak{H}$ but its domain is $\subset \mathfrak{M}$. If A_0 is bounded, then $A_0 E$ is everywhere defined in $\mathfrak{H}$ and so (being bounded) $\epsilon \mathbf{B}$. As its range is $\subset \mathfrak{M}$, $E(A_0 E) = A_0 E$. Obviously $(A_0 E)E = A_0 E$ so $A_0 E$ commutes with $E = P_{\mathfrak{M}}$ and is reduced by $\mathfrak{M}$. Its parts in $\mathfrak{M}$, $\mathfrak{H} - \mathfrak{M}$ are A_0, 0 resp.

Thus A_0, B_0 (in $\mathfrak{M}$) commute if and only if $A_0 E$, $B_0 E$ (in $\mathfrak{H}$) commute. $A_0 \epsilon (\mathbf{M}_{(\mathfrak{M})})'$ means therefore, that $A_0 E$ commutes with every $B_0 E$, $B_0 \epsilon \mathbf{M}_{(\mathfrak{M})}$.

The statement on B_0 means $B_0 = B_{(\mathfrak{M})}$, $B \in \mathbf{M}$ but then

$$EBE = EB = BE = B_0E.$$

(Note, that B_0E differs in general from $EB_0 = B_0$, their domains being $\mathfrak{H}$ resp. $\mathfrak{M}$). So A_0E must commute with every EBE where $B \in \mathbf{M}$, B commutes with E. Replacing B by EBE (which commutes with E and leaves EBE unaltered) shows that any $B \in \mathbf{M}$ will do. Now Lemma 11.3.1 applies to A_0E: $A_0E = EA' = A'E$ for some $A' \in \mathbf{M}'$. This means, $A_0 = (A_0E)_{(\mathfrak{M})} = A'_{\mathfrak{M}}$, $A_0 \in \mathbf{M}'_{(\mathfrak{M})}$.

Conversely: If $A_0 \in \mathbf{M}'_{(\mathfrak{M})}$, $B_0 \in \mathbf{M}_{(\mathfrak{M})}$ then $A_0 = A'_{(\mathfrak{M})}$, $B_0 = B_{(\mathfrak{M})}$ where $A' \in \mathbf{M}'$, $B \in \mathbf{M}$ so A', B commute, and A_0, B_0 too. So $A_0 \in \mathbf{M}'_{(\mathfrak{M})}$ implies $A_0 \in (\mathbf{M}_{(\mathfrak{M})})'$. So we have completely proved $(\mathbf{M}_{(\mathfrak{M})})' = \mathbf{M}'_{(\mathfrak{M})}$.

Lemma 11.3.3. *Let* $\mathbf{M}$, $\mathfrak{M}$, E *be as above. Consider the correspondence* $X \rightleftarrows X_{(\mathfrak{M})}$. *This is a one-to-one mapping and even algebraic ring-isomorphism of the following rings on each other:*

(i) *At any rate of the ring of all* $A \in \mathbf{M}$ *with* $EA = AE = A$ *on all* $\mathbf{M}_{(\mathfrak{M})}$.

(ii) *If* $\mathbf{M}$ *is a factor, of all* $\mathbf{M}'$ *on all* $\mathbf{M}'_{(\mathfrak{M})}$.

Proof: The invariance of αA, A^*, $A + B$, AB is obvious, only the one to one character must therefore be proved.

Ad (i): If $A_0 \in \mathbf{M}_{(\mathfrak{M})}$, form the A_0E from the proof of Lemma 11.3.2. If $A_0 = A_{(\mathfrak{M})}$, $A \in \mathbf{M}$, then as we saw, $A_0E = EAE$, so $A_0E \in \mathbf{M}$ too; and besides $A_0 = (A_0E)_{(\mathfrak{M})}$, $E(A_0E) = (A_0E)E = A_0E$. Furthermore $A_0 = (A_0E)_{(\mathfrak{M})}$ and A_0E determine each other uniquely. This completes the proof.

Ad (ii): Every $A' \in \mathbf{M}'$ commutes with $E = P_{\mathfrak{M}} \in \mathbf{M}$ so it is reduced by $\mathfrak{M}$ and $A'_{(\mathfrak{M})}$ can be formed. $A'_{(\mathfrak{M})} = B'_{(\mathfrak{M})}$ means that $A'f = B'f$ for all $f \in \mathfrak{M}$, that is for all $f = Eg$ so $A'E = B'E$. By the corollary to Theorem III this implies $A' = B'$ as $E \neq 0$. This completes the proof.

Lemma 11.3.4. *Let* $\mathbf{M}$, $\mathfrak{M}$, E *be as above. If* $\mathbf{M}$ *is a factor then* $\mathbf{M}_{(\mathfrak{M})}$ *is one too.*

Proof: Apply Lemma 11.3.2:

$$\mathbf{M}_{(\mathfrak{M})} \cdot (\mathbf{M}_{(\mathfrak{M})})' = \mathbf{M}_{(\mathfrak{M})} \cdot \mathbf{M}'_{(\mathfrak{M})} = (\mathbf{M} \cdot \mathbf{M}')_{(\mathfrak{M})} = (\alpha\, 1)_{(\mathfrak{M})} = (\alpha\, 1_{(\mathfrak{M})})$$

and $1_{(\mathfrak{M})}$ is the unity operator in $\mathfrak{M}$.

Lemma 11.3.5. *Let* $\mathbf{M}$ *be a factor,* $\mathfrak{M}$, E *as above. Then we have:*

(i) *If* F *runs over all projections* $F \in \mathbf{M}$, $F \leqq E$ (*cf.* (16), *p.* 76) *then* $F_{(\mathfrak{M})}$ *runs over all projections* $\in \mathbf{M}_{(\mathfrak{M})}$. $F_{(\mathfrak{M})} \sim G_{(\mathfrak{M})}$ $(\cdots \mathbf{M}_{(\mathfrak{M})})$, $(F, G \in \mathbf{M}, \leqq E)$, *is equivalent to* $F \sim G$ $(\cdots \mathbf{M})$.

(ii) *If* F' *runs over all projections* $F' \in \mathbf{M}'$ *then* $F'_{(\mathfrak{M})}$ *runs over all projections* $\in \mathbf{M}'_{(\mathfrak{M})}$. $F'_{(\mathfrak{M})} \sim G'_{(\mathfrak{M})}$ $(\cdots \mathbf{M}'_{(\mathfrak{M})})$, $(F', G' \in \mathbf{M}')$, *is equivalent to* $F' \sim G'$ $(\cdots \mathbf{M}')$.

Proof: Ad (i): The first part is a restatement of Lemma 11.3.3, (i) as $EF = FE = F$ means $F \leqq E$ (cf. (16), p. 76). As to the second part, note that $F \sim G$ $(\cdots \mathbf{M})$ means $F = U^*U$, $G = UU^*$ for some $U \in \mathbf{M}$ (cf. Lemma 4.3.1 and Definition 6.1.1) and $F_{(\mathfrak{M})} \sim G_{(\mathfrak{M})}$ $(\cdots \mathbf{M}_{(\mathfrak{M})})$ means similarly

$$F_{(\mathfrak{M})} = (U_{(\mathfrak{M})})^*\, U_{(\mathfrak{M})},\ G_{(\mathfrak{M})} = U_{(\mathfrak{M})}\, (U_{(\mathfrak{M})})^*$$

(cf. as above) for some $U \in \mathbf{M}$, $EU = UE = U$ (cf. Lemma 11.3.3, (i)). The

latter equation means again $F = U^*U$, $G = UU^*$ (by Lemma 11.3.3, (i), owing to the algebraic ring-isomorphism), so that all we need to prove is that the U of the first case fulfill automatically $EU = UE = U$. As the initial and final sets of that U have the projections $F, G \leqq E$ they are both $\subset \mathfrak{M}$ and this implies $EU = UE = U$.

Ad (ii): This follows immediately from Lemma 8.5.1 as $\mathbf{M}'$ and $\mathbf{M}'_{(\mathfrak{M})}$ are algebraically ring-isomorphic.

Lemma 11.3.6. *Let* $\mathbf{M}$ *be a factor,* $\mathfrak{M}$, E *as above. Let* $D_{\mathbf{M}}(F)$, $D_{\mathbf{M}'}(F')$ ($F \in \mathbf{M}$ *resp.* $F' \in \mathbf{M}'$) *be relative dimension functions in* $\mathbf{M}$ *resp.* $\mathbf{M}'$. *Then* $D^{(\mathfrak{M})}_{\mathbf{M}}(F_{(\mathfrak{M})}) = D_{\mathbf{M}}(F)$ *and* $D^{(\mathfrak{M})}_{\mathbf{M}'}(F'_{(\mathfrak{M})}) = D_{\mathbf{M}'}(F')$ *are relative dimension functions in* $\mathbf{M}_{(\mathfrak{M})}$ *resp.* $\mathbf{M}'_{(\mathfrak{M})}$.

Proof: Owing to Definition 9.1.1 this follows immediately from Lemma 11.3.5.

Lemma 11.3.7. *Let* $\mathbf{M}$ *be a factor,* $\mathfrak{M}$, E *as above. Let*

$$D_{\mathbf{M}}(F),\ D_{\mathbf{M}'}(F'),\ D^{(\mathfrak{M})}_{\mathbf{M}}(F),\ D^{(\mathfrak{M})}_{\mathbf{M}'}(F')$$

be the relative dimension functions of Lemma 11.3.6 *and* Δ, Δ', $\Delta_{(\mathfrak{M})}$, $\Delta'_{(\mathfrak{M})}$ *their resp. ranges. Then* $\Delta_{(\mathfrak{M})}$ *is the set of all* $\alpha \in \Delta, \alpha \leqq D_{\mathbf{M}}(E)$ *while* $\Delta'_{(\mathfrak{M})} = \Delta'$.

Proof: $\Delta_{(\mathfrak{M})}$ is by Lemma 11.3.5, (i) and Lemma 11.3.6, the set of all $D_{\mathbf{M}}(F)$, $F \leqq E$, that is the set of all $D_{\mathbf{M}}(\mathfrak{N})$, $\mathfrak{N} \subset \mathfrak{M}$. Replacing these $\mathfrak{N}$ by all $\mathfrak{N} \sim \mathfrak{N}' \subset \mathfrak{M}$, that is by all $\mathfrak{N} \lesssim \mathfrak{M}$, obviously does not affect the set of their $D_{\mathbf{M}}(\mathfrak{N})$. In other words: we have all $D_{\mathbf{M}}(\mathfrak{N})$ with $D_{\mathbf{M}}(\mathfrak{N}) \leqq D_{\mathbf{M}}(\mathfrak{M})$ that is all $\alpha \in \Delta$ with $\alpha \leqq D_{\mathbf{M}}(\mathfrak{M}) = D_{\mathbf{M}}(E)$.

$\Delta'_{(\mathfrak{M})} = \Delta_{(\mathfrak{M})}$ follows immediately from Lemma 11.3.5 (ii), and Lemma 11.3.6.

11.4. The treatment of § 11.3 was asymmetric, insofar as the $\mathfrak{M}$ occurring in it was assumed to be $\eta\ \mathbf{M}$. We will now obtain symmetric results by considering simultaneously an $\mathfrak{M}\ \eta\ \mathbf{M}$ and an $\mathfrak{M}'\ \eta\ \mathbf{M}'$ and applying the preceding Lemmas twice.

Lemma 11.4.1. *Let* $\mathbf{M}$ *be a factor;* $\mathfrak{M}$, $\mathfrak{M}'$ *closed linear sets both* $\neq (0)$; $\mathfrak{M}\ \eta\ \mathbf{M}$, $\mathfrak{M}'\ \eta\ \mathbf{M}'$, $E = P_{\mathfrak{M}}$, $E' = P_{\mathfrak{M}'}$. *Perform the operations of Definition* 11.3.1 *for the closed linear set* $\mathfrak{M}\cdot\mathfrak{M}'$. *The correspondence* $X \rightleftarrows X_{(\mathfrak{M}\cdot\mathfrak{M}')}$ *is then a one-to-one mapping and even an algebraic ring-isomorphism of the following rings on each other:*

(i) *Of the ring of all* $A \in \mathbf{M}$ *with* $EA = AE = A$ *on all* $\mathbf{M}_{(\mathfrak{M}\cdot\mathfrak{M}')}$.

(ii) *Of the ring of all* $A' \in \mathbf{M}'$ *with* $E'A' = A'E' = A'$ *on all* $\mathbf{M}'_{(\mathfrak{M}\cdot\mathfrak{M}')}$.

Besides we have

(iii) $(\mathbf{M}_{(\mathfrak{M}\cdot\mathfrak{M}')})' = \mathbf{M}'_{(\mathfrak{M}\cdot\mathfrak{M}')}$, $\mathbf{M}_{(\mathfrak{M}\cdot\mathfrak{M}')}$ *being a factor.*

(*These are rings of operators in the space* $\mathfrak{M}\cdot\mathfrak{M}' \neq (0)$.)

Proof: As $E \in \mathbf{M}$, $E' \in \mathbf{M}'$ therefore E, E' commute, and so $EE' = P_{\mathfrak{M}\cdot\mathfrak{M}'}$. The Corollary to Theorem III gives, owing to E, $E' \neq 0$ that $EE' \neq 0$, $\mathfrak{M}\cdot\mathfrak{M}' \neq (0)$. As $E' \in \mathbf{M}'$ therefore $E'_{(\mathfrak{M})} \in \mathbf{M}'_{(\mathfrak{M})}$ but $E'_{(\mathfrak{M})} = EE' = P_{\mathfrak{M}\cdot\mathfrak{M}'}$ so $\mathfrak{M}\cdot\mathfrak{M}'\ \eta\ \mathbf{M}'_{(\mathfrak{M})}$.

Now (i) results by applying first Lemma 11.3.3 (i) to $\mathbf{M}$, $\mathbf{M}'$, $\mathfrak{M}$ and then Lemma 11.3.3 (ii) to $\mathbf{M}'_{(\mathfrak{M})}$, $\mathbf{M}_{(\mathfrak{M})}$, $\mathfrak{M}\cdot\mathfrak{M}'$. (ii) follows from (i) by interchanging $\mathbf{M}$, $\mathfrak{M}$ with $\mathbf{M}'$, $\mathfrak{M}'$. (iii) follows by applying Lemmas 11.3.2 and 11.3.4 first to $\mathbf{M}$, $\mathbf{M}'$, $\mathfrak{M}$ and then to $\mathbf{M}'_{(\mathfrak{M})}$, $\mathbf{M}_{(\mathfrak{M})}$, $\mathfrak{M}\cdot\mathfrak{M}'$.

Lemma 11.4.2. *Let* $\mathbf{M}, \mathfrak{M}, \mathfrak{M}', E, E'$ *be as above. Then we have:*

(i) *If* F *runs over all projections* $F \in \mathbf{M}$, $F \leqq E$ *then* $F_{(\mathfrak{M}\cdot\mathfrak{M}')}$ *runs over all projections* $\in \mathbf{M}_{(\mathfrak{M}\cdot\mathfrak{M}')}$. *If* $D_{\mathbf{M}}(F)$ *is a relative dimension function in* $\mathbf{M}$ *then* $D_{\mathbf{M}}^{(\mathfrak{M}\cdot\mathfrak{M}')}(F_{(\mathfrak{M}\cdot\mathfrak{M}')}) = D_{\mathbf{M}}(F)$ *is one in* $\mathbf{M}_{(\mathfrak{M}\cdot\mathfrak{M}')}$. *Denote their ranges by* Δ *resp.* $\Delta_{(\mathfrak{M}\cdot\mathfrak{M}')}$ *then* $\Delta_{(\mathfrak{M}\cdot\mathfrak{M}')}$ *is the set of all* $\alpha \in \Delta, \alpha \leqq D_{\mathbf{M}}(E)$.

(ii) *If* F' *runs over all projections* $F' \in \mathbf{M}'$, $F' \leqq E'$, *then* $F'_{(\mathfrak{M}\cdot\mathfrak{M}')}$ *runs over all projections* $\in \mathbf{M}'_{(\mathfrak{M}\cdot\mathfrak{M}')}$. *If* $D_{\mathbf{M}'}(F')$ *is a relative dimension function in* $\mathbf{M}'$, *then* $D_{\mathbf{M}'}^{(\mathfrak{M}\cdot\mathfrak{M}')}(F'_{\mathfrak{M}\cdot\mathfrak{M}'}) = D_{\mathbf{M}'}(F')$ *is one in* $\mathbf{M}'_{(\mathfrak{M}\cdot\mathfrak{M}')}$. *Denote their ranges by* Δ' *resp.* $\Delta'_{(\mathfrak{M}\cdot\mathfrak{M})}$ *then* $\Delta'_{(\mathfrak{M}\cdot\mathfrak{M}')}$ *is the set of all* $\alpha' \in \Delta', \alpha' \leqq D_{\mathbf{M}'}(E')$.

Proof: (i) results by applying Lemmas 11.3.5 ((i) resp. (ii)), 11.3.6 and 11.3.7 first to $\mathbf{M}$, $\mathbf{M}'$, $\mathfrak{M}$ and then to $\mathbf{M}'_{(\mathfrak{M})}$, $\mathbf{M}_{(\mathfrak{M})}$ $\mathfrak{M}\cdot\mathfrak{M}'$. (ii) follows from (i) by interchanging $\mathbf{M}, \mathfrak{M}$ with $\mathbf{M}', \mathfrak{M}'$.

We see that the coupled factorisation $\mathbf{M}, \mathbf{M}'$ in $\mathfrak{H}$ generates a coupled factorisation $\mathbf{M}_{(\mathfrak{M}\cdot\mathfrak{M}')}, \mathbf{M}'_{(\mathfrak{M}\cdot\mathfrak{M}')}$ in $\mathfrak{M}\cdot\mathfrak{M}' \neq (0)$. Lemma 11.4.2 gives us the means to determine the classes of the latter factors in terms of those of the former ones.

Lemma 11.4.3. *Let* $\mathbf{M}, \mathbf{M}', \mathfrak{M}, \mathfrak{M}', E, E'$ *be as above. Then* $\mathbf{M}_{(\mathfrak{M}\cdot\mathfrak{M}')}, \mathbf{M}'_{(\mathfrak{M}\cdot\mathfrak{M}')}$ *belong to the same ones of the classes* (I), (II), (III), *as* $\mathbf{M}, \mathbf{M}'$ (*cf. Theorem X*). *In particular:*

(i) *If* $\mathbf{M}, \mathbf{M}'$ *are in class* (I), *and if* $D_{\mathbf{M}}(F), D_{\mathbf{M}'}(F')$ *are given in their standard normalisations, that the same is true for* $D_{\mathbf{M}}^{(\mathfrak{M}\cdot\mathfrak{M}')}(F_{\mathfrak{M}\cdot\mathfrak{M}'}), D_{\mathbf{M}'}^{(\mathfrak{M}\cdot\mathfrak{M}')}(F_{(\mathfrak{M}\cdot\mathfrak{M}')})$. *We have class* (I_n) *or* (I_∞) *for* $\mathbf{M}_{(\mathfrak{M}\cdot\mathfrak{M}')}$ (*resp.* $\mathbf{M}'_{(\mathfrak{M}\cdot\mathfrak{M}')}$) *corresponding to whether* $\mathfrak{M}$ (*resp.* $\mathfrak{M}'$) *is finite-n-dimensional or infinite-dimensional.*

(ii) *If* $\mathbf{M}, \mathbf{M}'$ *are in class* (II), *then* $\mathbf{M}_{(\mathfrak{M}\cdot\mathfrak{M}')}$ (*resp.* $\mathbf{M}'_{(\mathfrak{M}\cdot\mathfrak{M}')}$) *are in class* (II_1) *or* (II_∞) *corresponding to whether* $\mathfrak{M}$ (*resp.* $\mathfrak{M}'$) *is finite or infinite. So if* $\mathfrak{M}, \mathfrak{M}'$ *are both finite, the* C *of Theorem X* (*referred to the standard normalisations*) *has an invariant meaning. If we then vary* $\mathfrak{M}, \mathfrak{M}'$ *then* C *will be proportional to* $D_{\mathbf{M}}(\mathfrak{M})/D_{\mathbf{M}'}(\mathfrak{M})$.

(iii) *If* $\mathbf{M}, \mathbf{M}'$ *are in class* (III), *no further comment is necessary.*

Proof: Let first $\mathbf{M}, \mathbf{M}'$ be in class (I). Let $D_{\mathbf{M}}(F), D_{\mathbf{M}'}(F')$ be in standard normalisation. Put $D_{\mathbf{M}}(E) = m$, $D_{\mathbf{M}'}(E') = n$ then $m, n > 0$, that is $m, n, = 1, 2, \cdots, \infty$. So the ranges of $D_{\mathbf{M}}^{(\mathfrak{M}\cdot\mathfrak{M}')}(F_{(\mathfrak{M}\cdot\mathfrak{M}')})$ and $D_{\mathbf{M}'}^{(\mathfrak{M}\cdot\mathfrak{M}')}(F'_{(\mathfrak{M}\cdot\mathfrak{M}')})$ are $0, 1, \cdots, m$ resp. $0, 1, \cdots, n$. Thus $\mathbf{M}_{(\mathfrak{M}\cdot\mathfrak{M}')}, \mathbf{M}'_{(\mathfrak{M}\cdot\mathfrak{M}')}$ belong to class (I), and (i) is true.

Let second, $\mathbf{M}, \mathbf{M}'$ be in class (II). Choose $D_{\mathbf{M}}(F)$ and then $D_{\mathbf{M}'}(F')$ in any normalisations. Put $D_{\mathbf{M}}(E) = \alpha_0$, $D_{\mathbf{M}'}(E') = \alpha_0'$, then $\alpha_0, \alpha_0' > 0$ and the ranges of $D_{\mathbf{M}}^{(\mathfrak{M}\cdot\mathfrak{M}')}(F_{(\mathfrak{M}\cdot\mathfrak{M}')})$ and $D_{\mathbf{M}}^{(\mathfrak{M}\cdot\mathfrak{M}')}(F'_{(\mathfrak{M}\cdot\mathfrak{M}')})$ consist of all α with $0 \leqq \alpha \leqq \alpha_0$ resp. of all α' with $0 \leqq \alpha' \leqq \alpha_0'$. Thus $\mathbf{M}_{(\mathfrak{M}\cdot\mathfrak{M}')}, \mathbf{M}'_{(\mathfrak{M}\cdot\mathfrak{M}')}$ belong to class (II), and all parts of (ii), except those referring to C are proved.

Concerning C (cf. Theorem X) observe this: $\mathfrak{M}_f^{\mathbf{M}}{}_{(\mathfrak{M}\cdot\mathfrak{M}')}$ for $f \in \mathfrak{M}\cdot\mathfrak{M}'$ is $[A_{(\mathfrak{M}\cdot\mathfrak{M}')}f;\ A \in \mathbf{M}]$, that is $[AEE'f,\ A \in \mathbf{M},\ A$ commutes with $E]$. Then we may write as well $EAEE'f$ and replacing A by EAE (this is $\in \mathbf{M}$, commutes with E, and leaves $EAEE'$ unaltered), we can admit all $A \in \mathbf{M}$. Now $EAEE'f = EAf$ as $f \in \mathfrak{M}\cdot\mathfrak{M}'$ so we have $[EAf;\ A \in \mathbf{M}] = [Eg,\ g \in \mathfrak{M}_f^{\mathbf{M}}] = [EE_f^{\mathbf{M}}h,\ h \in \mathfrak{H}]$. But $E \in \mathbf{M}$, $E_f^{\mathbf{M}} \in \mathbf{M}'$ commute, so $E\cdot E_f^{\mathbf{M}} = P_{\mathfrak{M}\cdot\mathfrak{M}_f^{\mathbf{M}}}$; therefore $\mathfrak{M}_f^{\mathbf{M}}{}_{(\mathfrak{M}\cdot\mathfrak{M}')} =$

latter equation means again $F = U^*U$, $G = UU^*$ (by Lemma 11.3.3, (i), owing to the algebraic ring-isomorphism), so that all we need to prove is that the U of the first case fulfill automatically $EU = UE = U$. As the initial and final sets of that U have the projections $F, G \leqq E$ they are both $\subset \mathfrak{M}$ and this implies $EU = UE = U$.

Ad (ii): This follows immediately from Lemma 8.5.1 as $\mathbf{M}'$ and $\mathbf{M}'_{(\mathfrak{M})}$ are algebraically ring-isomorphic.

Lemma 11.3.6. *Let* $\mathbf{M}$ *be a factor,* $\mathfrak{M}$, E *as above. Let* $D_{\mathbf{M}}(F)$, $D_{\mathbf{M}'}(F')$ ($F \in \mathbf{M}$ *resp.* $F' \in \mathbf{M}'$) *be relative dimension functions in* $\mathbf{M}$ *resp.* $\mathbf{M}'$. *Then* $D^{(\mathfrak{M})}_{\mathbf{M}}(F_{(\mathfrak{M})}) = D_{\mathbf{M}}(F)$ *and* $D^{(\mathfrak{M})}_{\mathbf{M}'}(F'_{(\mathfrak{M})}) = D_{\mathbf{M}'}(F')$ *are relative dimension functions in* $\mathbf{M}_{(\mathfrak{M})}$ *resp.* $\mathbf{M}'_{(\mathfrak{M})}$.

Proof: Owing to Definition 9.1.1 this follows immediately from Lemma 11.3.5.

Lemma 11.3.7. *Let* $\mathbf{M}$ *be a factor,* $\mathfrak{M}$, E *as above. Let*

$$D_{\mathbf{M}}(F),\ D_{\mathbf{M}'}(F'),\ D^{(\mathfrak{M})}_{\mathbf{M}}(F),\ D^{(\mathfrak{M})}_{\mathbf{M}'}(F')$$

be the relative dimension functions of Lemma 11.3.6 *and* Δ, Δ', $\Delta_{(\mathfrak{M})}$, $\Delta'_{(\mathfrak{M})}$ *their resp. ranges. Then* $\Delta_{(\mathfrak{M})}$ *is the set of all* $\alpha \in \Delta, \alpha \leqq D_{\mathbf{M}}(E)$ *while* $\Delta'_{(\mathfrak{M})} = \Delta'$.

Proof: $\Delta_{(\mathfrak{M})}$ is by Lemma 11.3.5, (i) and Lemma 11.3.6, the set of all $D_{\mathbf{M}}(F)$, $F \leqq E$, that is the set of all $D_{\mathbf{M}}(\mathfrak{N})$, $\mathfrak{N} \subset \mathfrak{M}$. Replacing these $\mathfrak{N}$ by all $\mathfrak{N} \sim \mathfrak{N}' \subset \mathfrak{M}$, that is by all $\mathfrak{N} \precsim \mathfrak{M}$, obviously does not affect the set of their $D_{\mathbf{M}}(\mathfrak{N})$. In other words: we have all $D_{\mathbf{M}}(\mathfrak{N})$ with $D_{\mathbf{M}}(\mathfrak{N}) \leqq D_{\mathbf{M}}(\mathfrak{M})$ that is all $\alpha \in \Delta$ with $\alpha \leqq D_{\mathbf{M}}(\mathfrak{M}) = D_{\mathbf{M}}(E)$.

$\Delta'_{(\mathfrak{M})} = \Delta_{(\mathfrak{M})}$ follows immediately from Lemma 11.3.5 (ii), and Lemma 11.3.6.

11.4. The treatment of § 11.3 was asymmetric, insofar as the $\mathfrak{M}$ occurring in it was assumed to be $\eta\, \mathbf{M}$. We will now obtain symmetric results by considering simultaneously an $\mathfrak{M}\ \eta\ \mathbf{M}$ and an $\mathfrak{M}'\ \eta\ \mathbf{M}'$ and applying the preceding Lemmas twice.

Lemma 11.4.1. *Let* $\mathbf{M}$ *be a factor;* $\mathfrak{M}$, $\mathfrak{M}'$ *closed linear sets both* $\succnsim (0)$; $\mathfrak{M}\ \eta\ \mathbf{M}$, $\mathfrak{M}'\ \eta\ \mathbf{M}'$, $E = P_{\mathfrak{M}}$, $E' = P_{\mathfrak{M}'}$. *Perform the operations of Definition* 11.3.1 *for the closed linear set* $\mathfrak{M} \cdot \mathfrak{M}'$. *The correspondence* $X \rightleftarrows X_{(\mathfrak{M} \cdot \mathfrak{M}')}$ *is then a one-to-one mapping and even an algebraic ring-isomorphism of the following rings on each other:*

(i) *Of the ring of all* $A \in \mathbf{M}$ *with* $EA = AE = A$ *on all* $\mathbf{M}_{(\mathfrak{M} \cdot \mathfrak{M}')}$.

(ii) *Of the ring of all* $A' \in \mathbf{M}'$ *with* $E'A' = A'E' = A'$ *on all* $\mathbf{M}'_{(\mathfrak{M} \cdot \mathfrak{M}')}$.

Besides we have

(iii) $(\mathbf{M}_{(\mathfrak{M} \cdot \mathfrak{M}')})' = \mathbf{M}'_{(\mathfrak{M} \cdot \mathfrak{M}')}$, $\mathbf{M}_{(\mathfrak{M} \cdot \mathfrak{M}')}$ *being a factor.*

(*These are rings of operators in the space* $\mathfrak{M} \cdot \mathfrak{M}' \succnsim (0)$.)

Proof: As $E \in \mathbf{M}$, $E' \in \mathbf{M}'$ therefore E, E' commute, and so $EE' = P_{\mathfrak{M} \cdot \mathfrak{M}'}$. The Corollary to Theorem III gives, owing to E, $E' \succnsim 0$ that $EE' \succnsim 0$, $\mathfrak{M} \cdot \mathfrak{M}' \succnsim (0)$. As $E' \in \mathbf{M}'$ therefore $E'_{(\mathfrak{M})} \in \mathbf{M}'_{(\mathfrak{M})}$ but $E'_{(\mathfrak{M})} = EE' = P_{\mathfrak{M} \cdot \mathfrak{M}'}$ so $\mathfrak{M} \cdot \mathfrak{M}'\ \eta\ \mathbf{M}'_{(\mathfrak{M})}$.

Now (i) results by applying first Lemma 11.3.3 (i) to $\mathbf{M}$, $\mathbf{M}'$, $\mathfrak{M}$ and then Lemma 11.3.3 (ii) to $\mathbf{M}'_{(\mathfrak{M})}$, $\mathbf{M}_{(\mathfrak{M})}$, $\mathfrak{M} \cdot \mathfrak{M}'$. (ii) follows from (i) by interchanging $\mathbf{M}$, $\mathfrak{M}$ with $\mathbf{M}'$, $\mathfrak{M}'$. (iii) follows by applying Lemmas 11.3.2 and 11.3.4 first to $\mathbf{M}$, $\mathbf{M}'$, $\mathfrak{M}$ and then to $\mathbf{M}'_{(\mathfrak{M})}$, $\mathbf{M}_{(\mathfrak{M})}$, $\mathfrak{M} \cdot \mathfrak{M}'$.

Lemma 11.4.2. *Let* $\mathbf{M}, \mathfrak{M}, \mathfrak{M}', E, E'$ *be as above. Then we have:*

(i) *If* F *runs over all projections* $F \in \mathbf{M}$, $F \leqq E$ *then* $F_{(\mathfrak{M}\cdot\mathfrak{M}')}$ *runs over all projections* $\in \mathbf{M}_{(\mathfrak{M}\cdot\mathfrak{M}')}$. *If* $D_{\mathbf{M}}(F)$ *is a relative dimension function in* $\mathbf{M}$ *then* $D_{\mathbf{M}}^{(\mathfrak{M}\cdot\mathfrak{M}')}(F_{(\mathfrak{M}\cdot\mathfrak{M}')}) = D_{\mathbf{M}}(F)$ *is one in* $\mathbf{M}_{(\mathfrak{M}\cdot\mathfrak{M}')}$. *Denote their ranges by* Δ *resp.* $\Delta_{(\mathfrak{M}\cdot\mathfrak{M}')}$ *then* $\Delta_{(\mathfrak{M}\cdot\mathfrak{M}')}$ *is the set of all* $\alpha \in \Delta, \alpha \leqq D_{\mathbf{M}}(E)$.

(ii) *If* F' *runs over all projections* $F' \in \mathbf{M}'$, $F' \leqq E'$, *then* $F'_{(\mathfrak{M}\cdot\mathfrak{M}')}$ *runs over all projections* $\in \mathbf{M}'_{(\mathfrak{M}\cdot\mathfrak{M}')}$. *If* $D_{\mathbf{M}'}(F')$ *is a relative dimension function in* $\mathbf{M}'$, *then* $D_{\mathbf{M}'}^{(\mathfrak{M}\cdot\mathfrak{M}')}(F'_{\mathfrak{M}\cdot\mathfrak{M}'}) = D_{\mathbf{M}'}(F')$ *is one in* $\mathbf{M}'_{(\mathfrak{M}\cdot\mathfrak{M}')}$. *Denote their ranges by* Δ' *resp.* $\Delta'_{(\mathfrak{M}\cdot\mathfrak{M})}$ *then* $\Delta'_{(\mathfrak{M}\cdot\mathfrak{M}')}$ *is the set of all* $\alpha' \in \Delta', \alpha' \leqq D_{\mathbf{M}'}(E')$.

Proof: (i) results by applying Lemmas 11.3.5 ((i) resp. (ii)), 11.3.6 and 11.3.7 first to $\mathbf{M}$, $\mathbf{M}'$, $\mathfrak{M}$ and then to $\mathbf{M}'_{(\mathfrak{M})}$, $\mathbf{M}_{(\mathfrak{M})}$ $\mathfrak{M}\cdot\mathfrak{M}'$. (ii) follows from (i) by interchanging $\mathbf{M}, \mathfrak{M}$ with $\mathbf{M}', \mathfrak{M}'$.

We see that the coupled factorisation $\mathbf{M}, \mathbf{M}'$ in $\mathfrak{H}$ generates a coupled factorisation $\mathbf{M}_{(\mathfrak{M}\cdot\mathfrak{M}')}, \mathbf{M}'_{(\mathfrak{M}\cdot\mathfrak{M}')}$ in $\mathfrak{M}\cdot\mathfrak{M}' \neq (0)$. Lemma 11.4.2 gives us the means to determine the classes of the latter factors in terms of those of the former ones.

Lemma 11.4.3. *Let* $\mathbf{M}, \mathbf{M}', \mathfrak{M}, \mathfrak{M}', E, E'$ *be as above. Then* $\mathbf{M}_{(\mathfrak{M}\cdot\mathfrak{M}')}, \mathbf{M}'_{(\mathfrak{M}\cdot\mathfrak{M}')}$ *belong to the same ones of the classes* (I), (II), (III), *as* $\mathbf{M}, \mathbf{M}'$ (*cf. Theorem X*). *In particular:*

(i) *If* $\mathbf{M}, \mathbf{M}'$ *are in class* (I), *and if* $D_{\mathbf{M}}(F), D_{\mathbf{M}'}(F')$ *are given in their standard normalisations, that the same is true for* $D_{\mathbf{M}}^{(\mathfrak{M}\cdot\mathfrak{M}')}(F_{\mathfrak{M}\cdot\mathfrak{M}'}), D_{\mathbf{M}'}^{(\mathfrak{M}\cdot\mathfrak{M}')}(F_{(\mathfrak{M}\cdot\mathfrak{M}')})$. *We have class* (I_n) *or* (I_∞) *for* $\mathbf{M}_{(\mathfrak{M}\cdot\mathfrak{M}')}$ (*resp.* $\mathbf{M}'_{(\mathfrak{M}\cdot\mathfrak{M}')}$) *corresponding to whether* $\mathfrak{M}$ (*resp.* $\mathfrak{M}'$) *is finite-n-dimensional or infinite-dimensional.*

(ii) *If* $\mathbf{M}, \mathbf{M}'$ *are in class* (II), *then* $\mathbf{M}_{(\mathfrak{M}\cdot\mathfrak{M}')}$ (*resp.* $\mathbf{M}'_{(\mathfrak{M}\cdot\mathfrak{M}')}$) *are in class* (II_1) *or* (II_∞) *corresponding to whether* $\mathfrak{M}$ (*resp.* $\mathfrak{M}'$) *is finite or infinite. So if* $\mathfrak{M}, \mathfrak{M}'$ *are both finite, the* C *of Theorem X (referred to the standard normalisations) has an invariant meaning. If we then vary* $\mathfrak{M}, \mathfrak{M}'$ *then* C *will be proportional to* $D_{\mathbf{M}}(\mathfrak{M})/D_{\mathbf{M}'}(\mathfrak{M})$.

(iii) *If* $\mathbf{M}, \mathbf{M}'$ *are in class* (III), *no further comment is necessary.*

Proof: Let first $\mathbf{M}, \mathbf{M}'$ be in class (I). Let $D_{\mathbf{M}}(F), D_{\mathbf{M}'}(F')$ be in standard normalisation. Put $D_{\mathbf{M}}(E) = m$, $D_{\mathbf{M}'}(E') = n$ then $m, n > 0$, that is $m, n, = 1, 2, \cdots, \infty$. So the ranges of $D_{\mathbf{M}}^{(\mathfrak{M}\cdot\mathfrak{M}')}(F_{(\mathfrak{M}\cdot\mathfrak{M}')})$ and $D_{\mathbf{M}'}^{(\mathfrak{M}\cdot\mathfrak{M}')}(F'_{(\mathfrak{M}\cdot\mathfrak{M}')})$ are $0, 1, \cdots, m$ resp. $0, 1, \cdots, n$. Thus $\mathbf{M}_{(\mathfrak{M}\cdot\mathfrak{M}')}, \mathbf{M}'_{(\mathfrak{M}\cdot\mathfrak{M}')}$ belong to class (I), and (i) is true.

Let second, $\mathbf{M}, \mathbf{M}'$ be in class (II). Choose $D_{\mathbf{M}}(F)$ and then $D_{\mathbf{M}'}(F')$ in any normalisations. Put $D_{\mathbf{M}}(E) = \alpha_0$, $D_{\mathbf{M}'}(E') = \alpha_0'$, then $\alpha_0, \alpha_0' > 0$ and the ranges of $D_{\mathbf{M}}^{(\mathfrak{M}\cdot\mathfrak{M}')}(F_{(\mathfrak{M}\cdot\mathfrak{M}')})$ and $D_{\mathbf{M}}^{(\mathfrak{M}\cdot\mathfrak{M}')}(F'_{(\mathfrak{M}\cdot\mathfrak{M}')})$ consist of all α with $0 \leqq \alpha \leqq \alpha_0$ resp. of all α' with $0 \leqq \alpha' \leqq \alpha_0'$. Thus $\mathbf{M}_{(\mathfrak{M}\cdot\mathfrak{M}')}, \mathbf{M}'_{(\mathfrak{M}\cdot\mathfrak{M}')}$ belong to class (II), and all parts of (ii), except those referring to C are proved.

Concerning C (cf. Theorem X) observe this: $\mathfrak{M}_f^{\mathbf{M}}{}_{(\mathfrak{M}\cdot\mathfrak{M}')}$ for $f \in \mathfrak{M}\cdot\mathfrak{M}'$ is $[A_{(\mathfrak{M}\cdot\mathfrak{M}')}f;\ A \in \mathbf{M}]$, that is $[AEE'f,\ A \in \mathbf{M},\ A$ commutes with $E]$. Then we may write as well $EAEE'f$ and replacing A by EAE (this is $\in \mathbf{M}$, commutes with E, and leaves $EAEE'$ unaltered), we can admit all $A \in \mathbf{M}$. Now $EAEE'f = EAf$ as $f \in \mathfrak{M}\cdot\mathfrak{M}'$ so we have $[EAf;\ A \in \mathbf{M}] = [Eg,\ g \in \mathfrak{M}_f^{\mathbf{M}}] = [EE_f^{\mathbf{M}}h,\ h \in \mathfrak{H}]$. But $E \in \mathbf{M}$, $E_f^{\mathbf{M}} \in \mathbf{M}'$ commute, so $E \cdot E_f^{\mathbf{M}} = P_{\mathfrak{M}\cdot\mathfrak{M}_f^{\mathbf{M}}}$; therefore $\mathfrak{M}_f^{\mathbf{M}}{}_{(\mathfrak{M}\cdot\mathfrak{M}')} =$

$\mathfrak{M}\cdot\mathfrak{M}_f^{\mathbf{M}}$, $E_f^{\mathbf{M}(\mathfrak{M}\cdot\mathfrak{M}')} = (E_f^{\mathbf{M}})_{(\mathfrak{M})}$. Besides $f \epsilon \mathfrak{M}'\ \eta\ \mathbf{M}'$ implies $\mathfrak{M}_f^{\mathbf{M}} \subset \mathfrak{M}'$, $\mathfrak{M}\cdot\mathfrak{M}_f^{\mathbf{M}} \subset \mathfrak{M}\cdot\mathfrak{M}'$; therefore we may even write $E_f^{\mathbf{M}(\mathfrak{M}\cdot\mathfrak{M}')} = (E_f^{\mathbf{M}})_{(\mathfrak{M}\cdot\mathfrak{M}')}$. Similarly

$$E_f^{\mathbf{M}'(\mathfrak{M}\cdot\mathfrak{M}')} = (E_f^{\mathbf{M}'})_{(\mathfrak{M}\cdot\mathfrak{M}')}\,.$$

This proves that $D_{\mathbf{M}}^{(\mathfrak{M}\cdot\mathfrak{M}')}(F_{(\mathfrak{M}\cdot\mathfrak{M}')})$, $D_{\mathbf{M}'}^{(\mathfrak{M}\cdot\mathfrak{M}')}(F'_{(\mathfrak{M}\cdot\mathfrak{M}')})$ have the same C as $D_{\mathbf{M}}(F)$, $D_{\mathbf{M}'}(F)$. If $\mathfrak{M}$, $\mathfrak{M}'$ are finite, we must pass to the standard normalisations, that is divide by $D_{\mathbf{M}}^{(\mathfrak{M}\cdot\mathfrak{M}')}(1_{(\mathfrak{M}\cdot\mathfrak{M}')}) = D_{\mathbf{M}}^{(\mathfrak{M}\cdot\mathfrak{M}')}(E_{(\mathfrak{M}\cdot\mathfrak{M}')}) = D_{\mathbf{M}}(E) = D_{\mathbf{M}}(\mathfrak{M})$ resp. $D_{\mathbf{M}'}^{(\mathfrak{M}\cdot\mathfrak{M}')}(1_{(\mathfrak{M}\cdot\mathfrak{M}')}) = D_{\mathbf{M}'}^{(\mathfrak{M}\cdot\mathfrak{M}')}(E_{(\mathfrak{M}\cdot\mathfrak{M}')}) = D_{\mathbf{M}'}(E') = D_{\mathbf{M}'}(\mathfrak{M}')$. This multiplies C by $D_{\mathbf{M}}(\mathfrak{M})/D_{\mathbf{M}'}(\mathfrak{M}')$, proving the rest of (ii).

Let finally $\mathbf{M}$, $\mathbf{M}'$, be in class (III). Then $D_{\mathbf{M}}(F)$, $D_{\mathbf{M}'}(F')$ have both the range $0, \infty$. The ranges of $D_{\mathbf{M}}^{(\mathfrak{M}\cdot\mathfrak{M}')}(F_{(\mathfrak{M}\cdot\mathfrak{M}')})$, $D_{\mathbf{M}'}^{(\mathfrak{M}\cdot\mathfrak{M}')}(F_{(\mathfrak{M}\cdot\mathfrak{M}')})$ must therefore be 0 alone or $0, \infty$. The former is impossible (because $\mathfrak{M}\cdot\mathfrak{M}' \neq (0)$), so we have $0, \infty$ again, that is, $\mathbf{M}_{(\mathfrak{M}\cdot\mathfrak{M}')}$, $\mathbf{M}'_{(\mathfrak{M}\cdot\mathfrak{M}')}$ are in class (III). (iii) is void.

Lemma 11.4.3 show that if case (II) occurs at all, then combinations $\mathbf{M}$, $\mathbf{M}'$ of two cases (II_1) with an arbitrarily prescribed C (cf. Theorem X) exist. In fact: Choose $\mathbf{M}$ of class (II), then $\mathbf{M}'$ is of class (II) too. Choose $D_{\mathbf{M}}(\mathfrak{M})$, $D_{\mathbf{M}'}(\mathfrak{N})$ in such a normalisation that $D_{\mathbf{M}}(\mathfrak{H}) \geqq 1$, $D_{\mathbf{M}'}(\mathfrak{H}) \geqq 1$. Then $\mathfrak{M}\ \eta\ \mathbf{M}$, $\mathfrak{M}'\ \eta\ \mathbf{M}'$ can be chosen with arbitrarily prescribed $D_{\mathbf{M}}(\mathfrak{M}) = \alpha_0$, $D_{\mathbf{M}'}(\mathfrak{M}') = \alpha_0'$ if only $0 \leqq \alpha_0, \alpha_0' \leqq 1$. Thus α_0/α_0' can be prescribed arbitrarily, and with it C. Thus the invariant C exists effectively, as discussed at the end of § 10.2, if case (II) exists at all. That this is the case will be seen in Chapter XIII.

11.5. We now return to the question formulated at the end of §11.2. We will determine the class of $\mathbf{R}(\mathbf{M}, \mathbf{N})$ if $\mathbf{M}$, $\mathbf{N}$ are two factors which commute, $\mathbf{N}$ being of class (I).

Lemma 11.5.1. *Let* $\mathbf{M}, \mathbf{N}$ *be two factors which commute, form the factor* $\mathbf{R}(\mathbf{M}, \mathbf{N})$. *Let* φ *be minimal with respect to* $\mathbf{N}$ (*cf. Definition* 5.1.3). *Then* $\mathfrak{M}_\varphi^{\mathbf{N}'}\ \eta\ \mathbf{N}$ *and so* $\eta\ \mathbf{R}(\mathbf{M}, \mathbf{N})$ *and* $X \rightleftarrows X_{(\mathfrak{M}_\varphi^{\mathbf{N}'})}$ *is an algebraic ring-isomorphism of* $\mathbf{M}$ *and* $(\mathbf{R}(\mathbf{M}, \mathbf{N}))_{(\mathfrak{M}_\varphi^{\mathbf{N}'})}$.

Proof: $\mathfrak{M}_\varphi^{\mathbf{N}'}\ \eta\ \mathbf{N}$ is obvious; as $\mathbf{N} \subset \mathbf{R}(\mathbf{M}, \mathbf{N})$, $\mathbf{N}' \supset (\mathbf{R}(\mathbf{M}, \mathbf{N}))'$ it implies $\mathfrak{M}_\varphi^{\mathbf{N}'}\ \eta\ \mathbf{R}(\mathbf{M}, \mathbf{N})$. So $E_\varphi^{\mathbf{N}'} \epsilon \mathbf{N}$ and $\mathbf{R}(\mathbf{M}, \mathbf{N})$; thus it commutes with every $X \epsilon \mathbf{M}$.

$X \rightleftarrows X_{(\mathfrak{M}_\varphi^{\mathbf{N}'})}$ is an algebraic ring-isomorphism if it is a one to one mapping; besides it is a one to one mapping of all $Y \epsilon \mathbf{R}(\mathbf{M}, \mathbf{N})$ with $E_\varphi^{\mathbf{N}'}Y = YE_\varphi^{\mathbf{N}'} = Y$ on $(\mathbf{R}(\mathbf{M}, \mathbf{N}))_{(\mathfrak{M}_\varphi^{\mathbf{N}'})}$ (by Lemma 11.3.3 (i)). So we must only prove this: The correspondence $X_{(\mathfrak{M}_\varphi^{\mathbf{N}'})} = Y_{(\mathfrak{M}_\varphi^{\mathbf{N}'})}$ generates a one to one mapping of all $X \epsilon \mathbf{M}$ on all $Y \epsilon \mathbf{R}(\mathbf{M}, \mathbf{N})$ with $E_\varphi^{\mathbf{N}'}Y = YE_\varphi^{\mathbf{N}'} = Y$. If $X \epsilon \mathbf{M}$ is given, then $Y = E_\varphi^{\mathbf{N}'}X$ has the above properties, and clearly $Y_{(\mathfrak{M}_\varphi^{\mathbf{N}'})} = X_{(\mathfrak{M}_\varphi^{\mathbf{N}'})}$. Besides if X is given, our correspondence determines Y uniquely, because such a Y is uniquely determined by its $Y_{(\mathfrak{M}_\varphi^{\mathbf{N}'})}$ just because the correspondence $Y \rightleftarrows Y_{(\mathfrak{M}_\varphi^{\mathbf{N}'})}$, $Y \epsilon \mathbf{R}(\mathbf{M}, \mathbf{N})$, $E_\varphi^{\mathbf{N}'}Y = YE_\varphi^{\mathbf{N}'} = Y$ is one-to-one (by Lemma 11.3.3, (i)). So $X_{(\mathfrak{M}_\varphi^{\mathbf{N}'})} = Y_{(\mathfrak{M}_\varphi^{\mathbf{N}'})}$ is a one-to-one mapping of all $X \epsilon \mathbf{M}$ on a part of the $Y \epsilon \mathbf{R}(\mathbf{M}, \mathbf{N})$, $E_\varphi^{\mathbf{N}'}Y =$

$YE_\varphi^{\mathbf{N}'} = Y$. All we must prove now is that all these Y correspond to some X, that is, that they all have the form $Y = E_\varphi^{\mathbf{N}'} X$, $X \in \mathbf{M}$.

We may as well prove: If $Y \in \mathbf{R}(\mathbf{M}, \mathbf{N})$ then $E_\varphi^{\mathbf{N}'} Y E_\varphi^{\mathbf{N}'} = E_\varphi^{\mathbf{N}'} X$ for some $X \in \mathbf{M}$. Let $\mathbf{P}$ be the set of those Y_0 for which this holds for all $Y = AY_0B$, $A, B \in \mathbf{N}$. That is: For which these $E_\varphi^{\mathbf{N}'} AY_0BE_\varphi^{\mathbf{N}'} \in \mathbf{M}_{(\mathfrak{M}_\varphi^{\mathbf{N}'})}$, cf. Lemma 113.3 (i). (Observe $E_\varphi^{\mathbf{N}'} \in \mathbf{N} \subset \mathbf{M}'$.) Clearly $\mathbf{P}$ is weakly closed (along with $\mathbf{M}_{(\mathfrak{M}_\varphi^{\mathbf{N}'})}$) and $Y_1, Y_2 \in \mathbf{P}$ imply αY_1, Y_1^*, $Y_1 + Y_2 \in \mathbf{P}$ and $Y_1 Y_2 \in \mathbf{P}$, the last one because

$$\begin{aligned} &\textstyle\sum_{p=1}^{\infty} E_\varphi^{\mathbf{N}'} AY_1 U_{1,p} E_\varphi^{\mathbf{N}'} \cdot E_\varphi^{\mathbf{N}'} U_{p,1} Y_2 BE_\varphi^{\mathbf{N}'} \\ &= \textstyle\sum_{p=1}^{\infty} E_\varphi^{\mathbf{N}'} AY_1 U_{1,p} P_{[f_{1,1}, f_{1,2}, \cdots]} U_{p,1} Y_2 BE_\varphi^{\mathbf{N}'} \\ &= \textstyle\sum_{p=1}^{\infty} E_\varphi^{\mathbf{N}'} AY_1 P_{[f_{p,1}, f_{p,2}, \cdots]} Y_2 BE_\varphi^{\mathbf{N}'} = E_\varphi^{\mathbf{N}'} AY_1Y_2 BE_\varphi^{\mathbf{N}'} . \end{aligned}$$

(Choose the $U_{m,p}, f_{m,p}$ by Lemma 5.3.6 for $\mathbf{N}$ with $f_{1,1} = \varphi$.) So $\mathbf{P}$ is a ring.

If $Y_0 = XZ$, $X \in \mathbf{M}$, $Z \in \mathbf{N}$ then for $A, B \in \mathbf{N}$ $E_\varphi^{\mathbf{N}'} AXZBE_\varphi^{\mathbf{N}'} = E_\varphi^{\mathbf{N}'} AZBE_\varphi^{\mathbf{N}'} \cdot X$. As $E_\varphi^{\mathbf{N}'}$ is minimal with respect to $\mathbf{N}$, the last part of the proof of Lemma 5.1.3 applies:
$E_\varphi^{\mathbf{N}'} AZBE_\varphi^{\mathbf{N}'} = \alpha E_\varphi^{\mathbf{N}'}$ (replace there $\mathbf{M}$, $E = E_f^{\mathbf{M}'}$, B by $\mathbf{N}$, $E_\varphi^{\mathbf{N}'}$, $E_\varphi^{\mathbf{N}'} AZBE_\varphi^{\mathbf{N}'}$). Thus the above expression is $= \alpha E_\varphi^{\mathbf{N}'} X \in \mathbf{M}_{(\mathfrak{M}_\varphi^{\mathbf{N}'})}$. So $Y_0 = XZ \in \mathbf{P}$. This implies $\mathbf{P} \supset \mathbf{M}$ and $\mathbf{N}$, and so $\mathbf{P} \supset \mathbf{R}(\mathbf{M}, \mathbf{N})$. Application to $Y_0 \in \mathbf{R}(\mathbf{M}, \mathbf{N})$, $A = B = 1$ completes the proof.

Lemma 11.5.2. *Let* $\mathbf{M}$, $\mathbf{N}$ *be two factors which commute,* $\mathbf{N}$ *being of class* (I). *Then* $\mathbf{R}(\mathbf{M}, \mathbf{N})$ *belongs to the same class* (I), (II), (III) *as* $\mathbf{M}$. *In particular:*

(i) *If* $\mathbf{M}$ *is in a class* (I_m), $m = 1, 2, \cdots, \infty$, *and* $\mathbf{N}$ *is in a class* (I_n), $n = 1, 2, \cdots, \infty$ *then* $\mathbf{R}(\mathbf{M}, \mathbf{N})$ *is in class* $(\mathrm{I}_{m \cdot n})$.

(ii) *If* $\mathbf{M}$ *is in class* (II_m), $m = 1, \infty$ *and* $\mathbf{N}$ *is in class* (I_n), $n = 1, 2, \cdots, \infty$ *then* $\mathbf{R}(\mathbf{M}, \mathbf{N})$ *is in class* (II_p), *where* $p = 1$ *if* m, n *are both finite, and* $p = \infty$ *if* m *or* n *is infinite.*

(iii) *If* $\mathbf{M}$ *is in class* (III), *no further comment is necessary.*

Proof: As $\mathbf{N}$ is of class (I), minimal φ's (with respect to $\mathbf{N}$) exist by Theorem IV. Choose one, and put $\mathfrak{M} = \mathfrak{M}_\varphi^{\mathbf{N}'}$ then $\mathbf{M}$ is algebraically ring-isomorphic to $(\mathbf{R}(\mathbf{M}, \mathbf{N})_{(\mathfrak{M})}$ by Lemma 11.5.1. Thus it has the same class by Theorem IX. Now Lemma 11.4.3 shows (replacing there $\mathbf{M}$, $\mathfrak{M}$ by $\mathbf{R}(\mathbf{M}, \mathbf{N})$, $\mathfrak{M}$), that $\mathbf{M}$ and $\mathbf{R}(\mathbf{M}, \mathbf{N})$ belong to the same class (I), (II), (III).

Let us now consider (i)–(iii).

If 1 is infinite with respect to $\mathbf{M}$ then it is obviously so for $\mathbf{R}(\mathbf{M}, \mathbf{N})$ too; the same being true for $\mathbf{N}$ and $\mathbf{R}(\mathbf{M}, \mathbf{N})$. This takes care of (i) and of (ii) if $m = \infty$ or $n = \infty$, (iii) is void. So we need only to consider (i) and (ii) for m, n finite.

Construct for $\mathbf{N}$ the normalised orthogonal system $\varphi_1, \varphi_2, \cdots$ from Lemma 5.3.4, where all φ_i are minimal, $\mathfrak{M}_{\varphi_1}^{\mathbf{N}'}$, $\mathfrak{M}_{\varphi_2}^{\mathbf{N}'}$, $\cdots$ are mutually orthogonal, and $[\mathfrak{M}_{\varphi_1}^{\mathbf{N}'}, \mathfrak{M}_{\varphi_2}^{\mathbf{N}'}, \cdots] = \mathfrak{H}$. We know by Lemmas 5.3.5–5.3.8 that as $\mathbf{N}$ is in the case (I_n), the number of these φ_i's is n: $\varphi_1, \cdots, \varphi_n$ (n is finite!).

Lemma 11.5.1 applies to every φ_i. So $(\mathbf{R}(\mathbf{M}, \mathbf{N}))_{(\mathfrak{M}_{\varphi_i}^{\mathbf{N}'})}$ is in the same case

as $\mathbf{M}$, (I_m) resp. (II_1) (for (i) resp. (ii)). By this, $\mathfrak{M}_{\varphi_i}^{\mathbf{N}'}$ has the (relative) dimension m in the standard normalisation of $\mathbf{R}\ (\mathbf{M}, \mathbf{N})_{(\mathfrak{M}_{\varphi_i}^{\mathbf{N}'})}$ resp. a finite relative dimension in any normalisation of $\mathbf{R}\ (\mathbf{M}, \mathbf{N})_{(\mathfrak{M}_{\varphi_i}^{\mathbf{N}'})}$. The same is true by Lemma 11.4.3, (i) resp. (ii) for $\mathfrak{M}_{\varphi_i}^{\mathbf{N}'}$ and $\mathbf{R}\ (\mathbf{M}, \mathbf{N})$. As the $\mathfrak{M}_{\varphi_i}^{\mathbf{N}'}$ are mutually orthogonal and $[\mathfrak{M}_{\varphi_1}^{\mathbf{N}'}, \cdots, \mathfrak{M}_{\varphi_n}^{\mathbf{N}'}] = \mathfrak{H}$, the additivity of the relative dimension implies that it is mn (standard normalisation) resp. finite (any normalisation) for $\mathfrak{H}$ in $\mathbf{R}\ (\mathbf{M}, \mathbf{N})$. Thus we have case ($\mathrm{I}_{m\,n}$) resp. ($\mathrm{II}_1$) for $\mathbf{R}\ (\mathbf{M}, \mathbf{N})$.

This completes the proof.

Lemma 11.5.2 shows that if case (II) exists at all, then (II_∞) exists too. ((II_1) has been discussed in §11.4). It suffices to take an $\mathbf{M}_0$ of case (II) in an $\mathfrak{H}_1$, any $\mathbf{N}_0$ of case (I_∞) in an $\mathfrak{H}_2$ (for instance: $\mathfrak{H}_2$ a Hilbert space, $\mathbf{N}_0 = \mathbf{B}_2$), and then form $\mathfrak{H} = \mathfrak{H}_1 \otimes \mathfrak{H}_2$, $\mathbf{M} = \mathbf{M}_0^{(1)}$, $\mathbf{N} = \mathbf{N}_0^{(2)}$. Then $\mathbf{R}\ (\mathbf{M}, \mathbf{N})$ will certainly be of class (II_∞). But all this will be settled exhaustively in Chapter XIII.

Part IV: Examples of case (II)

Chapter XII: Construction of $\mathbf{M}$, $\mathbf{M}'$

12.1. The considerations at the end of §8.6 have established the existence of all cases (I_n), (I_∞) and of all their combinations for factors $\mathbf{M}$ resp. coupled factorisation $\mathbf{M}$, $\mathbf{M}'$. We are going to do now the same for the cases (II_1), (II_∞). We will thus answer a great part of the questions raised by Problems 3 and 4.

Our first objective is to construct certain coupled factorisations $\mathbf{M}$, $\mathbf{M}'$ which belong to the classes (II_1), (II_1) (the C of Theorem X being $= 1$) or (II_∞), (II_∞). These factorisations will contain many arbitrary parameters, and therefore represent a rather wide variety of examples. In fact, further constructions based on them will answer Problem 1 in the negative.

Definition 12.1.1. We will consider groups $\mathfrak{G}$ of the following kind:

(i) $\mathfrak{G}$ is a group, that is a composition rule ab, an inverse a^{-1} and a unity 1 defined in $\mathfrak{G}$, with the properties

$$(ab)c = a(bc), \qquad aa^{-1} = a^{-1}a = 1, \qquad a1 = 1a = a.$$

(So ab is associative but not necessarily commutative).

(ii) $\mathfrak{G}$ is finite or countably infinite.

Definition 12.1.2. We will consider spaces S of the following kind: A Lebesgue outer measure is defined in S, that is, for every subset T of S a real number $\mu^*(T) \geqq 0 \leqq \infty$ is given, with the following properties:

(i) $T_1 \subset T_2$ implies $\mu^*(T_1) \leqq \mu^*(T_2)$.

(ii) For every (finite or infinite) sequence $T_1, T_2, \cdots$.

$$\mu_1^*(T_1 \dotplus T_2 \dotplus \cdots) \leqq \mu_1^*(T_1) + \mu_2^*(T_2) + \cdots.$$

We now define measurability following Carathéordory (cf. (4), p. 246):

(*) A subset T of S is *measurable*, if and only if, for every subset T' of S, $\mu^*(T') = \mu^*(TT') + \mu^*(T' - TT')$.

We then write $\mu(T)$ for $\mu^*(T)$ and call it the *measure* of T.

We now continue the enumeration of our postulates.

iii) If $\mu^*(T) < \alpha$ then a measurable T_0 with $T_0 \supset T$, $\mu(T_0) < \alpha$ exists.

(iv) A (finite or infinite) sequence $T^{(1)}$, $T^{(2)}$ ... of measurable sets with finite measure exists, with the following property: If $x, y \in S$, and if $x \in T^{(i)}$ is equivalent to $y \in T^{(i)}$ (for all $i = 1, 2, \cdots$), then $x = y$.

Observe the following consequences of Definition 12.1.2: $\mu^*(T) \geqq 0, \leqq \infty$ and conditions (i)–(ii) correspond to Carathéodory's conditions (I)–(III), without (IV), (Cf. (4), pp. 238–239). This omission however is natural, because (IV) makes explicit use of the notion of distance, whereas we did not require that a distance (or any sort of a topology) be defined in S. (iii) is, (remembering (i)) equivalent to Carathéodory's condition (V), (cf. (4), p. 258), therefore our outer measure is a regular measure function (cf. loc. cit.) if we disregard the topological condition (IV). Closer inspection of Carathéodory's deductions shows that in consequence, all those results on outer measure, measurability, and measure loc. cit. remain true which are not explicitly topological in their statements. This applies to the considerations on pp. 246–250 loc. cit. (cf. the remark on p. 257 there). Thus 0 and S are measurable; if $T', T^{(\prime)}$ are measurable, then $T' - T'T''$ is too; if $T', T'', \cdots$ are measurable, then $T' \dotplus T'' \dotplus \cdots$ and $T' \cdot T'' \cdots$ are too. Besides, the well known convergence theorems on measure are valid.

As we see, the chief difference between the characteristics of the situation in S and those of the one which exists loc. cit., is this: The sets which are known to be measurable, and which serve as a basis for all constructions of measurable functions (cf. below Definition 12.1.3) are there the Borel-sets (a topological notion) while they are now the sets $T^{(1)}, T^{(2)}, \ldots$ from (iv).

In fact, this is essentially the meaning of (iv), which is the only one of our postulates without an analogue on Carathéodory's list.: It serves to replace the (absent or ignored) topology in S, and in particular the "topological separability" of S.

Let us mention the following obvious examples of spaces S:

Example a). Let S be a (finite or infinite) sequence $S = (x_1, x_2, \cdots)$. Let a corresponding sequence of real numbers $\bar{\alpha}_n \geqq 0, < \infty$ be given. Define

$$\mu^*(T) \equiv \sum_{x_n \in T} \bar{\alpha}_n.$$

Then (i), (ii) are obviously fulfilled; (iii) is fulfilled because all sets $T \subset S$ are measurable; (iv) is fulfilled, because we may choose for $T^{(1)}$, $T^{(2)}$ ... the set of all one-element $T \subset S$.

Example b). Let S be any measurable subset of a finite dimensional Euclidean space. Let $\mu^*(T)$ be the common Lebesgue measure. Then (i)–(iii) are obviously fulfilled; (iv) is fulfilled because we may choose for $T^{(1)}, T^{(2)}, \cdots$ the sequence $Ss_1, Ss_2, \cdots$, where $s_1, s_2, \cdots$ are all spheres with rational coordinates of the center and a rational radius.

We now introduce Lebesgue integration in the customary way:

Definition 12.1.3. A complex valued function $f(x)$, defined for all $x \in S$ is

called *measurable* if the sets $(x;\ \Re(f(x)) < \alpha)$, $(x;\ \Im(f(x) < \alpha)$ ($\Re$ = real part, $\Im$ = imaginary part) are measurable for every real α. The notions of summability and of the (Legesgue) integral $\int_S f(x)\,dx$ of a measurable function $f(x)$ are then defined in any of the several equivalent customary ways. (The procedure of Carathéodory, (4), pp. 418–427, which coincides with Lebesgue's "geometrical" definition, (10), pp. 116–120, would necessitate introducing first the notion of outer measure in a product space. Lebesgue's "analytical" definition, (10), pp. 112–116, however, is directly applicable; another very convenient definition is due to Bochner, (2), pp. 265–267).

We now form three functional spaces:

Definition 12.1.4. Let $\mathfrak{H}_{\mathfrak{G}}$ be the set of all complex valued functions $f(a)$, defined for all $a \in \mathfrak{G}$ with a finite $\sum_{a \in \mathfrak{G}} |f(a)|^2$. Define $(f, g)_{\mathfrak{G}} = \sum_{a \in \mathfrak{G}} f(a)\overline{g(a)}$.

Let $\mathfrak{H}_S$ be the set of all complex valued functions $f(x)$, defined for all $x \in S$ which are measurable in x, and with a finite $\int_S |f(x)|^2\,dx$. Consider f, g as identical if $f(x) = g(x)$ except for an x-set of measure 0. Define $(f, g)_S = \int_S f(x)\overline{g(x)}\,dx$.

Let $\mathfrak{H}_{S\mathfrak{G}}$ be the set of all complex valued functions $F(x, a)$ defined for all $x \in S$, $a \in \mathfrak{G}$, which are measurable in x (for every fixed $a \in \mathfrak{G}$) and with a finite $\sum_{a \in \mathfrak{G}} \int_S |F(x, a)|^2\,dx$. Consider F, G as identical, if $F(x, a) = G(x, a)$ except for an x-set of measure 0 (depending on a). Define

$$(F, G)_{S\mathfrak{G}} = \sum_{a \in \mathfrak{G}} \int_S (F(x, a)\ \overline{G(x, a)}\,dx.$$

$\alpha\cdot$ and $+$ are defined in the obvious way in all three spaces $\mathfrak{H}_{\mathfrak{G}}$, $\mathfrak{H}_S$, $\mathfrak{H}_{S\mathfrak{G}}$.

By definition 12.1.2 (iv), no two points $x \neq y$ can have the property

$$x, y \notin T^{(1)} \dotplus T^{(2)} \dotplus \cdots.$$

$S - (T^{(1)} \dotplus T^{(2)} \dotplus \cdots)$ is measurable, and if its measure is finite, we can add it to the sequence $T^{(1)}, T^{(2)}, \cdots$ thus forcing $T^{(1)} \dotplus T^{(2)} \dotplus \cdots = S$. So the only case where this is not possible is when $S - (T^{(1)} \dotplus T^{(2)} \dotplus \cdots) = (x_0)$, $\mu((x_0)) = \infty$. Then we must have $f(x_0) = 0$ and $F(x_0, a) = 0$ for all $f \in \mathfrak{H}_S$ resp. $F \in \mathfrak{H}_{S\mathfrak{G}}$ so we may omit x_0 from S. (If $x \in S$, $x \neq x_0$ then $x \in T^{(i)}$ for some $i = 1, 2, \cdots$, and so $\mu^*(x) \leqq \mu(T^{(i)}) < \infty$. Therefore Definition 12.1.5, cf. below, excludes $x_0 a \neq x_0$ that is, we have $x_0 a = x_0$. So the omission of x_0 does not affect the operation xa either). In what follows we can thus assume

$$T^{(1)} \dotplus T^{(2)} \dotplus \cdots = S.$$

If $\mu(S) = 0$, then $\mathfrak{H}_S$ and $\mathfrak{H}_{S\mathfrak{G}}$ would consist of 0 only. We therefore assume explicitly that $\mu(S) > 0$.

Lemma 12.1.1. *All three spaces $\mathfrak{H}_{\mathfrak{G}}$, $\mathfrak{H}_S$, $\mathfrak{H}_{\mathfrak{G}S}$ fulfill the postulates* A, B, C, E *of* (16), *pp.* 64–66; *thus they are finite dimensional Euclidean or Hilbert spaces* (*and* $\neq 0$).

Using the complete normalised orthogonal set of all functions

$$\varphi_{a_0}(a) = \begin{cases} 1 \text{ for } a = a_0 \\ 0 \text{ for } a \neq a_0; \end{cases} \qquad a_0 \in \mathfrak{G} \text{ in } \mathfrak{H}_{\mathfrak{G}},$$

and setting up the correspondence $F(x, a) \sim < F(x, \bar{a}_1), F(x, \bar{a}_2), \cdots > (\bar{a}_1, \bar{a}_2, \cdots$ *is some enumeration of* $\mathfrak{G}$*, so that the above* a_0 *runs over* $\bar{a}_1, \bar{a}_2, \cdots$*),* $\mathfrak{H}_{S\mathfrak{G}}$ *is isomorphic with* $\mathfrak{H}_{\mathfrak{G}} \otimes \mathfrak{H}_S$ *in the sense of Definition* 2.4.1 *and Lemma* 2.4.1.

We will use in what follows the more symmetric notation

$$F(x, a) \sim < F(x, a_0); a_0 \in \mathfrak{G} > .$$

Proof: If we use an enumeration $\bar{a}_1, \bar{a}_2, \cdots$ of $\mathfrak{G}$, and put $f(\bar{a}_n) = x_n$, $n = 1, 2, \cdots$ then $f(a) \sim (x_1, x_2, \cdots)$ and our definition coincides with the original definition of finite dimensional Euclidean or of Hilbert space (cf. (16) p. 69). Concerning $\mathfrak{H}_S$ it suffices to observe that the considerations of (16), pp. 108–111, on A, B, E apply immediately to it; while those on C apply, if the system of neighborhoods occurring there is replaced by $T^{(1)}, T^{(2)}, \cdots$.

As $\mu(S) = \mu(T^{(1)} \dotplus T^{(2)} \dotplus \cdots) > 0$ some $\mu(T^{(i)}) > 0$. At the same time it is $< \infty$. Then

$$f(x) \begin{cases} = 1, x \in T^{(i)} \\ = 0, x \notin T^{(i)} \end{cases}$$

belongs to $\mathfrak{H}_S$ and is $\neq 0$. So $\mathfrak{H}_S \neq (0)$. The last part of our Lemma results immediately by comparing our definitions with Definition 2.4.1 and Lemma 2.4.1. As $\mathfrak{H}_{S\mathfrak{G}}$ is isomorphic with $\mathfrak{H}_{\mathfrak{G}} \otimes \mathfrak{H}_S$ it must fulfill A, B, C, E, too. As $\mathfrak{H}_S \neq (0)$, therefore $\mathfrak{H}_{S\mathfrak{G}} \neq (0)$.

Heretofore the connection between S and $\mathfrak{G}$ was merely formal, due to our forming the space $\mathfrak{H}_{S\mathfrak{G}} = \mathfrak{H}_{\mathfrak{G}} \otimes \mathfrak{H}_S$. An intrinsic connection is established by the following assumptions:

Definition 12.1.5. $\mathfrak{G}$ is an m-group in S, if for every $a \in \mathfrak{G}$ there is defined a one to one mapping $x \rightleftarrows xa$ of S on itself, with the following properties:

(i) If $T \subset S$ and T_a is the $x \rightleftarrows xa$ image of T then $\mu^*(T) = \mu^*(T_a)$. (This implies that the measurability of T is equivalent to that of T_a.)

(ii) $(xa)b = x(ab)$. (This implies that $x \cdot 1 = x$ and that $x \rightleftarrows xa^{-1}$ is inverse to $x \rightleftarrows xa$.)

(iii) If $a \neq 1$ then $xa = x$ holds only for x-sets of measure 0.

$\mathfrak{G}$ is *ergodic* in S, if besides (i)–(iii) we have further:

(iv) If a measurable $T \subset S$ differs from each T_a, $a \in \mathfrak{G}$ only by a set of measure 0 (but depending on a, this means $\mu((T \dotplus T_a) - (T \cdot T_a)) = 0$) then either $\mu(T) = 0$ or $\mu(S - T) = 0$.

Observe that (iv) differs from the customary definitions of ergodicity in two ways: First $\mathfrak{G}$ is not necessarily the simple translation group, not even necessarily Abelian, second (which is essential if $\mu(S) = \infty$) we did not restrict the validity of (iv) to T's with a finite $\mu(T)$ only.

12.2. We introduce and discuss first some operators in $\mathfrak{H}_S$.

Lemma 12.2.1. *Let $\mathfrak{G}$ be an m-group in S. Define the operators:*

(i) $U_a f(x) = f(xa)$ *for any* $a \in \mathfrak{G}$.

(ii) $L_{\varphi(x)}\ f(x) = \varphi(x)\ f(x)$ *for any bounded and measurable complex valued function $\varphi(x)$ defined for all $x \in S$. These operators are bounded, U_a is even unitary.*

Proof: We have $\int |\ U_a(f(x))\ |^2\ dx = \int |\ f(xa)\ |^2\ dx = \int |\ f(x)\ |^2\ dx$ so U_a is bounded, and $\|\ U_a f\ \| = \|\ f\ \|$. Now clearly $U_{a^{-1}} U_a = U_a U_{a^{-1}} = 1$ so U_a has an inverse, thus it is unitary (cf. (16), p. 71).

If $|\ \varphi(x)\ | \leqq C$ for all $x \in S$, then $\int_S |\ L_{\varphi(x)} f(x)\ |^2 dx = \int |\ \varphi(x)\ |^2\ |\ f(x)\ |^2 dx$ $\leqq C^2 \int_S |\ f(x)\ |^2 dx$, $\|\ L_{\varphi(x)} f\ \| \leqq C\ \|\ f\ \|$. So $L_{\varphi(x)}$ too is bounded.

Lemma 12.2.2. *Let $\mathfrak{G}$ be an m-group in S. Let $\mathbf{L}$ be the set of all operators $L_{\varphi(x)}$ from Lemma* 12.2.1, (iii). *Then $\mathbf{L}$ is a ring and $\mathbf{L} = \mathbf{L}'$.*

Proof: As $\mathbf{L}'$ is necessarily a ring, it suffices to prove $\mathbf{L} = \mathbf{L}'$. As

$$(L_{\varphi(x)})^* = L_{\bar{\varphi}(x)},\ L_{\varphi(x)} \cdot L_{\psi(x)} = L_{\varphi(x)\,\psi(x)}$$

therefore every $L_{\psi(x)}$ commutes with every $L_{\varphi(x)}$ and $(L_{\varphi(x)})^*$ thus $L_{\psi(x)} \in \mathbf{L}'$ and so $\mathbf{L} \subset \mathbf{L}'$. So we must only prove $\mathbf{L}' \subset \mathbf{L}$.

Assume therefore $A \in \mathbf{L}'$. Then A commutes with every $L_{\varphi(x)}$, that is $AL_{\varphi(x)} f(x) = L_{\varphi(x)}\ Af(x)$. That is:

$$(\divideontimes) \qquad A(\varphi(x) f(x)) = \varphi(x)\ Af(x)\ .$$

The assumptions are: $\varphi(x)$, $f(x)$ are measurable, $\varphi(x)$ is bounded, $\int_S |\ f(x)\ |^2\ dx$ is finite, the equation holds with the exception of an x-set of measure 0. Such exceptions, by the way, are admitted in all the equations we are going to derive.

Let $T^{(1)}, T^{(2)}, \cdots$ be the sets from Definition 12.1.2 (iv); assume (following the remark before Lemma 12.1.1, that $T^{(1)} \dotplus T^{(2)} \dotplus \cdots = S$. Define

$$e_i(x) \begin{cases} = 1, \text{ for } x \in T^{(1)} \dotplus \cdots \dotplus T^{(i)} \\ = 0, \text{ for } x \notin T^{(1)} \dotplus \cdots \dotplus T^{(i)}\ . \end{cases}$$

Apply now $(\divideontimes)$ to $f = e_i$. Then $A(e_i(x)\varphi(x)) = (A\ e_i(x))\varphi(x)$ obtains. For $i \geqq j$, $e_i(x) e_j(x) = e_j(x)$, so $\varphi = e_j$ gives $Ae_j(x) = (Ae_i(x))\ e_j(x)$. Thus if $x \in T^{(1)} \dotplus \cdots \dotplus T^{(j)}$, $Ae_j(x) = Ae_i(x)$; in other words: for $x \in T^{(1)} \dotplus \cdots \dotplus T^{(j)}$ all $Ae_i(x)$, $i \geqq j$, agree. Thus for every x all $Ae_i(x)$ with a sufficiently great i have the same value. Call this value $\psi(x)$ as $\psi(x) = \lim_{i \to \infty} Ae_i(x)$ this is a measurable function. Now we have $Ae_j(x) = \psi(x)\ e_j(x)$. And our original equation becomes: $Ae_j(x)\varphi(x) = \psi(x) e_j(x)\varphi(x)$. That is: $Af(x) = \psi(x) f(x)$ where $f(x) = e_i(x)\varphi(x)$. Thus this equation is proved if $f(x)$ is bounded and 0 for all $x \notin T^{(1)} \dotplus \cdots \dotplus T^{(i)}$ for some $i = 1, 2, \cdots$.

A is bounded, say $\|\ Af\ \| \leqq C\ \|\ f\ \|$; therefore

$$\int_S |\ \psi(x)\ |^2\ |\ f(x)\ |^2\ dx \leqq C^2 \int_S |\ f(x)\ |^2\ dx$$

for these f's. Now put

$$f(x)\begin{cases} = 1 \text{ if } |\psi(x)| \geqq C+1,\ x \in T^{(1)} \dotplus \cdots \dotplus T^{(i)}. \\ = 0 \text{ otherwise}. \end{cases}$$

Denote the set of all x with $|\psi(x)| \geqq C+1$ by T_0. Then the above inequality gives:

$$(C+1)^2\,\mu(T_0(T^{(1)} \dotplus \cdots \dotplus T^{(i)})) \leqq C^2\mu(T_0(T^{(1)} \dotplus \cdots \dotplus T^{(i)})),$$

$$\mu(T_0(T^{(1)} \dotplus \cdots \dotplus T^{i})) = 0$$

and passing to the limit as $i \to \infty$, $\mu(T_0) = 0$. So we have everywhere $|\psi(x)| \leqq C+1$ (as T_0 does not matter), that is: $\psi(x)$ is bounded. We can therefore write: $Af = L_\psi f$ if $f(x)$ is as described above.

Now these f form an everywhere dense set in $\mathfrak{H}_S$. If any $f \in \mathfrak{H}_S$ is given, define

$$f_i(x)\begin{cases} = f(x) \text{ if } |f(x)| < i,\ x \in T^{(1)} \dotplus \cdots \dotplus T^{(i)} \\ = 0 \text{ otherwise} \end{cases}$$

for $i = 1, 2, \cdots$; then every f_i meets these requirements, and the f_i converge for $i \to \infty$ to f (in $\mathfrak{H}$). As A, L_ψ are both bounded, we have therefore $Af = L_\psi f$ for every f; that is $A = L_\psi \in \mathbf{L}$. So we have established $\mathbf{L}' \subset \mathbf{L}$ too, thus completing the proof.

Lemma 12.2.3. *Let $\mathfrak{G}$ be an m-group in S. The equation $L_{\varphi(x)} = L_{\psi(x)}\,U_a$ holds if and only if either $a = 1$, $\varphi(x) = \psi(x)$ (except for an x-set of measure 0) or $a \neq 1$, $\varphi(x) = \psi(x) = 0$ (except for an x-set of measure 0).*

Proof: The sufficiency of these conditions is obvious, so we need only to prove their necessity. Assume therefore $L_{\varphi(x)} = L_{\psi(x)}\,U_a$. So we have $\varphi(x)f(x) = \psi(x)f(ax)$ whenever $\int |f(x)|^2\,dx$ is finite. This equation, and similarly all those we are going to derive, holds with the exception of an x-set of measure 0. Use the e_i from the proof of Lemma 12.2.2, then we have: $e_i(x)\varphi(x) = e_i(xa)\psi(x)$, that is $\varphi(x) = \psi(x)$ if x and $xa \in T^{(1)} \dotplus \cdots \dotplus T^{i}$. Summing over all $i = 1, 2, \cdots$ we obtain $\varphi(x) = \psi(x)$ for $x \in S$. This settles the case $a = 1$. Assume now $a \neq 1$. Substituting $\varphi(x) = \psi(x)$ in our original equation gives $\varphi(x)f(x) = \varphi(x)f(xa)$ and so $\int_S |\varphi(x)|^2\,|f(x) - f(xa)|^2\,dx = 0$. Define now

$$e_i'(x)\begin{cases} = 1 \text{ for } x \in T_i \\ = 0 \text{ for } x \notin T_i, \end{cases}$$

then $e_i' \in \mathfrak{H}_S$ and we can put $f = e_i'$. Then $|f(x) - f(xa)|^2 = 1$ for

$$x \in (T^{(i)} \dotplus T_{a^{-1}}^{(i)}) - T^{(i)}T_{a^{-1}}^{(i)}$$

and so our integral formula becomes

$$\int_{S_i} |\varphi(x)|^2\,dx = 0,\ S_i = (T^{(i)} \dotplus T_{a^{-1}}^{(i)}) - T^{(i)}T_{a^{-1}}^{(i)}.$$

Put now $S' = \sum_{i=1}^{\infty} ((T^{(i)} \dotplus T_{a^{-1}}^{(i)}) - T^{(i)}T_{a^{-1}}^{(i)})$. As the integrand above is $|\varphi(x)|^2 \geqq 0$, adding of the above equations over all $i = 1, 2, \cdots$ gives $\int_{S'} |\varphi(x)|^2 dx = 0$.

Now $x \in S - S'$ means $x \notin (T^{(i)} \dotplus T_{a^{-1}}^{(i)}) - T^{(i)}T_{a^{-1}}^{(i)}$ for all $i = 1, 2, \cdots$; thus $x \in$ or $x \notin$ in both $T^{(i)}$ and $T_{a^{-1}}^{(i)}$ that is both x and $xa \in T^{(i)}$ or $\notin T^{(i)}$. By Definition 12.1.2, (iv), this implies $x = xa$. As $a \neq 1$, Definition 12.1.5, (iii) now necessitates $\mu(S - S') = 0$. Thus $\int_S |\varphi(x)|^2 dx = 0$ too, and therefore $\varphi(x) = 0$ (except for an x-set of measure 0). This completes the proof.

Lemma 12.2.4. *Let* $\mathfrak{G}$ *be ergodic in* S. *Let* $\mathbf{L}$ *be as in Lemma* 12.2.2 *and let* $\mathbf{U}$ *be the set of all* U_a, $a \in \mathfrak{G}$. *Then* $\mathbf{L} \cdot \mathbf{U}' = (\alpha 1)$.

Proof: $\alpha 1$ belongs obviously to $\mathbf{U}'$, and as $\alpha 1 = L_\alpha$, $\alpha 1 \in \mathbf{L}$ so $\alpha 1 \in \mathbf{L} \cdot \mathbf{U}'$, $\mathbf{L}\mathbf{U}' \supset (\alpha 1)$. So we need only prove $\mathbf{L}\mathbf{U}' \subset (\alpha 1)$. Assume therefore $A \in \mathbf{L} \cdot \mathbf{U}'$. Then $A = L_{\varphi(x)}$ and as $A \in \mathbf{U}'$ therefore $U_a A = A U_a$, $A = U_a^{-1} A U_a$, $L_{\varphi(x)} = L_{\varphi(xa^{-1})}$ for every $a \in \mathfrak{G}$. By Lemma 12.2.3 this means $\varphi(x) = \varphi(xa^{-1})$ except for an x-set of measure 0 (depending on $a \in \mathfrak{G}$).

Put $T_\alpha = (x;\ \Re(\varphi(x)) < \alpha)$. The above relation proves that $x \in T_\alpha$ and $xa^{-1} \in T_\alpha$ are equivalent, except for an x-set of measure 0; so

$$\mu((T_\alpha + T_\alpha a) - (T_\alpha T_\alpha a)) = 0 .$$

Therefore Definition 12.1.5, (iv) (the ergodicity) implies

$$\mu(T_\alpha) = 0 \quad \text{or} \quad \mu(S - T_\alpha) = 0 .$$

Denote the set of all (real) α's with $\mu(T_\alpha) = 0$ by N. If $\alpha \leqq \beta$ then $T_\alpha \subset T_\beta$ so $\beta \in N$ implies $\alpha \in N$. So if α_0 is the least upper bound of N, $-\infty \leqq \alpha_0 \leqq \infty$, then $\alpha < \alpha_0$ implies $\alpha \in N$, that is $\mu(T_\alpha) = 0$, and $\alpha > \alpha_0$ implies $\alpha \notin N$, that is $\mu(S - T_\alpha) = 0$.

So for every rational $\alpha < \alpha_0$, $\mu((x;\ \Re(\varphi(x)) < \alpha)) = 0$ and adding over them gives $\mu((x;\ \Re(\varphi(x)) < \alpha_0)) = 0$; for every rational $\alpha > \alpha_0$, $\mu((x;\ \Re(\varphi(x)) \geqq \alpha)) = 0$ and adding over them gives: $\mu((x;\ \Re(\varphi(x)) > \alpha_0)) = 0$. Adding again: $\mu((x;\ \Re(\varphi(x)) \neq \alpha_0)) = 0$. Similarly we obtain: $\mu((x;\ \Im(\varphi(x)) \neq \beta_0)) = 0$ for a suitable β_0, and if we put $\alpha_1 = \alpha_0 + i\beta_0$ then adding these two last equations gives:

$$\mu((x,\ \varphi(x) \neq \alpha_1)) = 0 .$$

So we have $\varphi(x) = \alpha_1$ except for an x-set of measure 0; and so $A = L_{\varphi(x)} = \alpha_1 1$, $A \in (\alpha 1)$. Thus $\mathbf{L}\mathbf{U}' \subset (\alpha 1)$, completing the proof.

12.3. We pass now to constructions in $\mathfrak{H}_{S\mathfrak{G}}$.

Lemma 12.3.1. *Let* $\mathfrak{G}$ *be an m-group in* S. *Define the operators*

(i) $\bar{U}_{a_0} F(x, a) = F(xa_0, aa_0)$

(ii) $\bar{V}_{a_0} F(x, a) = F(x, a_0^{-1} a)$

for any $a_0 \in \mathfrak{G}$.

(iii) $\overline{W}F(x, a) = F(xa^{-1}, a^{-1})$

(iv) $\overline{L}_{\varphi(x)}F(x, a) = \varphi(x)F(x, a)$

(v) $\overline{M}_{\varphi(x)}F(x, a) = \varphi(xa^{-1})F(x, a)$

for any bounded and measurable complex valued $\varphi(x)$ defined for all $x \in S$.

These operators are bounded, $\overline{U}_{a_0}$, $\overline{V}_{a_0}$, $\overline{W}$ are even unitary. Furthermore $\overline{W} = \overline{W}^{-1}$ and

$$\overline{W}\overline{U}_{a_0}\overline{W} = \overline{V}_{a_0}\,; \qquad \overline{W}\overline{L}_{\varphi(x)}\overline{W} = \overline{M}_{\varphi(x)}\,;$$

in other words: $F \to \overline{W}F$ is an involutory spatial isomorphism of $\mathfrak{H}_{S\mathfrak{G}}$ which interchanges $\overline{U}_{a_0}$, $\overline{L}_{\varphi(x)}$ with $\overline{V}_{a_0}$, $\overline{M}_{\varphi(x)}$.

Proof: The boundedness and unitarity are proved in the same way as in the proof of Lemma 12.2.1. $\overline{W}^2 = 1$ results from direct computation, and implies $\overline{W} = \overline{W}^{-1}$; the two last equations result from direct computation.

Lemma 12.3.2. *Let $\mathfrak{G}$ be an m-group in S. Let* **I** *be the set of all $\overline{U}_{a_0}$ and all $\overline{L}_{\varphi(x)}$ and* **J** *the set of all $\overline{V}_{a_0}$ and all $\overline{M}_{\varphi(x)}$ (cf. Lemma* 12.3.1). *Form for each bounded operator A in $\mathfrak{H}_{S\mathfrak{G}}$, ($A \in \mathbf{B}_{S\mathfrak{G}}$) the decomposition from Definition* 2.4.2, *$\bar{A} \sim < A_{ab} >_{a,b \in \mathfrak{G}}$ where every $A_{a,b} \in \mathbf{B}_S$. (Cf. the decomposition $\mathfrak{H}_{S\mathfrak{G}} = \mathfrak{H}_{\mathfrak{G}} \otimes \mathfrak{H}_S$ from Lemma* 12.1.1. *As described there, we replace the indices $t, s = 1, 2, \cdots$ by indices $a, b \in \mathfrak{G}$ as already the complete normalised orthogonal set φ_{a_0}, $a_0 \in \mathfrak{G}$ in $\mathfrak{H}_{\mathfrak{G}}$ has been indexed in this way.)*

Then $\bar{A} \in \mathbf{I}'$ if and only if $A_{a,b}$ has the form $A_{a,b} = L_{x_{ab^{-1}}(xb^{-1})}$ and $\bar{A} \in \mathbf{J}'$ if and only if $A_{a,b}$ has the form $A_{a,b} = L_{x_{a^{-1}b}(x)}\, U_{b^{-1}a}$. Here $\chi_c(x)$ must be a bounded and measurable function of x for every $c \in \mathfrak{G}$. (We do not determine for which systems $\chi_c(x)$, $c \in \mathfrak{G}$, a bounded $\bar{A}$, to which the given $\chi_c(x)$ corresponds, actually does exist.)

Proof: Remember the definition of the $A_{a,b}$ (Definition 2.4.2, with our present notations): $\bar{A} < f(x)\varphi_{a'}(a);\ a \in \mathfrak{G} > = < A_{a',b}f(x);\ b \in \mathfrak{G} >$; that is (if we replace a', a by a, b so that now $x \in S$ and $b \in \mathfrak{G}$ are the variables, and $a \in \mathfrak{G}$ is a parameter): $\bar{A}f(x)\varphi_a(b) = A_{a,b}f(x)$. Then the definitions of $\overline{U}_{a_0}$, $\overline{L}_{\varphi(x)}$ and $\overline{W}$ give: If $\bar{A} = < A_{a,b} >_{a,b \in \mathfrak{G}}$, then

$$\overline{U}_{a_0}^{-1} \cdot \bar{A} \cdot \overline{U}_{a_0} = < U_{a_0}^{-1} A_{aa_0^{-1}, ba_0^{-1}} U_{a_0} >_{a,b \in \mathfrak{G}},$$

$$\overline{W}\bar{A}\overline{W} = < U_b^{-1} A_{a^{-1}, b^{-1}} U_a >_{a,b \in \mathfrak{G}},$$

$$\overline{L}_{\varphi(x)}\bar{A} = < L_{\varphi(x)} A_{a,b} >_{a,b \in \mathfrak{G}},$$

$$\bar{A}\overline{L}_{\varphi(x)} = < A_{a,b} L_{\varphi(x)} >_{a,b \in \mathfrak{G}}.$$

Now $\bar{A} \in \mathbf{I}'$ means that $\bar{A}$ commutes with all $\overline{L}_{\varphi(x)}$, ($(L_{\varphi(x)})^* = L_{\overline{\varphi(x)}}$) and all $\overline{U}_{a_0}$, ($(\overline{U}_{a_0})^* = (\overline{U}_{a_0})^{-1} = \overline{U}_{a_0^{-1}}$). This means $A_{a,b} \in \mathbf{L}'$ that is $A_{a,b} \in \mathbf{L}$ (by Lemma 12.2.1), and $U_{a_0}^{-1} A_{aa_0^{-1}, ba_0^{-1}} U_{a_0} = A_{a,b}$. The first statement means $A_{a,b} = L_{\omega_{a,b}(x)}$ where $\omega_{a,b}(x)$ is a bounded and measurable function of x for every choice of $a, b \in \mathfrak{G}$. Now $U_{a_0}^{-1} A_{aa_0^{-1}, ba_0^{-1}} U_{a_0} = U_{a_0}^{-1} L_{\omega aa_0^{-1}, ba_0^{-1}(x)} U_{a_0} = L_{\omega_{aa_0^{-1}, ba_0^{-1}}(xa_0^{-1})}$ so the remaining requirement is: $\omega_{aa_0^{-1}, ba_0^{-1}}(xa_0^{-1}) = \omega_{a,b}(x)$. This equation, as the following ones, is valid with the exception of x-sets of measure 0.

Put $a_0 = b$ and $\chi_c(x) = \omega_{c,1}(x)$ then $\omega_{a,b}(x) = \chi_{ab^{-1}}(xb^{-1})$ obtains. Conversely: If this equation holds for a system of bounded measurable functions $\chi_c(x)$ then the $\omega_{a,b}(x)$ are such too, and $\omega_{aa_0^{-1},ba_0^{-1}}(xa_0^{-1}) = \omega_{a,b}(x)$. So the characterisation of the $\bar{A} \in \mathbf{I}'$ is given by $A_{a,b} = L_{\chi_{ab^{-1}}(xb^{-1})}$. This proves our statement concerning $\mathbf{I}'$.

As the spatial isomorphism $F \to \overline{W}F$ of $\mathfrak{H}_{S\mathfrak{G}}$ carries $\mathbf{I}$ into $\mathbf{J}$ it carries $\mathbf{I}'$ into $\mathbf{J}'$. So $\bar{B} \in \mathbf{J}'$ means $\bar{B} = \overline{W}\bar{A}\overline{W}$ for some $\bar{A} \in \mathbf{I}'$. The above result concerning the $\bar{A} \in \mathbf{I}'$, together with our formula $\overline{W}\bar{A}\overline{W}$ gives therefore:

$$B_{a,b} = U_b^{-1}A_{a^{-1},b^{-1}}U_a = U_b^{-1}L_{\chi_{a^{-1}b}(xb)}U_a = L_{\chi_{a^{-1}b}(x)}U_b^{-1}U_a = L_{\chi_{a^{-1}b}(x)}U_{b^{-1}a}.$$

This proves our statement concerning $\mathbf{J}'$.

Lemma 12.3.3. *Let $\mathfrak{G}$ be an m-group in S, and $\mathbf{I}$, $\mathfrak{G}$ as above. Then* $\mathbf{R}(\mathbf{I}) = \mathbf{J}'$, $\mathbf{R}(\mathbf{J}) = \mathbf{I}'$.

Proof: For $\bar{A} = \bar{V}_{a_0}$, $A_{a,b} = \delta_{a_0ab^{-1}} \cdot 1$ $\left(\text{define } \delta_c \begin{cases} = 1 \text{ for } c = 1 \\ = 0 \text{ for } c \neq 1 \end{cases}\right)$, and for $\bar{A} = \bar{M}_{\varphi(x)}$, $A_{a,b} = \delta_{ab^{-1}}L_{\varphi(xb^{-1})}$. So $\bar{V}_{a_0}$ and $\bar{M}_{\varphi(x)} \in \mathbf{I}'$ (with $\chi_c(x) = \delta_{a_0c}$ resp. $\delta_c\varphi(x)$), that is $\mathbf{J} \subset \mathbf{I}'$. As $\mathbf{I}'$ is a ring, this implies $\mathbf{R}(\mathbf{J}) \subset \mathbf{I}'$.

If $\bar{A} \in \mathbf{I}'$, $\bar{B} \in \mathbf{J}'$ then we have $A_{a,b} = L_{\chi_{ab^{-1}}(xb^{-1})}$, $B_{a,b} = L_{\xi_{a^{-1}b}(x)}U_{b^{-1}a}$. Now put $\bar{C} = \bar{A}\bar{B}$, $\bar{D} = \bar{B}\bar{A}$ then the formula (iv) of Lemma 2.4.3 gives:

$$C_{ab} = \sum_c A_{cb}B_{ac} = \sum_c \chi_{cb^{-1}}(xb^{-1})\xi_{a^{-1}c}(x)U_{c^{-1}a} = \sum_c \chi_{ac^{-1}b^{-1}}(xb^{-1})\xi_{c^{-1}}(x)U_c,$$

(we replaced the summation index c by ac^{-1}) and

$$\begin{aligned} D_{a,b} &= \sum_c B_{cb}A_{ac} = \sum_c \xi_{c^{-1}b}(x)U_{b^{-1}c}\chi_{ac^{-1}}(xc^{-1}) \\ &= \sum_c \xi_{c^{-1}b}(x)\chi_{ac^{-1}}(xb^{-1})U_{b^{-1}c} = \sum_c \chi_{ac^{-1}b^{-1}}(xb^{-1})\xi_{c^{-1}}(x)U_c, \end{aligned}$$

(we replaced the summation index c by bc, remember that the order of summation in $\sum_c$ does not matter by Lemma 2.4.3 (iv)). So we have $\bar{C} = \bar{D}$, $\bar{A}\bar{B} = \bar{B}\bar{A}$. As $\bar{B} \in \mathbf{J}'$ implies $\bar{B}^* \in \mathbf{J}'$, this means $\bar{A} \in \mathbf{J}''$; as $\bar{A}$ was an arbitrary element of $\mathbf{I}'$ we have $\mathbf{I}' \subset \mathbf{J}''$. As $1 \in \mathbf{J}$, therefore $\mathbf{J}'' = \mathbf{R}(\mathbf{J})$ (cf. (18), p. 397), so $\mathbf{I}' \subset \mathbf{R}(\mathbf{J})$.

Thus we have proved $\mathbf{R}(\mathbf{J}) = \mathbf{I}'$. The spatial isomorphism $F \to \overline{W}F$ of $\mathfrak{H}_{S\mathfrak{G}}$ interchanges $\mathbf{I}$ with $\mathbf{J}$, therefore we have proved $\mathbf{R}(\mathbf{I}) = \mathbf{J}'$ too. Thus the proof is complete.

Lemma 12.3.4. *Let $\mathfrak{G}$ be an m-group in S. Put* $\mathbf{M} = \mathbf{R}(\mathbf{I}) = \mathbf{J}'$ *then* $\mathbf{M}' = \mathbf{R}(\mathbf{J}) = \mathbf{I}'$. *If $\mathfrak{G}$ is ergodic in S, then $\mathbf{M}$ is a factor.*

Proof: Clearly $\mathbf{M}' = (\mathbf{R}(\mathbf{I}))' = \mathbf{I}'$. If $A \in \mathbf{M} \cdot \mathbf{M}'$ then we have by Lemma 12.3.2 $A_{a,b} = L_{\chi_{ab^{-1}}(xb^{-1})} = L_{\xi_{a^{-1}b}(x)}U_{b^{-1}a}$. So if $a \neq b$ that is $b^{-1}a \neq 1$ then Lemma 12.2.3 gives $\chi_{ab^{-1}}(xb^{-1}) = \xi_{a^{-1}b}(x) = 0$. That is: $\chi_c(x) = \xi_c(x) = 0$ if $c \neq 1$. For $a = b$, that is, $b^{-1}a = 1$ similarly $\chi_1(xb^{-1}) = \xi_1(x)$ obtains. This gives for $b = 1$, $\chi_1(x) = \xi_1(x)$ and then in general $\chi_1(xb^{-1}) = \chi_1(x)$. In other words: $U_b^{-1}L_{\chi_1(x)}U_b = L_{\chi_1(x)}$. So we found the following conditions: $\chi_c(x) = \xi_c(x)$ for all $c \in \mathfrak{G}$, if $c \neq 1$ then even $= 0$; if $c = 1$ then $L_{\chi_1(x)} \in \mathbf{L} \cdot \mathbf{U}'$.

Now if $\mathfrak{G}$ is ergodic in S, then Lemma 12.2.4 gives $\mathbf{L} \cdot \mathbf{U}' = (\alpha 1)$ so we have

$L_{\chi_1(x)} = \alpha 1$, $\chi_1(x) = \alpha$. This means $\chi_c(x) = \xi_c(x) = \alpha\delta_c$, that is $\bar{A} = \alpha 1$. Thus we have proved $\mathbf{M}\cdot\mathbf{M}' = (\alpha 1)$ that is: $\mathbf{M}$ is a factor.

12.4. From now on we assume that $\mathfrak{G}$ is ergodic in S.

Lemma 12.4.1. *Write for an* $\bar{A} \in \mathbf{M}$ *or* $\in \mathbf{M}'$ *that* $\bar{A} \approx [[\chi_c(x)]]_{x \in S,\, c \in \mathfrak{G}}$ *if* $A \sim < A_{a,b} >_{a,b \in \mathfrak{G}}$ *and* $A_{a,b} = L_{\chi_{ab^{-1}(x)} U_{b^{-1}a}}$ *resp.* $L_{\chi_{ab^{-1}(xb^{-1})}}$ (*cf. Lemma* 12.3.2). *If now* $\bar{A} \approx [[\chi_c(x)]]_{x \in S,\, c \in \mathfrak{G}}$ $\bar{B} \approx [[\xi_c(x)]]_{x \in S,\, c \in \mathfrak{G}}$, $\bar{C} \approx [[\eta_c(x)]]_{x \in S,\, c \in \mathfrak{G}}$ *then the following rules of computation hold:*

(i) *If* $\bar{C} = \alpha\bar{A}$ *then* $\eta_c(x) = \alpha\chi_c(x)$.

(ii) *If* $\bar{C} = \bar{A}^*$ *then* $\eta_c(x) = \overline{\chi_{c^{-1}}(xc^{-1})}$.

(iii) *If* $\bar{C} = \bar{A} + \bar{B}$ *then* $\eta_c(x) = \chi_c(x) + \xi_c(x)$.

(iv) *If* $\bar{C} = \bar{A}\bar{B}$ *then* $\eta_c(x) = \sum_a \chi_a(x)\xi_{ca^{-1}}(xa^{-1})$. *The* $\sum_a$ *converges "en mesure" (cf. the precise description of this notion at the end of the proof), irrespective of the order in which the* $a \in \mathfrak{G}$ *are gone through.*

Proof: Consider first $\bar{A} \in \mathbf{M}'$ (i), (iii) follow immediately from Lemma 2.4.3, (i), (iii); (ii) follows from Lemma 2.4.3, (ii), by the following consideration: $\bar{C}_{a,b} = L_{\eta_{ab^{-1}(xb^{-1})}}$, $\bar{C}_{a,b} = A^*_{b,a} = L_{\overline{\chi_{ba^{-1}(xa^{-1})}}}$, $\eta_{ab^{-1}}(xb^{-1}) = \overline{\chi_{ba^{-1}}(xa^{-1})}$, $\eta_c(x) = \overline{\chi_{c^{-1}(xc^{-1})}}$. Finally (iv) results from Lemma 2.4.3, (iv): $C_{a,b} = L_{\eta_{ab^{-1}(xb^{-1})}}$, $C_{ab} = \sum_c A_{cb}B_{ac} = \sum_c L_{\chi_{cb^{-1}(xb^{-1})}L_{\xi_{ac^{-1}(xc^{-1})}}} = \sum_c L_{\chi_{cb^{-1}(xb^{-1})}\xi_{ac^{-1}(xc^{-1})}}$, $L_{\eta_{ab^{-1}(xb^{-1})}} = \sum_c L_{\chi_{cb^{-1}(xb^{-1})}\xi_{ac^{-1}(xc^{-1})}}$ and putting $b = 1$, and writing c, a for a, c, $L_{\eta_c(x)} = \sum_a L_{\chi_a(x)\xi_{ca^{-1}(xa^{-1})}}$ is strongly convergent, irrespective of the order in which the $a \in \mathfrak{G}$ are gone through.

Let now $T \subset S$ be a measurable set of finite measure, put $e_T(x) \begin{cases} = 1 \text{ for } x \in T \\ = 0 \text{ for } x \notin T \end{cases}$ so that $e_T \in \mathfrak{H}_S$. Then if $\bar{a}_1, \bar{a}_2, \cdots$ is an enumeration of $\mathfrak{G}$ we have

$$\left\| L_{\eta_c(x)}e_T - \sum_1^i{}_j L_{\chi_{\bar{a}_j}(x)\xi_{c\bar{a}_j^{-1}(x\bar{a}_j^{-1})}} e_T \right\| \to 0 \text{ for } i \to \infty$$

so

$$\int_S \left| \eta_c(x)e_T(x) - \sum_1^i{}_j \chi_{\bar{a}_j^{-1}}(x)\xi_{c\bar{a}_j^{-1}}(x\bar{a}_j^{-1})e_T(x) \right|^2 dx \to 0\,,$$

$$\int_T \left| \eta_c(x) - \sum_1^i{}_j \chi_{\bar{a}_j}(x)\xi_{c\bar{a}_j^{-1}}(x\bar{a}_j^{-1}) \right|^2 dx \to 0\,.$$

This implies for every $\epsilon > 0$ that $\mu((x;\ |\ \eta_c(x) - \sum_{j=1}^i \chi_{\bar{a}_j}(x)\xi_{c\bar{a}_j^{-1}}(x\bar{a}_j^{-1})\ |\ \geqq \epsilon)\cdot T) \to 0$, for $i \to \infty$. That is: $\sum_{j=1}^\infty \chi_{\bar{a}_j}(x)\xi_{c\bar{a}_j^{-1}}(x\bar{a}_j^{-1})$ converges "en mesure" to $\eta_c(x)$. As $\bar{a}_1, \bar{a}_2, \cdots$ was an arbitrary enumeration of $\mathfrak{G}$ this proves (iv). (Of course, if $\mathfrak{G}$ is finite, the convergence considerations are unnecessary, the sum $\sum_a$ being simply equal to $\eta_c(x)$.)

Consider now $\bar{A} \in \mathbf{M}$. The spatial isomorphism $F \to \overline{W}F$ in $\mathfrak{H}_{S\mathfrak{G}}$ carries $\mathbf{I}$ into $\mathbf{J}$ and therefore $\mathbf{M}'$ into $\mathbf{M}$. So we have $\bar{A} = \overline{W}\bar{B}\overline{W}, \bar{B} \in \mathbf{M}'$. This $\bar{B} \approx [[\chi_c(x)]]_{x \in S,\, c \in \mathfrak{G}}$. Now the computation at the end of the proof of Lemma 12.3.2 shows that then $\bar{A} \approx [[\chi_c(x)]]_{x \in S,\, c \in \mathfrak{G}}$ with the same $\chi_c(x)$. Therefore, the rules of computation established in $\mathbf{M}'$ carry over immediately to $\mathbf{M}$.

After these preparations we are in the position to take the decisive step:

Definition 12.4.1. Assume $\bar{A} \in \mathbf{M}$ or $\in \mathbf{M}'$. Form the system of functions $\chi_c(x)$ with $\bar{A} \approx [[\chi_c(x)]]_{x \in S,\, c \in \mathfrak{G}}$ (cf. Lemma 12.4.1). Consider the two following cases:

(α) $\mu(S)$ is finite. Then, as $\chi_1(x)$ is bounded and measurable, $\int_S \chi_1(x)\, dx$ exists, and it is a finite complex number.

(β) $\chi_1(x) \geqq 0$ for all $x \in S$. Then $\int_S \chi_1(x)\, dx$ exists again, and it is a real number $\geqq 0, \leqq \infty$. In both cases define $t(\bar{A}) = \int_S \chi_1(x)\, dx$.

In what follows immediately, we will really make use of case (β) only; case (α) is needed for later applications (§15.4).

Lemma 12.4.2. *Assume* $\bar{A}, \bar{B} \in \mathbf{M}$ *or* $\mathbf{M}'$. *Then we have:*

(i) $t(\alpha\bar{A}) = \alpha t(\bar{A})$.

(ii) $t(\bar{A}^*) = t(\bar{A})$.

(iii) $t(\bar{A} + \bar{B}) = t(\bar{A}) + t(\bar{B})$.

These equations are to be so understood, that whenever case (α) *or* (β) *holds for the right sides, it holds for the left sides too (for* (*i*) *with case* (β) *however,* $\alpha \geqq 0$ *is needed), and both are equal. Finally we have (automatically with case* (β) *on both sides):*

(iv) $t(\bar{A}^*\bar{A}) = t(\bar{A}\bar{A}^*) \geqq 0$.

Proof: (i)–(iii) follow directly from Lemma 12.4.1, (i)–(iii) resp. In order to prove (iv), put

$$\bar{A}^*\bar{A} \approx [[\omega_c(x)]]_{x \in S,\, c \in \mathfrak{G}}, \qquad \bar{A}\bar{A}^* \approx [[\nu_c(x)]]_{x \in S,\, c \in \mathfrak{G}}.$$

As $\bar{A} \approx [[\chi_c(x)]]_{x \in S,\, c \in \mathfrak{G}}$, we have $\bar{A}^* \approx [[\overline{\chi_{c^{-1}}(xc^{-1})}]]_{x \in S,\, c \in \mathfrak{G}}$ and

$$\begin{aligned} \omega_c(x) &= \textstyle\sum_a \chi_{ca^{-1}}(xa^{-1})\, \overline{\chi_{a^{-1}}(xa^{-1})} \\ &= \textstyle\sum_a \chi_{ca}(xa)\, \overline{\chi_a(xa)}, \\ \nu_c(x) &= \textstyle\sum_a \chi_a(x)\, \overline{\chi_{ac^{-1}}(xc^{-1})}. \end{aligned}$$

(Use Lemma 12.4.1, (ii) and (iv). The sums converge "en mesure," irrespectively of the order.) Thus $\omega_1(x) = \sum_a |\chi_a(xa)|^2$, $\nu_1(x) = \sum_a |\chi_a(x)|^2$. We have case ($\beta$) for both, and

$$\begin{aligned} t(\bar{A}^*\bar{A}) &= \int_S \omega_1(x)\, dx = \textstyle\sum_a \int_S |\chi_a(xa)|^2\, dx \\ &= \textstyle\sum_a \int_S |\chi_a(x)|^2\, dx, \\ t(\bar{A}\bar{A}^*) &= \int_S \nu_1(x)\, dx = \textstyle\sum_a \int_S |\chi_a(x)|^2\, dx, \end{aligned}$$

proving our statements.

Lemma 12.4.3. *If $\bar{E}$ runs over all projections in* **M** *or in* **M**′ *then $t(E)$ is always defined (case (β) holds), and it is a relative dimension function for* **M** *resp.* **M**′. *In this normalisation the C of Theorem* X *is* $= 1$.

If $T \subset S$ is measurable, define $e_T(x)\begin{cases} = 1 \text{ for } x \in T \\ = 0 \text{ for } x \notin T \end{cases}$ *Then an* $\bar{E} \approx [[\delta_c e_T(x)]]_{x \in S,\, c \in \mathfrak{G}}$ *exists, in* **M** *as well as in* **M**′, *it is a projection, and* $t(\bar{E}) = \mu(T)$.

Proof: As $\bar{E}^*\bar{E} = \bar{E}^2 = \bar{E}$ for every projection $\bar{E}$, therefore Lemma 12.4.2 (iv) secures case (β) for $\bar{E}$ and $t(\bar{E}) \geqq 0, \leqq \infty$. If $T \subset S$ is measurable, then $\bar{L}_{e_T(x)} = < \delta_{a^{-1}b} L_{e_T(x)} >_{a,\, b \in \mathfrak{G}} = < L_{\delta_{ab^{-1}} e_T(x)} U_{b^{-1}a} >_{a,\, b \in \mathfrak{G}} \approx [[\delta_c e_T(x)]]_{x \in S,\, c \in \mathfrak{G}}$ in **M** and $\bar{M}_{e_T(x)} = < \delta_{ab^{-1}} L_{e_T(xa^{-1})} >_{a,\, b \in \mathfrak{G}} = < L_{\delta_{ab^{-1}} e_T(xb^{-1})} > \approx [[\delta_c e_T(x)]]_{x \in S,\, c \in \mathfrak{G}}$ in **M**′. So the $\bar{E}$'s required at the end of our Lemma exist; clearly $\bar{E}^* = \bar{E}$, $\bar{E}^2 = \bar{E}$ so that $\bar{E}$ is a projection; and $t(\bar{E}) = \int_S e_T(x)\, dx = \int_T 1 \cdot dx = \mu(T)$. Thus the last statement of our Lemma is true.

As we observed in the proof of Lemma 12.1.1, there exists at least one element of the sequence $T^{(1)}, T^{(2)}, \cdots$ from Definition 12.1.2, (iv), for which $\mu(T^{(i)}) > 0, < \infty$. So we have for its $\bar{E}$ (cf. above) $t(\bar{E}) > 0, < \infty$.

Now Lemma 8.3.5 implies that $t(\bar{E})$ is a relative dimension function for **M** resp. **M**′ if we can only establish for it the conditions (ii), (iii) in Definition 8.2.1. (iii) follows immediately from Lemma 12.4.2, (iii), because under those assumptions $P_{[\overline{\mathfrak{M}}, \overline{\mathfrak{N}}]} = P_{\overline{\mathfrak{M}}} + P_{\overline{\mathfrak{N}}}$. (ii) follows from Lemma 12.4.2, (iv), because $\bar{E} \sim \bar{F}$ (($\cdots$ **M**) resp. ($\cdots$ **M**′)) implies the existence of a $U(\in$ **M** resp. $\in$ **M**′) with $\bar{U}^*\bar{U} = \bar{E}$, $\bar{U}\bar{U}^* = \bar{F}$ (cf. Lemma 4.3.1).

Our last task is to determine the C of Theorem X. If $E \in$ **M**, $\bar{E} \approx [[\chi_c(x)]]_{x \in S,\, c \in \mathfrak{G}}$ then we saw at the end of the proof of Lemma 12.4.1 that $\overline{W}\bar{E}\overline{W} \in$ **M**′, $\overline{W}\bar{E}\overline{W} \approx [[\chi_c(x)]]_{x \in S,\, c \in \mathfrak{G}}$. Thus $t(\bar{E}) = t(\overline{W}\bar{E}\overline{W})$. Now, owing to the character of the spatial isomorphism $F \to \overline{W}F$ it carries every $\mathfrak{M}_F^{\mathbf{M}'}$ into the corresponding $\mathfrak{M}_{WF}^{\mathbf{M}}$ therefore $\overline{W}\bar{E}_F^{\mathbf{M}'}\overline{W} = \bar{E}_{WF}^{\mathbf{M}}$. This proves $t(\bar{E}_F^{\mathbf{M}'}) = t(\bar{E}_{WF}^{\mathbf{M}})$ and thus, if $\overline{W}F = \pm F$, $t(\bar{E}_F^{\mathbf{M}'}) = t(\bar{E}_F^{\mathbf{M}})$.

If besides $F \gneqq 0$, then $\bar{E}_F^{\mathbf{M}'} \gneqq 0$ and (as $t(\bar{E})$ is a relative weight function for **M**), $t(\bar{E}_F^{\mathbf{M}'}) > 0$. So the above relations imply, considering $t(\bar{E}_F^{\mathbf{M}}) = Ct(E_F^{\mathbf{M}'})$, that $C = 1$. So we need only to find an $F \gneqq 0$ with $WF = \pm F$. Choose any $G \gneqq 0$; as $W^2 = 1$ we have $\overline{W}(G \pm \overline{W}G) = \pm(G \pm \overline{W}G)$ and as $G = \frac{1}{2}((G + \overline{W}G) + (G - \overline{W}G))$ therefore both $G \pm \overline{W}G$ cannot be $= 0$. So either $F = G + \overline{W}G$ or $F = G - \overline{W}G$ meets our requirements.

Thus all parts of our Lemma are proved.

We restate the results obtained thus far:

Theorem XI. *Let $\mathfrak{G}$ be a finite or countably infinite group (Definition* 12.1.1), *S a space with a Lebesgue outer measure (Definitions* 12.1.2 *and* 12.1.3). *Let $\mathfrak{H}_{\mathfrak{G}}$, $\mathfrak{H}_S$, $\mathfrak{H}_{S\mathfrak{G}}$ be the spaces derived from them (Definition* 12.1.4). *Assume that $\mathfrak{G}$ is ergodic in S (Definition* 12.1.5).

Form the operators $\overline{U}_{a_0}$, $\overline{V}_{a_0}$, $\overline{W}$, $\overline{L}_{\varphi(x)}$, $\overline{M}_{\varphi(x)}$ in $\mathfrak{H}_{S\mathfrak{G}}$ (Lemma 12.3.1) *and with their help the rings* **M**, **M**′ (*Lemma* 12.3.4).

$F \to \overline{W}F$ is an involutory isomorphism of $\mathfrak{H}_{S\mathfrak{G}}$ which interchanges **M** *with* **M**′.

Certain systems of bounded and measurable x-functions $\chi_c(x)$, $c \in \mathfrak{G}$ *are in a one-to-one correspondence with all* $\bar{A} \in \mathbf{M}$ *and at the same time with all* $\bar{A} \in \mathbf{M}'$. *We denote both correspondences by* $\bar{A} \sim [[\chi_c(x)]]_{x \in S,\, c \in \mathfrak{G}}$ (*Lemma* 12.4.1). *Using these correspondences, the computation rules in* $\mathbf{M}$ *as well as those in* $\mathbf{M}'$ *can be expressed in terms of the* $\chi_c(x)$; *we get in both cases the same rules* (*Lemma* 12.4.1, (i)–(iv)).

$\mathbf{M}$, $\mathbf{M}'$ *are coupled factors. Relative dimension functions* $D_{\mathbf{M}}(\bar{E})$, $D_{\mathbf{M}'}(\bar{E})$ *for* $\mathbf{M}$ *resp.* $\mathbf{M}'$ ($\bar{E}$ *a projection* $\in \mathbf{M}$ *resp.* $\in \mathbf{M}'$) *can be defined as follows: Write* $\bar{E} \sim [[\chi_c(x)]]_{x \in S,\, c \in \mathfrak{G}}$ *then* $D_{\mathbf{M}}(\bar{E})$ *resp.* $D_{\mathbf{M}'}(\bar{E}) = \int_S \chi_1(x)\, dx$. (*We have* $\chi_1(x) \geqq 0$ *for all* $x \in S$, *therefore the integral on the right hand is always defined; it represents a real number* $\geqq 0$, $\leqq \infty$).

In this normalisation the C of Theorem X *is* $= 1$.

If $T \subset S$ *is measurable, and* $e_T(x) \begin{cases} = 1 \text{ for } x \in T \\ = 0 \text{ for } x \notin T \end{cases}$ *then we have for both projections* $\bar{E} = \bar{L}_{e_T(x)} \in \mathbf{M}$ *and* $\bar{E}' = \bar{M}_{e_T(x)} \in \mathbf{M}'$ *this:* $D_{\mathbf{M}}(\bar{E})$ *resp.* $D_{\mathbf{M}'}(\bar{E}) = \mu(T)$.

Chapter XIII: Properties of $\mathbf{M}$, $\mathbf{M}'$.

13.1. The characterisation of $\mathbf{M}$, $\mathbf{M}'$ by Theorem XI makes the determination of their classes easy. This determination is the object of the discussions which follow. If we wanted to use these $\mathbf{M}$, $\mathbf{M}'$ solely to obtain an example of case (II), then it could be shortened considerably: the examples (α) – (γ) after Lemma 13.2.1 could be obtained in a few lines, together with Lemma 13.1.2 for that special case, while Lemma 13.1.1 would be entirely unnecessary. But owing to the general interest of these $\mathbf{M}$, $\mathbf{M}'$ we prefer to give an exhaustive discussion.

In what follows we use throughout the notations and the assumptions of Theorem XI.

Lemma 13.1.1. *Case* (I) *holds for* $\mathbf{M}$, $\mathbf{M}'$ *if and only if a point* $x \in S$ *with* $\mu^*((x)) > 0$ *exists. If this is the case, S can be characterised, after the omission of a subset of measure* 0, (*which therefore is unessential*), *as follows:*

Every one-point set (x) *in* S ($x \in S$) *is measurable, and* $\mu((x))$ *has a value independent of* x: $\mu((x)) = \epsilon > 0$, $< \infty$, $\mathfrak{G}$ *is simply transitive in S, that is if* x_0 *is any fixed element of S, then* $a \rightleftarrows x_0 a$ *is a one to one mapping of* $\mathfrak{G}$ *on S.*

If $\mathfrak{G}$ *and S have* $n = 1, 2, \cdots, \infty$ *elements, then* $\mathbf{M}$, $\mathbf{M}'$ *are both in case* (I_n). *The* $D_{\mathbf{M}}(E)$, $D_{\mathbf{M}'}(E)$ *of Theorem* XI *go over into the standard normalisation, if both are multiplied by* $1/\epsilon$.

Proof: Assume first that case (I) holds. Then all values of $D_{\mathbf{M}}(\bar{E})$ are integer multiples of an $\epsilon_0 > 0$ and so all $\mu(T)$, $T \subset S$ and measurable, are so. Now a T with $\mu(T) > 0$, $< \infty$ exists (cf. the beginning of the proof of Lemma 12.4.3), therefore for one of these $T \subset S$, $\mu(T)$ assumes a minimal value. Denote it by T_0. Thus for any measurable $T \subset T_0$ we have, owing to $0 \leqq \mu(T_0) \leqq \mu(T)$ either $\mu(T_0) = \mu(T)$ or $\mu(T) = 0$.

Now consider the $T^{(1)}$, $T^{(2)}$, $\cdots$ of Definition 12.1.2, (iv), assuming that

$T^{(1)} \dotplus T^{(2)} \dotplus \cdots = S$ (cf. the remark before Lemma 12.1.1), and that 0 is one of them (it can be added to their system without altering its character. Consider an $x \in S$. Denote those n's for which $x \in T^{(n)}$ by $m_1, m_2, \cdots$ and those for which $x \notin T^{(n)}$ by $n_1, n_2, \cdots$. Owing to Definition 12.1.2 (iv), we have $(x) = T^{(m_1)} \cdot T^{(m_2)} \cdot \cdots \cdot (S - T^{(n_1)}) \cdot (S - T^{(n_2)}) \cdots$. As $x \in T_1^{(1)} \dotplus T^{(2)} \dotplus \cdots$ the sequence $m_1, m_2, \cdots$ is not empty; as 0 is a $T^{(n)}$ the sequence $n_1, n_2, \cdots$ is not empty either; by repeating m or n infinitely many times we can secure that both sequences are infinite. So we see: For every $x \in S$ there exist two infinite sequences $m_1, m_2, \cdots$ and $n_1, n_2, \cdots$ so that $(x) = \prod_{i=1}^{\infty} T^{(m_i)} \cdot (S - T^{(n_i)})$.

Now assume $x \in T_0$ and $\mu^*((x)) = 0$. Put $U_j = T_0 \cdot \prod_1^j T^{(m_i)} \cdot (S - T^{(n_i)})$. Then we have $T_0 \supset U_1 \supset U_2 \supset \cdots$, $(x) = U_1 U_2, \cdots$. So $\lim_{j \to \infty} \mu(U_j) = \mu((x)) = 0$ (all U_j are measurable along with T_0 and the $T^{(n)}$), and so a j with $\mu(U_j) < \mu(T_0)$ exists, (remember that $\mu(T_0) > 0$). By our assumptions on T_0 this implies $\mu(U_j) = 0$.

Now consider all sets U of this form: $U = T_0 \cdot \prod_{i=1}^{j} T^{(m_i)} (S - T^{(n_i)})$ $j = 1, 2, \cdots$; $m_1, \cdots m_j, n_1, \cdots n_j = 1, 2, \cdots$ for which $\mu(U) = 0$. Their set is countable: $U^{(1)}, U^{(2)}, \cdots$. So $\mu(U^{(1)} \dotplus U^{(2)} \dotplus \cdots) = 0$. Thus an $x \in T_0$ with $x \notin U^{(1)} \dotplus U^{(2)} \cdots$ exists. By what we proved above, this excludes $\mu^*((x)) = 0$. So an $x \in S$ with $\mu^*((x)) > 0$ exists.

Assume now conversely the existence of an $x \in S$ with $\mu^*((x)) > 0$. As $x = \prod_1^{\infty} T^{(m_i)}(S - T^{(n_i)})$ (cf. above), (x) is measurable, $\mu((x)) > 0$. Besides $(x) \subset T^{(m_1)}$, $\mu((x)) \leqq \mu(T^{(m_1)}) < \infty$. Put $\epsilon = \mu((x)) > 0, < \infty$ then $\mu((xa)) = \mu((x)) = \epsilon$ for every $a \in \mathfrak{G}$. Form now $S_1 = (xa; a \in \mathfrak{G})$, S_1 is exactly invariant under every transformation $x \to xb$, $b \in \mathfrak{G}$. Thus the ergodicity of $\mathfrak{G}$ in S (cf. Definition 12.1.5, (iv)) requires $\mu(S_1) = 0$ or $\mu(S - S_1) = 0$. But $\mu(S_1) \geqq \mu((x)) = \epsilon > 0$ so $\mu(S - S_1) = 0$. Omit the set $S - S_1$ from S, then we have $S_1 = S$, that is $S = (xa; a \in \mathfrak{G})$. So we have for every $x_0 \in S$, $\mu((x_0)) = \epsilon$. By Def. 12.1.5 (iii), $a \neq 1$ implies $x_0 \neq x_0 a$, so $a \neq b$ implies $x_0 a \neq x_0 b$. Thus all statements of the second section of our Lemma are established for this case.

Choose an $x_0 \in S$ and define $e_{(x_0)}(x) \begin{cases} = 1 \text{ for } x = x_0 \\ = 0 \text{ for } x \gtrless x_0 \end{cases}$. Then $\bar{L}_{e_{(x_0)}} \in \mathbf{M}$, it is a projection, and $D_{\mathbf{M}'}(\bar{L}_{e_{(x_0)}(x)}) = \mu((x_0)) = \epsilon$. We claim that $L_{e_{(x_0)}(x)}$ is minimal in $\mathbf{M}$. This is the case, if for every F with $\bar{L}_{e_{(x_0)}} F = F \gtrless 0$, $E_F^{\mathbf{M}'} = \bar{L}_{e_{(x_0)}(x)}$ (cf. Lemma 5.1.2, obviously $L_{e_{(x_0)}(x)} \gtrless 0$). Now $L_{e_{(x_0)}(x)} F = F$ means $e_{x_0}(x) F(x, a) = F(x, a)$ that is $F(x, a) = 0$ if $x \gtrless x_0$. We must show: One such F, say F_0, gives if it is $\gtrless 0$ by application of operators $\bar{A} \in \mathbf{M}'$ all others. As $F_0 \gtrless 0$ but $F_0(x, a) = 0$ for $x \gtrless x_0$ an $a_0 \in \mathfrak{G}$ with $F_0(x_0, a_0) \gtrless 0$ exists. Now $\dfrac{1}{F_0(x_0, a_0)} \bar{M}_{e_{(x_0 a_1^{-1})}(x)} \bar{V}_{a_1 a_0^{-1}} \in \mathbf{M}'$ and it transforms $F_0(x, a)$ into $F'_{a_1}(x, a) \begin{cases} = 1 \text{ for } x = x_0, a = a_1 \\ = 0 \text{ for otherwise} \end{cases}$ and all desired $F(x, a)$ are linear aggregates of these.

Thus $\bar{L}_{e_{(x_0)}(x)}$ is minimal in $\mathbf{M}$ and therefore case (I) holds for $\mathbf{M}$, $\mathbf{M}'$ (cf. Theorem IV). So the necessary and sufficient character of our condition is established.

As the second section of our Lemma has already been verified, all we need to consider is the last one. As $\overline{L}_{e(x_0)(x)}$ is minimal in $\mathbf{M}$ and $D_{\mathbf{M}'}(\overline{L}_{e(x_0)(x)}) = \epsilon$ the standard normalisation of $D_{\mathbf{M}'}(\bar{E}')$ obtains by multiplying it by $1/\epsilon$. As the spatial isomorphism $F \to \overline{W}F$ interchanges $\mathbf{M}$, $D_{\mathbf{M}}(\bar{E})$ with $\mathbf{M}'$, $D_{\mathbf{M}'}(\bar{E})$ the same is true for $D_{\mathbf{M}}(E)$.

Finally, if S (that is $\mathfrak{G}$) has $n = 1, 2, \cdots, \infty$ elements, say $x_1, \cdots, x_n$ (if $n = \infty$ omit x_n), then $D_{\mathbf{M}'}(1) = D_{\mathbf{M}'}(\overline{L}_1) = D_{\mathbf{M}'}(\overline{L}_{e(x_1)(x)+\cdots+e(x_n)(x)}) = D_{\mathbf{M}'}(\overline{L}_{e(x_1)(x)} + \cdots + \overline{L}_{e(x_n)(x)}) = D_{\mathbf{M}'}(L_{e(x_1)(x)}) + \cdots + D_{\mathbf{M}'}(\overline{L}_{e(x_n)(x)}) = n\epsilon$. In the standard normalisation this gives n, so we have case (I_n) for $\mathbf{M}'$. The spatial isomorphism gives the same for $\mathbf{M}$.

Lemma 13.1.2. *Case* (II) *holds for* $\mathbf{M}$, $\mathbf{M}'$ *if and only if there is for every* $x \in S$, $\mu^*((x)) = 0$. *If* $\mu(S)$ *is finite, then case* (II_1) *holds for both* $\mathbf{M}$, $\mathbf{M}'$ *and the* $D_{\mathbf{M}}(E)$, $D_{\mathbf{M}'}(E)$ *of Theorem* XI *go over into the standard normalisation, if both are multiplied by* $1/\mu(S)$. *If* $\mu(S)$ *is infinite, then case* (II_∞) *holds for both* $\mathbf{M}, \mathbf{M}'$.

Proof: As an $E' \in \mathbf{M}'$ with $D_{\mathbf{M}'}(\bar{E}') > 0, < \infty$ exists (cf. the beginning of the proof of Lemma 12.4.3), case (III) cannot hold. So either case (I) or case (II) holds, and therefore Lemma 13.1.1 implies that our present condition is necessary and sufficient for case (II).

As $D_{\mathbf{M}}(1) = D_{\mathbf{M}}(\bar{M}_1) = D_{\mathbf{M}}(\bar{M}_{e_S(x)}) = \mu(S)$ and $D_{\mathbf{M}'}(1) = D_{\mathbf{M}'}(\overline{L}_1) = D_{\mathbf{M}'}(\overline{L}_{e_S(x)}) = \mu(S)$, the other statements of our Lemma follow immediately.

The two preceding Lemmas characterise $\mathbf{M}$, $\mathbf{M}'$ completely. We see (using the terminology of Theorem VIII): The $\mathbf{M}$, $\mathbf{M}'$ as constructed in Theorem XI are always of the same class; this class is finite or infinite if $\mu(S)$ is finite resp. infinite; it is discrete or continuous if the measure $\mu(T)$, $T \subset S$ is discrete resp. continuous (that is, if an $x \in S$ with $\mu^*((x)) > 0$ does resp. does not exist); it is never purely infinite. And the difference between the discrete and the continuous cases corresponds to that one between effective transitivity of $\mathfrak{G}$ in S, and ergodicity without such transitivity.

13.2. Examples of case (I), based on Lemma 13.1.1, are so obvious that we need not be concerned with them. (Owing to the one-to-one correspondence between S and $\mathfrak{G}$ we may even put, without any loss of generality, $S = \mathfrak{G}$. Then we can make $\epsilon = 1$ and then $\mu^*(T)$, $T \subset S$ becomes $\mu^*(T) =$ (number of elements in T).) We wish however to give effective examples of case (II), based on Lemma 13.1.2. These examples will be of a very special type, as we will only consider Abelian groups $\mathfrak{G}$, and even those highly specialised.

Lemma 13.2.1. *Let* S *be either the set of all real numbers,* S_∞, *or the set of all real numbers* $\geqq 0, < 1$, *which can be (and in what follows will be) looked at as the set of all real numbers* (mod 1), S_1. *Use the common Lebesgue measure in* S. *Let* $\mathfrak{G}$ *be a module of real numbers, that is a finite or countably infinite set with these properties:* $0 \in \mathfrak{G}$, $a, b \in \mathfrak{G}$ *imply* $-a \in \mathfrak{G}$, $a + b \in \mathfrak{G}$. $\mathfrak{G}$ *is a group, if we define* unit $= 0$, $a^{-1} = -a$, $ab = a + b$.

$\mathfrak{G}$ *is an m-group in* S, *if we define* $xa = x + a$. *For* $S = S_1$ *we insist that* $1 \in \mathfrak{G}$.

Excepting $\mathfrak{G} = 0$ *and* $\mathfrak{G} = (\ , \cdots, -2\alpha_0, -\alpha_0, 0, \alpha_0, \cdots)$ *(for a fixed*

$a_0 > 0$ *if,* $S = S_1$, *then necessarily* $a_0 = 1/n$, $n = 1, 2, \cdots$), $\mathfrak{G}$ *is always ergodic in* S.

Proof: $\mathfrak{G}$ satisfies obviously Definition 12.1.1. S satisfies Definition 12.1.2 because it is a special case of Example b) given after it; it is easy to verify Definition 12.1.5, (i)–(iii), for $\mathfrak{G}$ and S, that is the m-group character. It remains for us to decide about Definition 12.1.5 (iv), that is ergodicity.

Every $T \subset S$ is invariant under $\mathfrak{G} = (0)$ and the set

$$T = x;\ pa_0 \leqq x < (p + \tfrac{1}{2})a_0\,, \qquad p = 0 \pm 1, \pm 2, \cdots) \subset S$$

(for $S = S_1$ take it (mod 1)) is invariant under

$$\mathfrak{G} = (\cdots, -2a_0, -a_0, 0, a_0, 2a_0, \cdots)\,.$$

Thus these $\mathfrak{G}$ are not ergodic.

Assume now that $\mathfrak{G}$ is not of this form. If $S = S_\infty$ and some $a_0 \neq 0$ is $\epsilon\ \mathfrak{G}$ we can map S_∞ and $\mathfrak{G}$ by the one-to-one transformation $x \rightleftarrows (1/a_0)\,x$, $a \rightleftarrows (1/a_0)a$. This is an isomorphism for everything that is important now and it carries a_0 into 1. For this reason we may assume $1 \epsilon \mathfrak{G}$ originally. If $1 \epsilon \mathfrak{G}$, then all sets T which are invariant under $\mathfrak{G}$ can be looked at (mod 1); then we can do this for $S = S_\infty$, that is we obtain $S = S_1$. So we need to discuss $S = S_1$ only.

Now $\mu(S) = \mu(S_1) = 1$ is finite, so every $e_T(x)$ $\begin{cases} = 1 \text{ for } x \epsilon T \\ = 0 \text{ for } x \notin T \end{cases}$ is $e_T \epsilon \mathfrak{H}_S$.

Ergodicity means: $U_a(e_T) = e_T$ for all $a \epsilon \mathfrak{G}$ means $e_T = 0$ or 1 or more generally: $f \epsilon \mathfrak{H}_S$, $U_a f = f$ for all $a \epsilon \mathfrak{G}$ means $f =$ constant.

The $\varphi_n(x) = e^{2\pi i n x}$, $n = 0, \pm 1, \pm 2, \cdots$ form a complete normalised orthogonal set in $\mathfrak{H}_S$, $U_a\varphi_n = e^{2\pi i n a}\varphi_n$. Put $f = \sum_{-\infty}^{\infty} \alpha_n\varphi_n$, ($\sum_{-\infty}^{\infty} |\,\alpha_n\,|^2$ finite; this orthogonal expansion in the space $\mathfrak{H}_S$ is numerically the "in the mean" convergent Fourier expansion of f), then $U_a f = \sum_{-\infty}^{\infty} e^{2\pi i n a}\alpha_n\varphi_n$, $U_a f = f$ means $\alpha_n = e^{2\pi i n a}\alpha_n$, that is $\alpha_n = 0$, except if na is an integer for every $a \epsilon \mathfrak{G}$. If this happens for any $n \neq 0$ then it is the case for $|\,n\,| > 0$ too, so we may assume $n = 1, 2, \cdots$.

Denote the set of all na, $a \epsilon \mathfrak{G}$ by $\mathfrak{G}'$. $\mathfrak{G}'$ then consists of integers. $0 \epsilon \mathfrak{G}'$; $p, q \epsilon \mathfrak{G}'$ imply $-p$, $p + q \epsilon \mathfrak{G}'$. Thus $\mathfrak{G}'$ consists of all multiples of a fixed $q = 1, 2, \cdots$. As $1 \epsilon \mathfrak{G}$, $n \epsilon \mathfrak{G}'$, q is a divisor of n, say $n = qm$. Thus

$$\mathfrak{G} = \left(\ , \cdots, -\frac{2q}{n}, -\frac{q}{n}, 0, \frac{q}{n}, \frac{2q}{n}, \cdots\right)$$

$$= \left(\cdots, \cdots, -\frac{2}{m}, -\frac{1}{m}, 0, \frac{1}{m}, \frac{2}{m}, \cdots\right)$$

contradicting our assumption concerning $\mathfrak{G}$.

So $\alpha_n = 0$ whenever $n \neq 0$ and therefore $f = \alpha_0\varphi_0 = \alpha_0 =$ constant. This completes the proof.

Now as $\mu(S_\infty) = \infty$ and $\mu(S_1) = 1$ we see by Lemma 13.1.2 that every

ergodic $\mathfrak{G}$ gives an example of $\mathbf{M}$, $\mathbf{M}'$ in class (II_∞) resp. (II_1). Some simple $\mathfrak{G}'$ which are obviously ergodic by Lemma 13.2.1 are these:

(α) $\mathfrak{G}$ is the set $\mathfrak{G}_\theta$ of all $m + n\,\theta$, $m, n = 0, \pm 1, \pm 2, \cdots$ where θ is any given irrational number.

(β) $\mathfrak{G}$ is the set $\mathfrak{G}_{\text{rat.}}$ of all rational numbers.

(γ) $\mathfrak{G}$ is the set $\mathfrak{G}_{\text{rat. }p}$ of all rational numbers of the form

$$m/p^n,\ m = 0, \pm 1, \pm\,, 2, \cdots, \qquad n = 0, 1, 2, \cdots$$

where p is any given number 2, 3, $\cdots$ (not necessarily a prime!).

13.3. The above examples (with $S = S_\infty$) show that factorisations $\mathbf{M}$, $\mathbf{M}'$ of the classes (II_∞), (II_∞) exist. Then Lemma 11.4.3, (ii) and the remarks after this Lemma show that the combinations of classes (II_1), (II_∞); (II_∞), (II_1); (II_1), (II_1) exist too, and that in the last case the C of Theorem X can be prescribed arbitrarily. That any combination of cases (I_m), (I_n), $m, n = 1, 2, \cdots$, exists, has been remarked at the end of §8.6. All other combinations, except (III_∞), (III_∞), have been excluded by Theorem X. By Theorem X again (III_∞), (III_∞) exist if and only if factors in case (III) exist at all. So we have:

Theorem XII. *Factorisations* $\mathbf{M}$, $\mathbf{M}'$ *belonging to the following classes do exist:* (I_m), (I_n), $m, n = 1, 2, \cdots, \infty$; (II_m), (II_n), $m, n = 1, \infty$, *and in case* $m, n = 1$ *even the* C *of Theorem* X *can be prescribed arbitrarily.* (III_∞), (III_∞) *exists if and only if factors in case* (III) *exist, a problem as yet unsolved. All other combinations of classes do not exist.*

This answers Problem 2, negatively (considering Lemma 8.6.1), and answers Problems 3, 4 as completely as it is possible at the present state of our knowledge.

13.4. We can use our $\mathbf{M}$, $\mathbf{M}'$ to construct a factorisation in which the factors $\mathbf{M}$ and $\mathbf{N}$ are not coupled ($\mathbf{M} \not\gtrless \mathbf{N}'$), and thus answer Problem 1 negatively. This leads to the partial answer to Problem 10 discussed in §11.2.

In what follows we use throughout the notations and the assumptions of Theorem XI.

Lemma 13.4.1. *Let* $\mathfrak{G}^0$ *be a proper subgroup of* $\mathfrak{G}$ *which is ergodic in* S. *Every* $\bar{A} \in \mathbf{M}'$ *is* $A \approx [[\chi_c(x)]]_{x \in S,\, c \in \mathfrak{G}}$, *consider those for which* $\chi_c(x) = 0$ *whenever* $c \notin \mathfrak{G}^0$. *Denote their set by* $\mathbf{N}$. *Then* $\mathbf{N}$ *is a ring,* $\mathbf{N} \subsetneqq \mathbf{M}'$, $\mathbf{M}' \cdot \mathbf{N}' = (\alpha 1)$.

Proof: $\mathbf{N} \subset \mathbf{M}'$ is obvious. As $\bar{V}_{a_0} \in \mathbf{M}'$, $\bar{V}_{a_0} \approx [[\delta_{ca_0^{-1}}]]_{x \in S,\, c \in \mathfrak{G}}$, therefore $\bar{V}_{a_0} \in \mathbf{N}$ if and only if $a_0 \in \mathfrak{G}^0$, so $\mathfrak{G}^0 \subsetneqq \mathfrak{G}$ implies $\mathbf{N} \not\gtrless \mathbf{M}'$.

The computation rules of Lemma 12.4.1 show that $A, B \in \mathbf{N}$ imply αA, A^*, $A + B$, $AB \in \mathbf{N}$. So we must only prove that $\mathbf{N}$ is weakly closed in order to show that it is a ring. As $\mathbf{M}'$ is weakly closed (being a ring) even relative weak closure of $\mathbf{N}$ in $\mathbf{M}'$ suffices.

Now an $\bar{A} \in \mathbf{M}'$ belongs to $\mathbf{N}$ if and only if $a_0 \notin \mathfrak{G}^0$ implies $\chi_{a_0}(x) = 0$ for $\bar{A} \approx [[\chi_c(x)]]_{x \in S,\, c \in \mathfrak{G}}$; that is if $ab^{-1} \notin \mathfrak{G}^0$ implies $A_{a,b} = 0$ for $\bar{A} \sim < A_{a,b} >_{a,b \in \mathfrak{G}}$. So we need only to prove: If two $a, b \in \mathfrak{G}$ are given, then the set of all $\bar{A} \sim < A_{a',b'} >_{a',b' \in \mathfrak{G}}$ with $A_{a,b} = 0$ is weakly closed. Now $(A_{a,b}f, g) = (\bar{A}f^{[a]}, g^{[b]})$ where $h^{[c]}(x) = \delta_c h(x)$, $(h \in \mathfrak{H}_S, h^{[c]} \in \mathfrak{H}_{S\mathfrak{G}})$ and so our condition is $(\bar{A}f^{[a]}, g^{[b]}) = 0$ for every pair $f, g \in \mathfrak{H}_S$. This describes obviously a weakly closed set.

Thus $\mathbf{N}$ is a ring and $\subsetneqq \mathbf{M}'$.

Assume now $\bar{A} \in \mathbf{M}', \mathbf{N}'$. As $\bar{M}_{\chi(x)} \in \mathbf{M}'$, $\bar{M}_{\chi(x)} \approx [[\delta_c X^{(x)}]]_{x \in S,\, c \in \mathfrak{G}}$ and as $c \notin \mathfrak{G}^0$ implies $c \gneqq 1$, $\delta_c = 0$ so we have $\bar{M}_{\chi(x)} \in \mathbf{N}$. So $\bar{A}$ must commute with every $\bar{M}_{\chi(x)}$ and (cf. above) with every $\bar{V}_{a_0}$, $a_0 \in \mathfrak{G}^0$. Now $\bar{A} \in \mathbf{M}'$, $A \approx [[\xi_c(x)]]_{x \in S,\, c \in \mathfrak{G}}$ so our requirements are (by Lemma 2.4.3, (iv)) $\xi_c(x)\chi(xc^{-1}) = \xi_c(x)\chi(x)$ and $\xi_{a_0^{-1}ca_0}(xa_0) = \xi_c(x)$.

The first equation reads $L_{\xi_c(x)\chi} = L_{\xi_c(x)} U_{c^{-1}} \chi$ for all bounded $\chi \in \mathfrak{H}_S$. Thus it is for all $\chi(x) = e_T(x) \begin{cases} = 1 \text{ for } x \in T \\ = 0 \text{ for } x \notin T \end{cases}$, $T \subset S$, $\mu(T)$ finite; and for their linear aggregates and their condensation points; that is for all $\chi \in \mathfrak{H}_S$ (cf. the second section of the proof of Lemma 12.1.1). So $L_{\xi_c(x)} = L_{\xi_c(x)} U_{c^{-1}}$ which implies by Lemma 12.2.3, that $\xi_c(x) = 0$ if $c \gneqq 1$ (and so $c^{-1} \gneqq 1$).

The second equation reads: $U_{a_0}^{-1} L_{\xi_1(x)} U_{a_0} = L_{\xi_1(x)}$. Denote the set of all U_{a_0}, $a_0 \in \mathfrak{G}^0$ by $\mathbf{U}^0$. Then we have $L_{\xi_1(x)} \in \mathbf{L} \cdot \mathbf{U}^{0\prime}$. As $\mathfrak{G}^0$ is ergodic in S, Lemma 12.2.4 applies: $L_{\xi_1(x)} = \alpha 1 = L_\alpha$, $\xi_1(x) = \alpha$. So we have $\xi_c(x) = \delta_c \alpha$, $\bar{A} = \alpha 1$. This completes the proof of $\mathbf{M}' \cdot \mathbf{N}' = (\alpha 1)$.

Lemma 13.4.2. *Let* $\mathfrak{G}^0$, $\mathbf{N}$ *be as above. Then* $\mathbf{M}$, $\mathbf{N}$ *is a factorisation, but* $\mathbf{M}$, $\mathbf{N}$ *are not coupled factors.*

Proof: As $\mathbf{N} \subset \mathbf{M}'$ and $\mathbf{R}(\mathbf{M}, \mathbf{N}) = (\mathbf{R}(\mathbf{M}, \mathbf{N}))'' = (\mathbf{M}' \cdot \mathbf{N}')' = (\alpha 1)' = \mathbf{B}$, therefore $\mathbf{M}$, $\mathbf{N}$ is a factorisation. As $\mathbf{N} \gneqq \mathbf{M}'$ they are not coupled.

It is easy to give examples of groups $\mathfrak{G}$, $\mathfrak{G}^0$ as required by Lemmas 13.4.1 and 13.4.2: We use the examples (α) – (γ) at the end of §13.1. So $\mathfrak{G} = \mathfrak{G}_\theta$, $\mathfrak{G}^0 = \mathfrak{G}_{k\theta}$, θ irrational, $k = 2, 3, \cdots$, will do; or $\mathfrak{G} = \mathfrak{G}_{\text{rat}}$, $\mathfrak{G}^0 = \mathfrak{G}^0_{\text{rat } q}$, $q = 2, 3, \cdots$ or $\mathfrak{G} = \mathfrak{G}_{\text{rat } p}$, $\mathfrak{G}^0 = \mathfrak{G}_{\text{rat } q}$, $p, q = 2, 3, \cdots, q$ such a divisor of p no power of which is divisible by p (that is: q does not contain all prime factors of p, for instance $p = 6$, $q = 2$).

Thus Lemma 11.1.2 implies that the $\mathbf{M}$, $\mathbf{M}'$ of Theorem XI are not normal for $\mathfrak{G} = \mathfrak{G}_\theta$, $\mathfrak{G}_{\text{rat}}$, $\mathfrak{G}_{\text{rat } p}$ if p is not the power of a prime. (For $\mathbf{M}'$ this results directly, for $\mathbf{M}$ it follows then from the spatial isomorphism $F \to \overline{W}F$ in $\mathfrak{H}_{S\mathfrak{G}}$ which interchanges $\mathbf{M}$ with $\mathbf{M}'$). So we now have the examples of not normal factors of class (II) we wanted. The question whether all factors of class (II) are not normal, remains nevertheless open; we can even not decide whether $\mathbf{M}$, $\mathbf{M}'$ for $\mathfrak{G} = \mathfrak{G}_{\text{rat } p}$, p power of a prime, are normal. (Cf. note at end.)

This is the extent to which we can answer Problem 10.

Part V: The finite cases

Chapter XIV: Generalised additivity of $D_{\mathbf{M}}(\mathfrak{M})$

14.1. The object of the chapters which follow is the investigation of the finite cases (cf. Theorem VIII), that is, of all factors $\mathbf{M}$ which belong to cases (I_n), $n = 1, 2, \cdots$ or (II_1). The outstanding fact is that they all behave very similarly. This is remarkable, because an $\mathbf{M}$ in a case (I_n) is ring-isomorphic to the ring of all (bounded) operators of an n-dimensional Euclidean space $\mathfrak{H}_n$ and therefore completely elementary; while an $\mathbf{M}$ in case (II_1) lies in a Hilbert

space, unbounded operators can be η **M**, operators in **M** can have continuous spectra, etc.

All our results will be valid for all finite cases, that is, both for (I_n) and (II_1). The really interesting application is of course (II_1), while most of our results are void or trivial or well known facts from linear geometry, when applied to (I_n). Our reason for including the (I_n) too in all our statements is only the desire to stress the analogy between (I_n) and (II_1), and to put every result concerning (II_1) at once in the right light by showing what it would mean for (I_n).

Our notations will be these: $\mathfrak{H}$ is a space; **M** a factor in $\mathfrak{H}$ belonging to a finite case, (I_n), $n = 1, 2, \cdots$, or (II_1); $D_{\mathbf{M}}(\mathfrak{M})$ a relative dimension function for **M** with the normalisation $D_{\mathbf{M}}(\mathfrak{H}) = 1$. For (I_n) this is the standard normalisation multiplied by $1/n$ for (II_1) it is the standard normalisation itself.

In the next § (and only there) however these assumptions will not be used, the class of **M** and the normalisation of $D_{\mathbf{M}}(\mathfrak{H})$ being arbitrary.

14.2. We discuss the behavior of $D_{\mathbf{M}}(\mathfrak{M})$ for such $\mathfrak{M}$, $\mathfrak{N}$ which need not to be orthogonal, nor comparable ($\mathfrak{M} \subsetneqq \mathfrak{N}$), thus generalising the statements of Definition 8.2.1.

Lemma 14.2.1. *Assume* $\mathfrak{M}$, $\mathfrak{N}$ η **M**. *Then*

$$D_{\mathbf{M}}([\mathfrak{M}, \mathfrak{N}] - \mathfrak{N}) \leqq D_{\mathbf{M}}(\mathfrak{M}).$$

Proof: This follows immediately from Lemma 7.3.4.

Lemma 14.2.2. *Assume* $\mathfrak{M}$, $\mathfrak{N}$ η **M** *and* $D_{\mathbf{M}}(\mathfrak{M}) > D_{\mathbf{M}}(\mathfrak{N})$. *Then* $\mathfrak{M}(\mathfrak{H} - \mathfrak{N}) \neq (0)$.

Proof: We have, using Lemma 14.2.1

$$D_{\mathbf{M}}([\mathfrak{N}; \mathfrak{H} - \mathfrak{M}]\cdot\mathfrak{M}) = D_{\mathbf{M}}([\mathfrak{N}, \mathfrak{H} - \mathfrak{M}] - (\mathfrak{H} - \mathfrak{M})) \leqq D_{\mathbf{M}}(\mathfrak{N}) < D_{\mathbf{M}}(\mathfrak{M})$$

so $[\mathfrak{N}, \mathfrak{H} - \mathfrak{M}]\cdot\mathfrak{M} \neq \mathfrak{M}$. But $[\mathfrak{N}; \mathfrak{H} - \mathfrak{M}]\cdot\mathfrak{M} \subset \mathfrak{M}$, therefore $\mathfrak{M} - [\mathfrak{N}, \mathfrak{H} - \mathfrak{M}]\cdot\mathfrak{M} \neq (0)$. Now $\mathfrak{M} - [\mathfrak{N}, \mathfrak{H} - \mathfrak{M}]\,\mathfrak{M} = \mathfrak{M}\cdot(\mathfrak{H} - [\mathfrak{N}, \mathfrak{H} - \mathfrak{M}]) = \mathfrak{M}(\mathfrak{H} - \mathfrak{N})(\mathfrak{H} - (\mathfrak{H} - \mathfrak{M})) = \mathfrak{M}\cdot(\mathfrak{H} - \mathfrak{N})\cdot\mathfrak{M} = \mathfrak{M}\cdot(\mathfrak{H} - \mathfrak{N})$. So we proved $\mathfrak{M}\cdot(\mathfrak{H} - \mathfrak{N}) \neq (0)$.

Lemma 14.2.3. *Assume* $\mathfrak{M}$, $\mathfrak{N}$ η **M**. *Then* $D_{\mathbf{M}}([\mathfrak{M}, \mathfrak{N}]) \leqq D_{\mathbf{M}}(\mathfrak{M}) + D_{\mathbf{M}}(\mathfrak{N})$ *and the equality holds if* $\mathfrak{M}\cdot\mathfrak{N} = (0)$.

Proof: Put $\mathfrak{M}' = [\mathfrak{M}, \mathfrak{N}] - \mathfrak{N}$. Then $\mathfrak{M}'$ η **M**, $\mathfrak{M}'$, $\mathfrak{N}$ are orthogonal, $[\mathfrak{M}', \mathfrak{N}] = [\mathfrak{M}, \mathfrak{N}]$ and so Definition 8.2.1, (iii), applies. $D_{\mathbf{M}}([\mathfrak{M}, \mathfrak{N}]) = D_{\mathbf{M}}([\mathfrak{M}', \mathfrak{N}']) = D_{\mathbf{M}}(\mathfrak{M}') + D_{\mathbf{M}}(\mathfrak{N}) \leqq D_{\mathbf{M}}(\mathfrak{M}) + D_{\mathbf{M}}(\mathfrak{N})$, considering $D_{\mathbf{M}}(\mathfrak{M}') \leqq D_{\mathbf{M}}(\mathfrak{M})$ by Lemma 14.2.1.

We see that we have $=$ if $D_{\mathbf{M}}(\mathfrak{M}') = D_{\mathbf{M}}(\mathfrak{M})$. If this is not the case, then $D_{\mathbf{M}}(\mathfrak{M}') < D_{\mathbf{M}}(\mathfrak{M})$ and so Lemma 14.2.2 applies: $\mathfrak{M}\cdot(\mathfrak{H} - \mathfrak{M}') \neq (0)$. Now $f \in \mathfrak{M}(\mathfrak{H} - \mathfrak{M}')$ means $f \in \mathfrak{M}$ and the orthogonality of f to $\mathfrak{M}' = [\mathfrak{M}, \mathfrak{N}] - \mathfrak{N}$. But as $f \in \mathfrak{M} \subset [\mathfrak{M}, \mathfrak{N}]$ the second statement means $f \in [\mathfrak{M}, \mathfrak{N}] - (([\mathfrak{M}, \mathfrak{N}]) - \mathfrak{N}) = \mathfrak{N}$. So we have $f \in \mathfrak{M}\cdot\mathfrak{N}$ and therefore the above result states $\mathfrak{M}\cdot\mathfrak{N} \neq (0)$.

This completes the proof.

Lemma 14.2.4. *Assume* $\mathfrak{M}, \mathfrak{N}\ \eta\ \mathbf{M}$. *Then*

$$D_{\mathbf{M}}([\mathfrak{M}, \mathfrak{N}]) + D_{\mathbf{M}}(\mathfrak{M}\cdot\mathfrak{N}) = D_{\mathbf{M}}(\mathfrak{M}) + D_{\mathbf{M}}(\mathfrak{N}).$$

Proof: Put $\mathfrak{M}_1 = \mathfrak{M} - \mathfrak{M}\cdot\mathfrak{N}$. Then $\mathfrak{M}_1, \mathfrak{M}\cdot\mathfrak{N}$ are orthogonal, $[\mathfrak{M}_1, \mathfrak{M}\cdot\mathfrak{N}] = \mathfrak{M}$, so by Definition 8.2.1, (iii), $D_{\mathbf{M}}(\mathfrak{M}) = D_{\mathbf{M}}(\mathfrak{M}_1) + D_{\mathbf{M}}(\mathfrak{M}\cdot\mathfrak{N})$.

Furthermore $[\mathfrak{M}_1, \mathfrak{N}] = [\mathfrak{M}_1, [\mathfrak{N}, \mathfrak{M}\cdot\mathfrak{N}]] = [[\mathfrak{M}_1, \mathfrak{M}\cdot\mathfrak{N}], \mathfrak{N}] = [\mathfrak{M}, \mathfrak{N}]$ and if $f\ \epsilon\ \mathfrak{M}_1\cdot\mathfrak{N}$ then $f\ \epsilon\ \mathfrak{M}_1 \subset \mathfrak{M}$, $f\ \epsilon\ \mathfrak{N}$, $f\ \epsilon\ \mathfrak{M}\cdot\mathfrak{N}$ but $f\ \epsilon\ \mathfrak{M}_1 = \mathfrak{M} - \mathfrak{M}\cdot\mathfrak{N} \subset \mathfrak{H} - \mathfrak{M}\cdot\mathfrak{N}$ so $f = 0$, which proves $\mathfrak{M}_1\cdot\mathfrak{N} = (0)$. So Lemma 14.2.3 applies:

$$D_{\mathbf{M}}([\mathfrak{M}, \mathfrak{N}]) = D_{\mathbf{M}}([\mathfrak{M}_1, \mathfrak{N}]) = D_{\mathbf{M}}(\mathfrak{M}_1) + D_{\mathbf{M}}(\mathfrak{N}).$$

Adding $D(\mathfrak{M}\cdot\mathfrak{N})$ to both sides gives, considering our previous equation, the desired formula

$$D_{\mathbf{M}}([\mathfrak{M}, \mathfrak{N}]) + D_{\mathbf{M}}(\mathfrak{M}\cdot\mathfrak{N}) = D_{\mathbf{M}}(\mathfrak{M}) + D_{\mathbf{M}}(\mathfrak{N}).$$

Lemma 14.2.5. *Let* $\mathfrak{M}_1, \mathfrak{M}_2, \cdots$ *be a (finite or infinite) sequence, all* $\mathfrak{M}_i\ \eta\ \mathbf{M}$. *Then* $D_{\mathbf{M}}([\mathfrak{M}_1, \mathfrak{M}_2, \cdots]) \leqq \sum_i D_{\mathbf{M}}(\mathfrak{M}_i)$. *The equality holds if and only if one of these two conditions is satisfied:*

(i) $D_{\mathbf{M}}([\mathfrak{M}_1, \mathfrak{M}_2, \cdots])$ *is infinite.*

(ii) $D_{\mathbf{M}}([\mathfrak{M}_1, \mathfrak{M}_2, \cdots])$ *is finite, and* $[\mathfrak{M}_1, \mathfrak{M}_2, \cdots \mathfrak{M}_{i-1}]\cdot\mathfrak{M}_i = (0)$ *for* $i = 2, 3, \cdots$.

In case (ii) *there is even* $[\mathfrak{M}_{m_1}, \mathfrak{M}_{m_2}, \cdots]\cdot[\mathfrak{M}_{n_1}, \mathfrak{M}_{n_2}, \cdots] = (0)$ *if* $m_1, m_2, \cdots$ *and* $n_1, n_2, \cdots$ *are any two sequences without common elements.*

Proof: We can assume that the sequence $\mathfrak{M}_1, \mathfrak{M}_2, \cdots$ is infinite, as we could otherwise insert infinitely many terms $\mathfrak{M}_i = (0)$.

Lemma 8.3.3 can be applied to $\mathfrak{M}_1 \subset [\mathfrak{M}_1, \mathfrak{M}_2] \subset [\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_3] \subset \cdots$ and it gives: $D_{\mathbf{M}}([\mathfrak{M}_1, \mathfrak{M}_2, \cdots]) = \lim_{i\to\infty} D_{\mathbf{M}}([\mathfrak{M}_1, \cdots, \mathfrak{M}_i])$. Now Lemma 14.2.3 can be applied to $[\mathfrak{M}_1, \cdots, \mathfrak{M}_{i-1}], \mathfrak{M}_i$ and it gives: $D_{\mathbf{M}}([\mathfrak{M}_1, \cdots, \mathfrak{M}_i]) \leqq D_{\mathbf{M}}([\mathfrak{M}_1, \cdots, \mathfrak{M}_{i-1}]) + D_{\mathbf{M}}(\mathfrak{M}_i)$ with an equality if (ii) holds; therefore $D_{\mathbf{M}}([\mathfrak{M}_1, \cdots, \mathfrak{M}_j]) \leqq \sum_{i=1}^{j} D_{\mathbf{M}}(\mathfrak{M}_i)$ and passing to the limit

$$D_{\mathbf{M}}([\mathfrak{M}_1, \mathfrak{M}_2, \cdots]) \leqq \sum_{i=1}^{\infty} D_{\mathbf{M}}(\mathfrak{M}_i)\,,$$

again with an equality if (ii) holds. So we have proved the relation with a $\leqq$ in general, and the sufficience of (ii) for the equality. (i) too implies equality, because if $D_{\mathbf{M}}([\mathfrak{M}_1, \mathfrak{M}_2, \cdots])$ is infinite, then

$$\sum_{i=1}^{\infty} D_{\mathbf{M}}(\mathfrak{M}_i) \geqq D([\mathfrak{M}_1, \mathfrak{M}_2, \cdots])$$

is infinite too.

The necessity of (i) or (ii) means that if $D_{\mathbf{M}}([\mathfrak{M}_1, \mathfrak{M}_2, \cdots]) = \sum_1^{\infty} D_{\mathbf{M}}(\mathfrak{M}_i)$ and finite, then (ii) holds. We prove that in this case

$$[\mathfrak{M}_{m_1}, \mathfrak{M}_{m_2}, \cdots]\,[\mathfrak{M}_{n_1}, \mathfrak{M}_{n_2}, \cdots] = (0)$$

if $m_1, m_2, \cdots$ and $n_1, n_2, \cdots$ have no common elements: This proves (ii) (put $m_1 = 1, \cdots, m_{i-1} = i - 1$ and $n_1 = i$), and at the same time it justifies the last statement of our Lemma too. As we can add to the sequence $n_1, n_2, \cdots$ all

$i = 1, 2, \cdots$ which differ from $m_1, m_2, \cdots$ and $n_1, n_2, \cdots$ we may assume that $n_1, n_2, \cdots$ is the complementary set to $m_1, m_2, \cdots$ in $1, 2, \cdots$.

Now we have

$$D_{\mathbf{M}}([\mathfrak{M}_{m_1}, \mathfrak{M}_{m_2}, \cdots]) \leqq \sum_j D_{\mathbf{M}}(\mathfrak{M}_{m_j}),\ D_{\mathbf{M}}([\mathfrak{M}_{n_1}, \mathfrak{M}_{n_2}, \cdots]) \leqq \sum_j D_{\mathbf{M}}(\mathfrak{M}_{n_j})$$

and so by Lemma 14.2.4

$$\begin{aligned} &D_{\mathbf{M}}([\mathfrak{M}_1, \mathfrak{M}_2, \cdots]) + D_{\mathbf{M}}([\mathfrak{M}_{m_1}, \mathfrak{M}_{m_2}, \cdots]\,[\mathfrak{M}_{n_1}, \mathfrak{M}_{n_2}, \cdots]) \\ &\quad = D_{\mathbf{M}}([[\mathfrak{M}_{m_1}, \mathfrak{M}_{m_2}, \cdots], [\mathfrak{M}_{n_1}, \mathfrak{M}_{n_2}, \cdots]]) \\ &\qquad\qquad + D_{\mathbf{M}}([\mathfrak{M}_{m_1}, \mathfrak{M}_{m_2}, \cdots] \cdot [\mathfrak{M}_{n_1}, \mathfrak{M}_{n_2}, \cdots]) \\ &\quad = D_{\mathbf{M}}([\mathfrak{M}_{m_1}, \mathfrak{M}_{m_2}, \cdots]) + D_{\mathbf{M}}([\mathfrak{M}_{n_1}, \mathfrak{M}_{n_2}, \cdots]) \\ &\quad \leqq \sum_j D_{\mathbf{M}}(\mathfrak{M}_{m_j}) + \sum_j D_{\mathbf{M}}(\mathfrak{M}_{n_j}) = \sum_1^\infty D_{\mathbf{M}}(\mathfrak{M}_i) \\ &\quad = D_{\mathbf{M}}([\mathfrak{M}_1, \mathfrak{M}_2, \cdots]). \end{aligned}$$

As $D_{\mathbf{M}}([\mathfrak{M}_1, \mathfrak{M}_2, \cdots])$ is finite by assumption, this implies

$$D_{\mathbf{M}}([\mathfrak{M}_{m_1}, \mathfrak{M}_{m_2}, \cdots]\,[\mathfrak{M}_{n_1}, \mathfrak{M}_{n_2}, \cdots]) \leqq 0,$$

so $= 0$, and therefore $[\mathfrak{M}_{m_1}, \mathfrak{M}_{m_2}, \cdots] \cdot [\mathfrak{M}_{n_1}, \mathfrak{M}_{n_2}, \cdots] = (0)$.

This completes the proof.

Chapter XV: The relative trace

15.1. From now on we make the assumption of §14.1: $\mathbf{M}$ is a factor in a finite case ((I_n), $n = 1, 2, \cdots$, or (II_1)), and $D_{\mathbf{M}}(\mathfrak{M})$ is normalised by $D_{\mathbf{M}}(\mathfrak{H}) = 1$.

We define an analogue of the trace of common (finite dimensional) matrix theory, more precisely: We define a notion which shares most essential properties of the trace, and coincides with it in the discrete cases (I_n), $n = 1, 2, \cdots$. The importance of this notion lies of course in its validity in the continuous case (II_1).

Definition 15.1.1. Let $A \in \mathbf{M}$ be Hermitian. Form its resolution of unity $E(\lambda)$, $-\infty < \lambda < \infty$. (Cf. (16), p. 92. As A is bounded this coincides with Hilbert's spectral form. The description which we will use is discussed in (18), pp. 389–390, footnote 42 and p. 418). It is characterised and uniquely determined as discussed loc. cit., by the following properties:

(α) $E(\lambda)$ is a projection, defined for all $-\infty < \lambda < \infty$.

(β) $\lambda \leqq \mu$ implies $E(\lambda)$ part of $E(\mu)$.

(γ) $\lambda \geqq \lambda_0$, $\lambda \to \lambda_0$ implies $E(\lambda) \to E(\lambda_0)$ in the strong topology.

(δ) There exists a c so that $E(\lambda) = 0$ if $\lambda \leqq -c$ and $E(\lambda) = 1$ if $\lambda \geqq c$.

(ϵ) For all $f, g \in \mathfrak{H}$

$$(Af, g) = \int_{-\infty}^{\infty} \lambda\, d(E(\lambda)f, g).$$

(This is a numerical Stieltjes integral, certainly convergent, because we can

replace $\int_{-\infty}^{\infty}$ by $\int_{-c}^{c}$ owing to (δ); and $(E(\lambda)f, g)$ is of bounded variation owing to (β), cf. (16), p. 93.)

This equation may be written symbolically as

$$(*) \qquad A = \int_{-\infty}^{\infty} \lambda \, dE(\lambda) .$$

As $1 \in \mathbf{M}$ therefore $A \in \mathbf{M}$ has the consequence that all $E(\lambda) \in \mathbf{M}$ (cf. (18), p. 390). Therefore all $D_{\mathbf{M}}(E(\lambda))$ are defined, and we can form the numerical Stieltjes integral

$$(**) \qquad T_{\mathbf{M}}(A) = \int_{-\infty}^{\infty} \lambda d \, D_{\mathbf{M}}(E(\lambda)) .$$

(This integral is convergent because we can replace $\int_{-\infty}^{\infty}$ by $\int_{-c}^{c}$ owing to (δ), as $D_{\mathbf{M}}(E(\lambda)) = 0$ resp. $= 1$ if $\lambda < -c$ resp. $\geqq c$ and $D_{\mathbf{M}}(E(\lambda))$ is monotone owing to (β)).

We call this $T_{\mathbf{M}}(A)$ which is thus defined for all $A \in \mathbf{M}$ the relative trace of A.

It is clear how the symbolic equation (*) suggests the definition (**). We shall see in §15.2 after Lemma 15.2.2, that in the discrete cases (I_n), $n = 1, 2, \cdots$, our $T_{\mathbf{M}}(A)$ is $1/n$ times the trace of A taken in the usual sense for matrices of degree n.

15.2. An alternative definition of the trace can be obtained by using an extension of the "Minimax principle" (cf. (5), pp. 26–29). This extension is as follows:

Lemma 15.2.1. *Let $A \in \mathbf{M}$ be Hermitian, and $E(\lambda)$, $-\infty < \lambda < \infty$ its resolution of unity. Form the quantity*

$$(\ddagger) \qquad \epsilon(\alpha) = \underset{\left[D_{\mathbf{M}}(\mathfrak{M}) \geqq \alpha\right]}{\text{g.l.b.}} \; \underset{\left[\|f\| = 1\right]}{\{\text{l.u.b.}\ (Af, f)\}}$$

for every $\alpha > 0, \leqq 1$. Then at the same time

$$(\ddagger\ddagger) \qquad \epsilon(\alpha) = \text{g.l.b.}_{[D_{\mathbf{M}}(E(\lambda)) \geqq \alpha]} \lambda .$$

Proof: Consider the set of all λ with $D_{\mathbf{M}}(E(\lambda)) \geqq \alpha$. As it contains $\lambda = c$ it is not empty; as it does not contain $\lambda = -c$ it is not the set of all real numbers; it contains with λ every $\lambda' > \lambda$ by Definition 15.1.1 (β); it contains its g.l.b. λ_0 by Definition 15.1.1, (γ). Thus it is the set of all $\lambda \geqq \lambda_0$. So we must prove $\epsilon(\alpha) = \lambda_0$.

Define $\mathfrak{M}_0$ by $P_{\mathfrak{M}_0} = E(\lambda_0)$, $\mathfrak{M}_0 \eta \mathbf{M}$ because $E(\lambda_0) \in \mathbf{M}$ and $D_{\mathbf{M}}(\mathfrak{M}_0) = D_{\mathbf{M}}(E(\lambda_0)) \geqq \alpha$. For $f \in \mathfrak{M}_0$ we have $E(\lambda_0)f = f$ and so $E(\lambda)f = f$ if $\lambda \geqq \lambda_0$. Thus

$$(Af, f) = \int_{-\infty}^{\infty} \lambda(d(E(\lambda)f, f) = \int_{-c}^{\lambda_0} \lambda d(E(\lambda)f, f) \leqq \int_{-c}^{\lambda_0} \lambda_0 \, d(E(\lambda)f, f)$$

$$= \lambda_0((E\lambda_0)f, f) - (E(-c)f, f)) = \lambda_0(f, f) = \lambda_0 \| f \|^2,$$

and so l.u.b. $(Af, f) \leqq \lambda_0$. As $\mathfrak{M}_0 \eta \mathbf{M}$, $D_{\mathbf{M}}(\mathfrak{M}_0) \geqq \alpha$ this proves $\epsilon(\alpha) \leqq \lambda_0$. $\left[{}^{f \epsilon \mathfrak{M}_0}_{\| f \| = 1}\right]$

Assume now $\epsilon(\alpha) < \lambda_0$. Then by (‡) an $\mathfrak{M} \eta \mathbf{M}$, $D_{\mathbf{M}}(\mathfrak{M}) \geqq \alpha$ with l.u.b. $\left[{}^{f \epsilon \mathfrak{M}}_{\| f \| = 1}\right]$ $(Af, f) < \lambda_0$ must exist. Choose a $\lambda' >$ l.u.b. (Af, f), $< \lambda_0$. By the definition $\left[{}^{f \epsilon \mathfrak{M}}_{\| f \| = 1}\right]$ of λ_0 the relation $\lambda' < \lambda_0$ has the consequence $D_{\mathbf{M}}(E(\lambda')) < \alpha$. Define $\mathfrak{M}'$ by $P_{\mathfrak{M}'} = E(\lambda')$; then $\mathfrak{M}' \eta \mathbf{M}$ because $E(\lambda') \epsilon \mathbf{M}$. We have

$$D_{\mathbf{M}}(\mathfrak{M}') = D_{\mathbf{M}}(E(\lambda')) < \alpha \leqq D_{\mathbf{M}}(\mathfrak{M})$$

so by Lemma 14.2.2 $\mathfrak{M} \cdot (\mathfrak{H} - \mathfrak{M}') \neq (0)$. Therefore an $f_0 \epsilon \mathfrak{M}(\mathfrak{H} - \mathfrak{M}')$ with $f_0 \neq 0$ exists.

Now as $f_0 \epsilon \mathfrak{H} - \mathfrak{M}'$ we have $E(\lambda')f_0 = 0$ and so $E(\lambda)f_0 = 0$ if $\lambda \leqq \lambda'$. Thus

$$(Af_0, f_0) = \int_{-\infty}^{\infty} \lambda d(E(\lambda)f_0, f_0) = \int_{\lambda'}^{c} \lambda d(E(\lambda)f_0, f_0) \geqq \int_{\lambda'}^{c} \lambda' d(E(\lambda)f_0, f_0)$$

$$= \lambda'((E(c)f_0, f_0) - (E(\lambda')f_0, f_0)) = \lambda'(f_0, f_0) = \lambda' \| f_0 \|^2.$$

Multiplying f_0 with $1/\| f_0 \|$ we can obtain $\| f_0 \| = 1$. But at the same time $f_0 \epsilon \mathfrak{M}$ too, therefore we have l.u.b. $(Af, f) \geqq \lambda'$, contradicting our original as- $\left[{}^{f \epsilon \mathfrak{M}}_{\| f \| = 1}\right]$ sumptions on λ'.

Thus there must be $\epsilon(\alpha) = \lambda_0$ and the proof is completed.

Lemma 15.2.2. *Let* $A \epsilon \mathbf{M}$ *be Hermitian, and* $\epsilon(\alpha)$ *as in Lemma* 15.2.1. *Then*

$$T_{\mathbf{M}}(A) = \int_0^1 \epsilon(\alpha)\, d\alpha\,.$$

Proof: We have $T_{\mathbf{M}}(A) = \int_{-\infty}^{\infty} \lambda d\,(D_{\mathbf{M}}(E(\lambda)))$. It suffices to remember the definition of the Riemann-Stieltjes integral in order to see that this coincides with the Riemann integral $\int_0^1 (\text{g.l.b. } \lambda)_{[D_{\mathbf{M}}(E(\lambda)) \geqq \alpha]}\, d\alpha$ (cf. (19), p. 198). But this is, by formula (‡‡) of Lemma 15.2.1, equal to $\int_0^1 \epsilon(\alpha)\, d\alpha$.

Observe that in the case (I_n), $D_{\mathbf{M}}(\mathfrak{M})$ assumes no other values than $0, 1/n, 2/n, \cdots, (n-1)/n, 1$ therefore $\epsilon(\alpha)$ is constant in each interval $(p-1)/n < \alpha \leqq p/n$, $p = 1, \cdots, n$. Thus Lemma 15.2.2 gives

$$T_{\mathbf{M}}(A) = \int_0^1 \epsilon(\alpha)\, d\alpha = \frac{1}{n} \sum_{p=1}^{n} \epsilon\left(\frac{p}{n}\right),$$

and one verifies easily that $\epsilon(p/n)$ is the proper value No. p (from below) of A. Thus $T_{\mathfrak{M}}(A)$ is $1/n$ times the trace of A. A more direct proof is given in §15.4 after Theorem XIII. In the case (II_1) $\int_0^1 \epsilon(\alpha)\, d\alpha$ becomes an integral with a really continuous domain. Owing to the analogy with the case (I_n) it seems natural to introduce the following terminology:

Definition 15.2.1. Let $A \in \mathfrak{M}$ be Hermitian, and $\epsilon(\alpha)$ as in Lemma 15.2.1. Then we call $\epsilon(\alpha)$ the proper value No. α (from below) on A, on the continuous scale $0 < \alpha \leqq 1$.

The immediate application of our new way to express $T_{\mathfrak{M}}(A)$ is this

Lemma 15.2.3. *Let A, $B \in \mathfrak{M}$ be Hermitian, and $A - B$ (semi-) definite, Then $T_{\mathfrak{M}}(A) \geqq T_{\mathfrak{M}}(B)$.*

Proof: Form the $\epsilon(\alpha)$ of Lemma 15.2.1 for A, and for B, denoting it by $\epsilon(\alpha)$ resp. $\eta(\alpha)$. As the definiteness of $A - B$ means $((A - B)f, f) \geqq 0$, $(Af, f) \geqq (Bf, f)$ for every f, therefore Lemma 15.2.1 (‡) gives $\epsilon(\alpha) \geqq \eta(\alpha)$. Now Lemma 15.2.2 gives $T_{\mathfrak{M}}(A) \geqq T_{\mathfrak{M}}(B)$.

Note that $T_{\mathfrak{M}}(A) \geqq 0$ for a definite A is obvious, and that this would imply directly our Lemma, if we knew that $T_{\mathfrak{M}}(A - B) = T_{\mathfrak{M}}(A) - T_{\mathfrak{M}}(B)$. But this relation is only established at present if A, B commute (cf. Lemma 15.3.4 and the remarks after it), and this makes the above simplification impracticable.

15.3. We derive now the main formal properties of $T_{\mathfrak{M}}(A)$.

Lemma 15.3.1. *Let $E \in \mathfrak{M}$ be a projection. Then $T_{\mathfrak{M}}(E) = D_{\mathfrak{M}}(E)$.*

Proof: $E(\lambda)\begin{cases} = E \text{ for } \lambda \geqq 1 \\ = 0 \text{ for } \lambda < 1 \end{cases}$ is E's resolution of unity, as conditions (α)–(ϵ) from Definition 15.1.1 are easily verified for them. So

$$D_{\mathfrak{M}}(E(\lambda))\begin{cases} = D_{\mathfrak{M}}(E) & \text{for } \lambda \geqq 1 \\ = 0 & \text{for } \lambda < 1 \end{cases}$$

and so

$$T_{\mathfrak{M}}(E) = \int_{-\infty}^{\infty} \lambda d D_{\mathfrak{M}}(E(\lambda)) = 1 \cdot D_{\mathfrak{M}}(E) = D_{\mathfrak{M}}(E).$$

Lemma 15.3.2. *Let $A \in \mathfrak{M}$ be Hermitian, and a, b two real numbers. Then*

$$T_{\mathfrak{M}}(aA + b \cdot 1) = a T_{\mathfrak{M}}(A) + b.$$

Proof: We first prove $T_{\mathfrak{M}}(-A) = -T_{\mathfrak{M}}(A)$ because this will permit us to restrict ourselves afterwards to the case $a \geqq 0$.

Let $E(\lambda)$ be A's resolution of unity. Then $1 - E(-\lambda)$ fulfills for $-A$ all conditions (α)–(ϵ) of Definition 15.1.1 except (γ). $E_1(\lambda) = \lim_{\lambda' > \lambda, \lambda' \to \lambda} 1 - E(-\lambda)$ fulfills, as a simple argument shows, all of them. So $E_1(\lambda)$ is $-A$'s resolution of unity. Now $E_1(\lambda) \neq 1 - E(-\lambda)$ only in the points of discontinuity of $E(-\lambda)$, that is in the points $-\lambda$ where λ runs over all point proper values of A. This is a countable set. *A fortiori* $D_{\mathfrak{M}}(E_1(\lambda)) \neq D_{\mathfrak{M}}(1 - E(\lambda))$ holds only in a countable set. Therefore we can, by the definition of Riemann-Stieltjes inte-

grals, replace $d(D_{\mathfrak{M}}(E_1(\lambda)))$ by $d(D_{\mathfrak{M}}(1 - E(-\lambda))$ in every Riemann-Stieltjes integral in which it occurs. Thus

$$T_{\mathfrak{M}}(-A) = \int_{-\infty}^{\infty} \lambda dD_{\mathfrak{M}}(E_1(\lambda)) = \int_{-\infty}^{\infty} \lambda dD_{\mathfrak{M}}(1 - E(-\lambda)) =$$

$$\int_{-\infty}^{\infty} (-\lambda) dD_{\mathfrak{M}}(E(-\lambda)) = \int_{\infty}^{-\infty} \lambda dD_{\mathfrak{M}}(E(\lambda)) = -\int_{-\infty}^{\infty} \lambda dD_{\mathfrak{M}}(E(\lambda)) = -T_{\mathfrak{M}}(A).$$

Assume now $a \geqq 0$. Denote the $\epsilon(\alpha)$ of A and of $aA + b1$ by $\epsilon(\alpha)$ resp. $\eta(\alpha)$. For $\| f \| = 1$, $((aA + b1)f, f) = a(Af, f) + b(f, f) = a(Af, f) + b$ and as $a \geqq 0$ this formula can be used when forming g.l.b.'s and l.u.b.'s. Therefore Lemma 15.2.1 (‡) gives $\eta(\alpha) = a\epsilon(\alpha) + b$ and then Lemma 15.2.2 gives $T_{\mathfrak{M}}(aA + b1) = aT_{\mathfrak{M}}(A) + b$. This completes the proof.

Lemma 15.3.3. *$T_{\mathfrak{M}}(A)$ is a continuous function of A if the uniform topology for operators (cf.* (18), *p.* 384) *is used.*

Proof: We will show: If $\| (A - B)f \| \leqq \epsilon \| f \|$ for every $f \in \mathfrak{H}$ then

$$| T_{\mathfrak{M}}(A) - T_{\mathfrak{M}}(B) | \leqq \epsilon.$$

Under the above assumption we have

$$| ((A - B)f, f) | \leqq \| (A - B)f \| \cdot \| f \| \leqq \epsilon \| f \|^2,$$

$$(Af, f) \leqq (Bf, f) + \epsilon \| f \|^2 = ((Bf + \epsilon 1)f, f).$$

So $(B + \epsilon 1) - A$ is definite, and by Lemma 15.2.3 $T_{\mathfrak{M}}(B + \epsilon 1) \geqq T_{\mathfrak{M}}(A)$. But by Lemma 15.3.2 $T_{\mathfrak{M}}(B + \epsilon 1) = T_{\mathfrak{M}}(B) + \epsilon$ so we have $T_{\mathfrak{M}}(A) - T_{\mathfrak{M}}(B) \leqq \epsilon$. Interchanging A, B (the assumption was symmetric in A, B) we obtain $T_{\mathfrak{M}}(A) - T_{\mathfrak{M}}(B) \geqq -\epsilon$. So $| T_{\mathfrak{M}}(A) - T_{\mathfrak{M}}(B) | \leqq \epsilon$, completing the proof.

We will prove elsewhere considerably more about the continuity of $T_{\mathfrak{M}}(A)$. (Cf. note at the end of this paper.)

Lemma 15.3.4. *Let A, $B \in \mathbf{M}$ be Hermitian, and commute with each other. Then*

$$T_{\mathfrak{M}}(A + B) = T_{\mathfrak{M}}(A) + T_{\mathfrak{M}}(B).$$

Proof: Let $E(\lambda)$ and $F(\lambda)$ be the resolutions of unity belonging to A resp. B. By (18), pp. 390–391 (in particular footnote 43)) A, B are limits of uniformly convergent sequences of linear aggregates of the $E(\lambda)$ resp. the $F(\lambda)$. Therefore it suffices, by Lemma 15.3.3 to prove

$$T_{\mathfrak{M}}\left(\sum_1^m a_i E(\lambda_i) + \sum_1^n b_j F(\mu_j)\right) = T_{\mathfrak{M}}\left(\sum_1^m a_i E(\lambda_i)\right) + T_{\mathfrak{M}}\left(\sum_1^n b_j F(\mu_j)\right).$$

Observe, that if A, B, commute, all $E(\lambda)$, $F(\mu)$ commute with each other (cf. (16), p. 115).

So it suffices to prove this:

$$T_{\mathfrak{M}}\left(\sum_1^p (c_\rho + d_\rho)E_\rho\right) = T_{\mathfrak{M}}\left(\sum_1^p c_\rho E_\rho\right) + T_{\mathfrak{M}}\left(\sum_1^p d_\rho E_\rho\right)$$

if $E_1, \cdots, E_p$ are commutative projections. (Substitute $m + n$, $E(\lambda_1), \cdots,$

$E(\lambda_m), F(\mu_1), \cdots, F(\mu_n), a_1, \cdots a_m, 0, \cdots, 0; 0, \cdots, 0, b_1, \cdots, b_n$ for $p, E_1, \cdots, E_p, c_1, \cdots, c_p, d_1, \cdots, d_p$.)

This would follow immediately from

$$(\divideontimes) \qquad T_{\mathbf{M}}\left(\sum_1^p c_\rho E_\rho\right) = \sum_1^p c_\rho D_{\mathbf{M}}(E_\rho)$$

(for all choices of $c_1, \cdots, c_p$ provided that $E_1, \cdots, E_p$ commute).

Consider the 2^p terms arising when computing

$$(E_1 + (1 - E_1)) \cdots (E_p + (1 - E_p)) .$$

They are products of commutative projections $\epsilon \mathbf{M}$, thus projections $\epsilon \mathbf{M}$, their sum is 1, and the sum of those which arise from the term E_ρ in the factor $E_\rho + (1 - E_\rho)$ is E_ρ. Denote them by $E'_1, \cdots, E'_q$, $q = 2^p$; as

$$E'_1 + \cdots + E'_q = 1$$

they are mutually orthogonal. Write $E_\rho = E'_{\pi_{\rho,1}} + \cdots + E'_{\pi_{\rho,s}}$. We claim: If $(\divideontimes)$ holds for $E'_1, \cdots, E'_q$ (and all $c_1, \cdots, c_q$), then it holds for $E_1, \cdots, E_p$ too. Indeed:

$$T_{\mathbf{M}}\left(\sum_1^p c_\rho E_\rho\right) = T_{\mathbf{M}}\left(\sum_{\rho=1}^p c_\rho \left(\sum_{\sigma=1}^s E'_{\pi_{\rho,\sigma}}\right)\right) = T_{\mathbf{M}}\left(\sum_{\tau=1}^q \left(\sum_{\rho,\sigma;\, \pi_{\rho,\sigma}=\tau} c_\rho\right) E'_\tau\right)$$

$$= \sum_{\tau=1}^q \left(\sum_{\rho,\sigma;\, \pi_{\rho,\sigma}=\tau} c_\rho\right) D_{\mathbf{M}}(E'_\tau) = \sum_{\rho=1}^p c_\rho \left(\sum_{\sigma=1}^s D_{\mathbf{M}}(E'_{\pi_{\rho,\sigma}})\right)$$

$$= \sum_1^p c_\rho D_{\mathbf{M}}(E_\rho) .$$

In other words: We may assume in $(\divideontimes)$ that $E_1, \cdots, E_p$ are mutually orthogonal, and $E_1 + \cdots + E_p = 1$.

If now two c_ρ in $(\divideontimes)$ are equal, say $c_\rho = c_\sigma$ for some $\rho \neq \sigma$ then E_ρ, E_σ coalesce to $E_\rho + E_\sigma$ in both sides of $(\divideontimes)$. Se we may assume that all c_ρ are different. As a permutation does not matter, we may even assume that $c_1 < c_2 < \cdots < c_p$.

Define now

$$G(\lambda) \begin{cases} = 0 & \text{for } \lambda < c_1 . \\ = E_1 + \cdots + E_\rho & \text{for } \lambda \geqq c_\rho, < c_{\rho+1} \text{ if } \rho = 1, \cdots, p-1 . \\ = 1 & \text{for } \lambda \geqq c_p . \end{cases}$$

One verifies immediately the conditions (α)–(ϵ) of Definition 15.1.1 for these $G(\lambda)$ and $\sum_1^p c_\rho E_\rho$, therefore it is the resolution of unity which belongs to $\sum_1^p c_\rho E_\rho$. Now Definition 15.1.1 gives

$$T_{\mathbf{M}}\left(\sum_1^p c_\rho E_\rho\right) = \int_{-\infty}^{\infty} \lambda d(D_{\mathbf{M}}(G(\lambda))$$

$$= \sum_1^p c_\rho (D_{\mathbf{M}}(E_1 + \cdots + E_\rho) - D_{\mathbf{M}}(E_1 + \cdots + E_{\rho-1})) = \sum_1^p c_\rho D_{\mathbf{M}}(E_\rho) .$$

This completes the proof.

Observe that we should expect from the analogy of the trace of finite dimensional matrices, that the additivity of $T_{\mathbf{M}}(A)$ is expressed in Lemma 15.3.4 holds even without assuming the commutativity of A, B. We will come back to this question at the end of §15.4, Problem 11.

15.4. The relation $\mathfrak{M} \sim \mathfrak{N}$ can be expressed in a new way, owing to $D_{\mathbf{M}}(\mathfrak{H}) = 1$.

Lemma 15.4.1. $\mathfrak{M} \sim \mathfrak{N}$ ($\cdots$ **M**) *is equivalent to* $\mathfrak{M}$ η **M** *and the existence of a unitary* $V \in \mathbf{M}$ *which maps* $\mathfrak{M}$ *on* $\mathfrak{N}$.

Proof: If $\mathfrak{M}$ η **M** and such a $V \in \mathbf{M}$ exists, then $U = VP_{\mathfrak{M}}$ is $\in \mathbf{M}$, partially isometric, with the initial and final sets $\mathfrak{M}$ resp. $\mathfrak{N}$; so $\mathfrak{M} \sim \mathfrak{N}$ ($\cdots$ **M**).

Conversely: If $\mathfrak{M} \sim \mathfrak{N}$ ($\cdots$ **M**) then $D_{\mathbf{M}}(\mathfrak{M}) = D_{\mathbf{M}}(\mathfrak{N})$, $D_{\mathbf{M}}(\mathfrak{H} - \mathfrak{M}) = D_{\mathbf{M}}(\mathfrak{H}) - D_{\mathbf{M}}(\mathfrak{M}) = D_{\mathbf{M}}(\mathfrak{H}) - D_{\mathbf{M}}(\mathfrak{N}) = D_{\mathbf{M}}(\mathfrak{H} - \mathfrak{N})$ (this step is possible, because $D_{\mathbf{M}}(\mathfrak{H}) = 1$ is finite), $\mathfrak{H} - \mathfrak{M} \sim \mathfrak{H} - \mathfrak{N}$. Let U_1, U_2 be $\in \mathbf{M}$, partially isometric, and have the resp. initial sets $\mathfrak{M}$, $\mathfrak{H} - \mathfrak{M}$ and the resp. final sets $\mathfrak{N}$, $\mathfrak{H} - \mathfrak{N}$. One verifies easily that $V = U_1 + U_2 \in \mathbf{M}$ is unitary, and that it maps $\mathfrak{M}$ on $\mathfrak{N}$. $\mathfrak{M}$ η **M** is obvious.

This proof was based on the finiteness of $D_{\mathbf{M}}(\mathfrak{H})$ that is of $\mathfrak{H}$ (with respect to **M**). In fact the finiteness of $\mathfrak{H}$ is necessary and sufficient for the Lemma itself: Because if $\mathfrak{H}$ is infinite, then $\mathfrak{H} \sim \mathfrak{M}$ ($\cdots$ **M**) to some $\mathfrak{M} \subsetneqq \mathfrak{H}$ and no unitary U maps $\mathfrak{H}$ on an $\mathfrak{M} \subsetneqq \mathfrak{H}$.

We now state the essential invariance property of $T_{\mathbf{M}}(A)$:

Lemma 15.4.2. *Let* $A \in \mathbf{M}$ *be Hermitian and* $U \in \mathbf{M}$ *be unitary. Then*

$$T_{\mathbf{M}}(U^{-1}AU) = T_{\mathbf{M}}(A).$$

Proof: If A has the resolution of unity $E(\lambda)$, then $U^{-1}AU$ has clearly $U^{-1}E(\lambda)U$. Put $E(\lambda) = P_{\mathfrak{M}(\lambda)}$, $U^{-1}E(\lambda)U = P_{\mathfrak{N}(\lambda)}$ then $\mathfrak{N}(\lambda)$ is $\mathfrak{M}(\lambda)$'s image by U^{-1}. So $\mathfrak{M}(\lambda) \sim \mathfrak{N}(\lambda)$($\cdots$ **M**), $D_{\mathbf{M}}(\mathfrak{M}(\lambda)) = D_{\mathbf{M}}(\mathfrak{N}(\lambda))$, $D_{\mathbf{M}}(E(\lambda)) = D_{\mathbf{M}}(U^{-1}E(\lambda)U)$. Now Definition 15.1.1 gives $T_{\mathbf{M}}(A) = T_{\mathbf{M}}(U^{-1}AU)$.

We are now able to characterise the relative trace by inner properties:

Lemma 15.4.3. *Assume that a (real and finite) numerical function* $T'(A)$ *is defined for all Hermitian* $A \in \mathbf{M}$ *and that it has the following properties:*

(i) $T'(1) = 1$.

(ii) $T'(aA) = aT'(A)$ *if* a *is real.*

(iii) $T'(A + B) = T'(A) + T'(B)$ *if* A, B *commute.*

(iv) $T'(A) \geqq 0$ *if* A *is (semi-) definite.*

(v) $T(U^{-1}AU) = T'(A)$ *if* $U \in \mathbf{M}$ *is unitary.*

This is the case if and only if $T'(A)$ *is the relative trace:* $T'(A) = T_{\mathbf{M}}(A)$ *for all Hermitian* $A \in \mathbf{M}$.

Proof: $T_{\mathbf{M}}(A)$ has the properties (i)–(v) by Lemmas 15.3.2, 15.3.4, 15.2.3, and 15.4.2. Therefore only the converse problem remains: to determine $T'(A)$ if it fulfills (i)–(v). Assume therefore that such a $T'(A)$ is given.

If E is a projection $\in \mathbf{M}$ then, as E is definite, we have $T'(E) \geqq 0$ by (iv). If $E \sim F$($\cdots$ **M**) then by Lemma 15.4.1 a unitary $U \in \mathbf{M}$ with $F = U^{-1}EU$ exists,

and so by condition (v) $T'(E) = T'(F)$. If E, F are orthogonal, that is $EF = FE = 0$ then they commute, and therefore $T'(E + F) = T'(E) + T'(F)$. So the conditions (ii), (iii) of Definition 8.2.1 are fulfilled, and as $T'(1) = 1 > 0$, $< \infty$ therefore Lemma 8.3.5 applies: $T'(E)$ is a relative dimension function (with respect to $\mathbf{M}$). So $T'(E) = cD_{\mathbf{M}}(E)$ with a suitable constant c; but as $T'(1) = D_{\mathbf{M}}(1) = 1$ by (ii), therefore $c = 1$, that is: $T'(E) = D_{\mathbf{M}}(E)$.

Thus if $E_1, \cdots, E_p$ are mutually orthogonal projections, $E_1 + \cdots + E_p = 1$ and $c_1 < c_2 < \cdots < c_p$ then we have: If $\rho \gtrless \sigma$ then $E_\rho E_\sigma = E_\sigma E_\rho = 0$ so E_ρ, E_σ commute, and so by (ii), (iii) $T'(\sum_1^p c_\rho E_\rho) = \sum_1^p c_\rho T'(E_\rho) = \sum_1^p c_\rho D_{\mathbf{M}}(E_\rho)$ and this, by the considerations at the end of the proof of Lemma 15.3.4 is $= T_{\mathbf{M}}(\sum_1^p c_\rho E_\rho)$. So the desired equation $T'(A) = T_{\mathbf{M}}(A)$ holds for all A's of the above form $\sum_1^p c_\rho E_\rho$.

(i)–(iii) imply in general $T'(aA + b1) = aT'(A) + b$. (iii), (iv) imply $T'(A) \geqq T'(B)$ if $A - B$ is (semi-) definite and if A, B commute. Thus the argument of the proof of Lemma 15.3.3 applies to $T'(A)$ too, if we restrict ourselves to some commutative set of operators: Then $T'(A)$ is a continuous function of A, if the uniform topology for operators is used. We know from Lemma 15.3.3 that this holds for $T_{\mathbf{M}}(A)$ too.

Now form the operators mentioned in (18), pp. 390–391 (in particular footnote 43)): They were denoted by $A^{\kappa\epsilon} = \sum_{\nu=1}^{n} \lambda_\nu(E(\lambda_\nu) - (E(\lambda_{\nu-1}))$. As we pointed out there, they are uniformly convergent with the limit A; and as all $E(\lambda)$ commute with each other and with A, the same is true for the $A^{\kappa\epsilon}$. Thus $T'(A) = T_{\mathbf{M}}(A)$ follows, if we can prove $T'(A^{\kappa\epsilon}) = T_{\mathbf{M}}(A^{\kappa\epsilon})$. But this equation holds, because $A^{\kappa\epsilon}$ has the above discussed form. Put $p = n$, $c_\nu = \lambda_\nu$, $E_\nu = E(\lambda_\nu) - E(\lambda_{\nu-1})$.

Thus the proof is completed.

We restate the results obtained thus far:

Theorem XIII. *Let* $\mathbf{M}$ *be a factor of a finite class* ((I_n), *or* (II_1)). *Then there exists one and only one (real and finite) numerical function* $T'(A)$, *defined for all Hermitian* $A \in \mathbf{M}$ *which has the following properties:*

(i) $T'(1) = 1$.

(ii) $T'(aA) = aT'(A)$ *if* a *is real.*

(iii) $T'(A + B) = T'(A) + T'(B)$ *if* A, B *commute.*

(iv) $T'(A) \geqq 0$ *if* A *is (semi-) definite.*

(v) $T'(U^{-1}AU) = T'(A)$ *if* $U \in \mathbf{M}$ *is unitary.*

This unique $T'(A)$ *is the relative trace of* A, $T_{\mathbf{M}}(A)$. *Further essential properties of this* $T'(A)$ *are given in Lemmas* 15.2.3, 15.3.1, *and* 15.3.3. *Equivalent definitions are contained in Definition* 15.1.1 *and in Lemma* 15.2.2.

In particular we have (this follows from Lemma 15.3.1, *so from* (i)–(v)):

(vi) *For a projection* $E \in \mathbf{M}$, $T'(E) = 0$ *implies* $E = 0$.

In case (I_n) we obtain by this unicity theorem an easy determination of $T_{\mathbf{M}}(A)$: $\mathbf{M}$ in case (I_n) means $\mathfrak{H} = \mathfrak{H}_1 \otimes \mathfrak{H}_2$, $\mathbf{M} = B_1^{(1)}$, $\mathfrak{H}_1$, has dimension n. Now Lemma 2.4.6 show, that using the symbolism $A^0 \sim < A_{t,s} >_{t,s=1,\cdots,n}$ of Definition 2.4.2, the $A^0 \in \mathbf{M}$ are characterised by $A^0 \sim < \alpha_{t,s} 1 >_{t,s=1,\cdots,n}$ and

thus correspond to the numerical matrices $\{\alpha_{t,s}\}_{t,s=1,\cdots,n}$. Now $T'(A^0) = 1/n \sum_{t=1}^{n} \alpha_{t,t}$ fulfills (i)–(v), as one verifies easily, therefore $T_{\mathbf{M}}(A^0) = 1/n \sum_{t=1}^{n} \alpha_{t,t}$.

The examples of case (II_1) given in Theorem XI, considering Lemma 13.1.2, can be discussed too. If $\bar{A} \in \mathbf{M}$ or $\bar{A} \in \mathbf{M}'$ write $\bar{A} \approx [\chi_c(x)]_{x \in S,\, c \in \mathfrak{S}}$ and form $t(\bar{A}) = \int_S \chi_1(x)dx$ (cf. loc. cit. and Lemma 12.4.2). Then $T'(\bar{A}) = (1/\mu(S))\, t(\bar{A})$ fulfills (i)–(iii), as one verifies easily with the help of Lemma 12.4.2, (i)–(iii).

As to (iv) observe, that every definite operator $\bar{A}$ has the form $\bar{B}^2$, where $\bar{B} = \alpha(\bar{A})$, $\alpha(x) = |x|^{\frac{1}{2}}$. Thus $\bar{B}$ belongs to our ring ($\mathbf{M}$ resp. $\mathbf{M}'$) and is Hermitian. A fortiori $\bar{A} = \bar{B}^*\bar{B}$, and now $t(\bar{A}) = t(\bar{B}^*\bar{B}) \geqq 0$, $T'(\bar{A}) \geqq 0$ follow from Lemma 12.4.2, (iv).

(v) is proved as follows: As $A = \frac{1}{2}(A + A^*) + i \cdot 1/2i\,(A - A^*)$, it suffices to consider Hermitian operators A; and as

$$A = (\tfrac{1}{2}(A + 1))^2 - (\tfrac{1}{2}(A - 1))^2,$$

we may even assume $A = B^2$, B Hermitian. So we must prove

$$T'(U^{-1}B^2U) = T'(B^2),$$

that is $t(U^{-1}B^2U) = t(B^2)$. This coincides with Lemma 12.4.2, (iv), if we put there $A = BU$.

In all these cases, (iii), that is $T_{\mathbf{M}}(A + B) = T_{\mathbf{M}}(A) + T_{\mathbf{M}}(B)$ holds even without the restriction that A, B commute. This raises the following problem:

Problem 11. *Is* $T_{\mathbf{M}}(A + B) = T_{\mathbf{M}}(A) + T_{\mathbf{M}}(B)$ *always true, without further restrictions on* A, B?

The answer is certainly yes in case (I_n), and for all known examples of case (II_1). (Cf. also note at the end of this paper.)

15.5. The statements of Theorem XIII concerning the unique determination of $T'(A)$ by the properties enumerated there, can be formulated and discussed, even if $\mathbf{M}$ is not assumed to be a factor, but only a ring with $1 \in \mathbf{M}$.

So if $\mathbf{M}$ is a ring with $1 \in \mathbf{M}$ we may ask: When do the conditions (i)–(vi) of Theorem XIII have a unique solution? This is the case if $\mathbf{M}$ is a factor and in a finite case. Let us now consider the converse problem. Assume that $\mathbf{M}$ is a ring with $1 \in \mathbf{M}$ and that (i)–(vi) have a unique solution, say $T^0(A)$.

Consider $\mathbf{M} \cdot \mathbf{M}'$. This is a ring, and therefore it is the ring generated by all projections $E \in \mathbf{M} \cdot \mathbf{M}'$ (cf. (18), p. 392). Consider a projection $E \in \mathbf{M} \cdot \mathbf{M}'$. If $A \in \mathbf{M}$ the A and E commute, as $E \in \mathbf{M}'$ and so $AE = AEE = EAE$. Thus if A is Hermitian resp. (semi-) definite, AE is too. Besides A, $E \in \mathbf{M}$ so $AE \in \mathbf{M}$. Now take two constants a_1, $a_2 > 0$ and define

$$T'(A) = a_1T^0(A) + a_2T^0(AE).$$

Then $T'(A)$ behaves with respect to the conditions (i)–(vi) of Theorem XIII as follows: (i) holds if $a_1 + a_2T^0(E) = 1$. (ii) holds at any rate. (iii) holds, because $E \in \mathbf{M}'$ commutes with the A, $B \in \mathbf{M}$ so if these A, B commute, then AE, BE do too. (iv) holds, because AE is (semi-) definite along with A. (v)

holds, because $E \in \mathbf{M}'$ commutes with the $U \in \mathbf{M}$ so $(U^{-1}AU)E = U^{-1}(AE)U$. (vi) holds, because if F is a projection, therefore (semi-) definite, then FE is it too; thus $T^0(F) \geqq 0$, $T^0(FE) \geqq 0$ and $T'(F) = 0$ implies $T^0(F) = 0$, $F = 0$.

Under these conditions our assumption necessitates $T'(A) = T^0(A)$ for each Hermitian $A \in \mathbf{M}$. So in particular $T'(E) = T^0(E)$ but the definition gives $T'(E) = (a_1 + a_2)\, T^0(E)$. Therefore $(a_1 + a_2 - 1)T^0(E) = 0$. Thus we have proved: $a_1, a_2 > 0$, $a_1 + a_2 T^0(E) = 1$ must imply $(a_1 + a_2 - 1)T^0(E) = 0$.

This is clearly not the case if $T^0(E) \neq 0, 1$; therefore we have either $T^0(E) = 0$, $E = 0$ (by (vi)), or $T^0(E) = 1$, $T^0(1 - E) = 1 - T^0(E) = 0$, $1 - E = 0$, $E = 1$ (by (i), (iii), (vi)).

Thus $\mathbf{M} \cdot \mathbf{M}' = \mathbf{R}(0, 1) = (\alpha 1)$ and so $\mathbf{M}$ is a factor.

Now the argument made at the beginning of the proof of Lemma 15.4.3 shows, remembering Lemma 8.3.5, that $T^0(E)$ is a relative dimension function with respect to $\mathbf{M}$ (E runs over all projections $E \in \mathbf{M}$), $T^0(1) = 1$ (by (i)) being finite, 1 is finite with respect to $\mathbf{M}$, and so is $\mathfrak{H}$. Therefore $\mathbf{M}$ is in one of the finite cases: (I_n), $n = 1, 2, \cdots$, or (II_1).

Consequently we have proved this:

Theorem XIV: *Let* $\mathbf{M}$ *be a ring with* $1 \in \mathbf{M}$. *The conditions* (i)–(vi) *of Theorem* XIII *admit a unique solution if and only if* $\mathbf{M}$ *is a factor, and belongs to one of the finite cases:* (I_n), $n = 1, 2, \cdots$, *or* (II_1).

Note that while (i)–(v) imply (vi) if $\mathbf{M}$ is a factor, they may not imply it if $\mathbf{M}$ is merely a ring with $1 \in \mathbf{M}$ and therefore the unicity of their solution may not guarantee the factor character of $\mathbf{M}$. This is the reason why (i)–(v) where sufficient in Theorem XIII, but (i)–(vi) were needed in Theorem XIV. In fact there exist rings $\mathbf{M}$ with $1 \in \mathbf{M}$ for which (i)–(v) have a unique solution, and which nevertheless are not factors. Easy considerations, which we will not detail here, show in fact that the ring $\mathbf{M} = (P_{[\varphi]})'$ ($\mathfrak{H}$ any space of ∞ dimensions, $\varphi \in \mathfrak{H}$, $\| \varphi \| = 1$) is such; the unique solution of (i)-(v) being $T^0(A) = (A\varphi, \varphi)$ and $\mathbf{M}$ is not a factor, because $P_{[\varphi]} \in \mathbf{M} \cdot \mathbf{M}'$.

We have not attempted in this chapter to give anything like a complete theory of the trace. We wanted to give only a brief summary of the most characteristic features.

Many further problems could be handled with our present methods. For instance, Lemma 15.3.3 could be strengthened so as to apply even to the strong topology of operators (cf. (18), p. 381); connections could be established between $T_{\mathbf{M}}(A)$ and the (Af, f) etc. We propose to come back to this subject, as well as to Problem 11, in subsequent papers.

Chapter XVI: Unbounded operators

16.1. $\mathbf{M}$ is assumed throughout this chapter too to be a factor in a finite case ((I_n), $n = 1, 2, \cdots$, or (II_1)), and that $D_{\mathbf{M}}(\mathfrak{M})$ is normalised by $D_{\mathbf{M}}(\mathfrak{H}) = 1$.

The following Lemma is a consequence of Lemma 6.2.1, valid in the finite cases only:

Lemma 16.1.1. *If* X *is a linear, closed operator with an everywhere dense do-*

main, and $X \eta \mathbf{M}$ *then* $(f; Xf = 0)$ *and* $(f; X^*f = 0)$ *are linear closed sets* $\eta \mathbf{M}$ *and*

$$D_{\mathbf{M}}((f; Xf = 0)) = D_{\mathbf{M}}((f, X^*f = 0)).$$

Proof: $X^*f = 0$ means $(X^*f, g) = (f, Xg) = 0$ for all $g \epsilon$ Domain X, that is, that f is orthogonal to Range X. So $(f; X^*f = 0) = \mathfrak{H} - [\text{Range } X]$ therefore $(f; X^*f = 0) \eta \mathbf{M}$, $D_{\mathbf{M}}((f; X^*f = 0)) = D_{\mathbf{M}}(\mathfrak{H}) - D_{\mathbf{M}}([\text{Range } X]) = 1 - D_{\mathbf{M}}([\text{Range } X])$. Replacing X by X^* we see that $(f; Xf = 0) \eta \mathbf{M}$, $D_{\mathbf{M}}((f; Xf = 0)) = 1 - D_{\mathbf{M}}([\text{Range } X^*])$. Now Lemma 6.2.1 states $D_{\mathbf{M}}([\text{Range } X]) = D_{\mathbf{M}}([\text{Range } X^*])$ therefore $D_{\mathbf{M}}((f; Xf = 0)) = D_{\mathbf{M}}((f; X^*f = 0))$.

In the cases (I_n), $n = 1, 2, \cdots$, this is the statement that the number of linearly independent solutions of a linear equation $Xf = 0$ coincides with the one for the adjoint equation $X^*f = 0$. We see now, that owing to our notion of relative dimension, it is true in the case (II_1) too. Observe, that these are the only cases where it holds: In any infinite case $\mathfrak{H}$ is infinite, so $\mathfrak{H} \sim \mathfrak{M} \subsetneqq \mathfrak{H}$ and if $U \epsilon \mathbf{M}$ is the partially isometric operator which maps $\mathfrak{H}$ on $\mathfrak{M}$ then $(f; Uf = 0) = (0)$, $(f; U^*f = 0) = \mathfrak{H} - \mathfrak{M} \neq (0)$.

16.2. The key to the deductions which follow is contained in this definition:

Definition 16.2.1. Let $\mathfrak{A}$ be an arbitrary linear set in $\mathfrak{H}$ ($\mathfrak{A}$ is not necessarily closed, and not necessarily $\eta \mathbf{M}$). If a sequence $\mathfrak{M}_1, \mathfrak{M}_2, \cdots$ of linear closed sets exists which has the following properties:

(i) $\mathfrak{M}_i \eta \mathbf{M}$.

(ii) $\mathfrak{M}_1 \subset \mathfrak{M}_2 \subset \cdots \subset \mathfrak{A}$.

(iii) $[\mathfrak{M}_1, \mathfrak{M}_2, \cdots] = \mathfrak{H}$.

then $\mathfrak{A}$ is called *essentially dense.*

We prove:

Lemma 16.2.1. *Every essentially dense* $\mathfrak{A}$ *is everywhere dense too.*

Proof: As all $\mathfrak{M}_i \subset \mathfrak{A}$ and as $\mathfrak{A}$ is a linear set, $\{\mathfrak{M}_1, \mathfrak{M}_2, \cdots\} \subset \mathfrak{A} \subset \mathfrak{H}$. $\mathfrak{H} = [\mathfrak{M}_1, \mathfrak{M}_2, \cdots] \subset [\mathfrak{A}] \subset \mathfrak{H}$, $[\mathfrak{A}] = \mathfrak{H}$. As $\mathfrak{A}$ is linear, $[\mathfrak{A}]$ consists of its condensation points only, therefore $\mathfrak{A}$ is everywhere dense.

Lemma 16.2.2. *If* $\mathfrak{A}_1, \mathfrak{A}_2, \cdots$ *is a (finite or infinite) sequence of essentially dense sets, then* $\mathfrak{A}_1 \cdot \mathfrak{A}_2 \cdot \cdots$ *is essentially dense too.*

Proof: We can assume that the sequence $\mathfrak{A}_1, \mathfrak{A}_2, \cdots$ is infinite, as we could otherwise insert infinitely many terms $\mathfrak{A}_i = \mathfrak{H}$.

Form the sequences from Definition 16.2.1: $\mathfrak{M}_{i,1}, \mathfrak{M}_{i,2}, \cdots$ for $\mathfrak{A}_i$, $i = 1, 2, \cdots$. By Lemma 8.3.3, $\lim_{\rho \to \infty} D_{\mathbf{M}}(\mathfrak{M}_{i,\rho}) = D_{\mathbf{M}}([\mathfrak{M}_{i,1}, \mathfrak{M}_{i,2}, \cdots]) = D_{\mathbf{M}}(\mathfrak{H}) = 1$ and so we can form a sequence $\rho_{i,1} < \rho_{i,2} < \cdots$ with $D_{\mathbf{M}}(\mathfrak{M}_{i,\rho_{i,\nu}}) > 1 - (1/2^{i+\nu})$, $D_{\mathbf{M}}(\mathfrak{H} - \mathfrak{M}_{i,\rho_{i,\nu}}) = D_{\mathbf{M}}(\mathfrak{H}) - D_{\mathbf{M}}(\mathfrak{M}_{i,\rho_{i,\nu}}) < 1/2^{i+\nu}$.

Form $\overline{\mathfrak{M}}_\nu = \mathfrak{M}_{1,\rho_{1,\nu}} \cdot \mathfrak{M}_{2,\rho_{2,\nu}} \cdots$. Clearly $\overline{\mathfrak{M}}_\nu$ is a closed linear set, and $\overline{\mathfrak{M}}_\nu \eta \mathbf{M}$. As $\rho_{i,\nu} < \rho_{i,\nu+1}$ therefore $\mathfrak{M}_{i,\rho_{i,\nu}} \subset \mathfrak{M}_{i,\rho_{i,\nu+1}}$ and so $\overline{\mathfrak{M}}_\nu \subset \overline{\mathfrak{M}}_{\nu+1}$; as $\mathfrak{M}_{i,\rho_{i,\nu}} \subset \mathfrak{A}_i$ so $\overline{\mathfrak{M}}_\nu \subset \mathfrak{A}_1 \cdot \mathfrak{A}_2 \cdots$ thus we have $\overline{\mathfrak{M}}_1 \subset \overline{\mathfrak{M}}_2 \subset \cdots \subset \mathfrak{A}_1 \cdot \mathfrak{A}_2 \cdots$. Finally we have $D_{\mathbf{M}}(\mathfrak{H} - \overline{\mathfrak{M}}_\nu) = D_{\mathbf{M}}([\mathfrak{H} - \mathfrak{M}_{1,\rho_{1,\nu}}, \mathfrak{H} - \mathfrak{M}_{2,\rho_{2,\nu}}, \cdots])$ by Lemma 14.2.5 this is $\leqq \sum_{i=1}^{\infty} D_{\mathbf{M}}(\mathfrak{H} - \mathfrak{M}_{i,\rho_{i,\nu}}) < \sum_{i=1}^{\infty} 1/2^{i+\nu} = 1/2^\nu$, and so $D_{\mathbf{M}}(\mathfrak{H} - [\overline{\mathfrak{M}}_1, \overline{\mathfrak{M}}_2, \cdots]) \leqq D_{\mathbf{M}}(\mathfrak{H} - \overline{\mathfrak{M}}_\nu) < 1/2^\nu$. This holds for every

$\nu = 1, 2, \cdots$, therefore $D_{\mathbf{M}}(\mathfrak{H} - [\overline{\mathfrak{M}}_1, \overline{\mathfrak{M}}_2, \cdots]) = 0$, $\mathfrak{H} - [\overline{\mathfrak{M}}_1, \overline{\mathfrak{M}}_2, \cdots] = (0)$, $[\overline{\mathfrak{M}}_1, \overline{\mathfrak{M}}_2, \cdots] = \mathfrak{H}$.

Thus the sequence $\overline{\mathfrak{M}}_1, \overline{\mathfrak{M}}_2, \cdots$ fulfills all conditions (i)–(iii) of Definition 16.2.1 for the linear set $\mathfrak{A}_1 \cdot \mathfrak{A}_2 \cdot \cdots$; therefore $\mathfrak{A}_1 \cdot \mathfrak{A}_2 \cdots$ is essentially dense.

Lemma 16.2.3. *Let $\mathfrak{A}$ be an essentially dense set, and X a linear closed operator with an everywhere dense domain, and $X \eta \mathbf{M}$. Then the set $(f; f \in$ Domain X, $Xf \in \mathfrak{A})$ is essentially dense.*

Proof: Let W, B be the canonical decomposition of X: B self adjoint and definite, W partially isometric, $W \in \mathbf{M}$, $B \eta \mathbf{M}$ and $X = WB$ (cf. Definition 4.4.1, and Lemma 4.4.1).

Let $E(\lambda)$, $-\infty < \lambda < \infty$, be the resolution of unity which belongs to B. This means, as B is not necessarily bounded, that $E(\lambda)$ fulfills the conditions (α)–(γ) of Definition 15.1.1 unaltered, but instead of (δ)–(ϵ) these modifications:

(δ') If $\lambda \to -\infty$ resp. $+\infty$ then $E(\lambda) \to 0$ resp. 1 in the sense of strong operator convergence.

(ϵ') $f \in$ Domain B if and only if $\int_{-\infty}^{\infty} \lambda^2 d \parallel E(\lambda) f \parallel^2$ is finite, and then $(Bf, g) = \int_{-\infty}^{\infty} \lambda d(E(\lambda)f, g)$, (cf. (15), p. 92).

The (semi-) definiteness of B means that $E(\lambda) = 0$ for $\lambda < 0$ (cf. (21), p. 302). Put $E(\lambda) = P_{\mathfrak{M}(\lambda)}$.

As $B \eta \mathbf{M}$, B is invariant under every unitary transformation $U' \in \mathbf{M}'$ and so is $E(\lambda)$, thus $E(\lambda) \in \mathbf{M}$, $\mathfrak{M}(\lambda) \eta \mathbf{M}$, $E(\lambda)$ and $\mathfrak{M}(\lambda)$ obviously reduce B. If $f \in \mathfrak{M}(\lambda)$ then $E(\lambda)f = f$ so $E(\lambda')\, f \begin{cases} = f, \lambda' \geqq \lambda \\ = 0, \lambda' < 0 \end{cases}$. Therefore (δ') shows that Bf is defined, and (ϵ') that $0 \leqq (Bf, f) \leqq \lambda \parallel f \parallel^2$. Thus the part of B in $\mathfrak{M}(\lambda)$ is bounded and everywhere defined (in $\mathfrak{M}(\lambda)$) or: $BE(\lambda)$ is bounded and everywhere defined (in $\mathfrak{H}$).

We know that $\mathfrak{M}(1) \subset \mathfrak{M}(2) \subset \cdots$ and as $\lim_{i \to \infty} E(i) = 1$ we have $[\mathfrak{M}(1), \mathfrak{M}(2), \cdots] = \mathfrak{H}$. Thus by Lemma 8.3.3

$$\lim_{i \to \infty} D_{\mathbf{M}}(\mathfrak{M}(i)) = D_{\mathbf{M}}(\mathfrak{H}) = 1,$$
$$\lim_{i \to \infty} D_{\mathbf{M}}(\mathfrak{H} - \mathfrak{M}(i)) = \lim_{i \to \infty} (D_{\mathbf{M}}(\mathfrak{H}) - D_{\mathbf{M}}(\mathfrak{M}(i)) = 0.$$

Choose next the sequence $\mathfrak{M}_1, \mathfrak{M}_2, \cdots$ from Definition 16.2.1 for $\mathfrak{A}$. Then $\mathfrak{M}_1 \subset \mathfrak{M}_2 \subset \cdots \subset \mathfrak{A}$, $[\mathfrak{M}_1, \mathfrak{M}_2, \cdots] = \mathfrak{H}$, therefore as above, $\lim_{i \to \infty} D_{\mathbf{M}}(\mathfrak{H} - \mathfrak{M}_i) = 0$.

Form now the set $\mathfrak{M}^{(i)} = (f; f \in \mathfrak{M}(i), WBE(i)f \in \mathfrak{M}_i)$. As $BE(i)$ is everywhere defined and bounded, $\mathfrak{M}^{(i)}$ is linear and closed. As $f \in \mathfrak{M}(i)$ implies $WBE(i)f = WBf = Xf$ we have $\mathfrak{M}^{(i)} \subset (f; f \in$ Domain X, $Xf \in \mathfrak{M}_i) \subset (f, f \in$ Domain X, $Xf \in \mathfrak{A})$. Finally $f \in \mathfrak{M}(i)$ implies $f \in \mathfrak{M}(i+1)$ and $E(i+1)f = E(i)f = f$ and so $\mathfrak{M}^{(i)} \subset \mathfrak{M}^{(i+1)}$. Thus we have $\mathfrak{M}^{(1)} \subset \mathfrak{M}^{(2)} \subset \cdots \subset (f; f \in$ Domain X, $Xf \in \mathfrak{A})$.

The definition of $\mathfrak{M}^{(i)}$ makes it obvious that it is invariant under all unitary transformations $U' \in \mathbf{M}'$ therefore $\mathfrak{M}^{(i)} \eta \mathbf{M}$.

Now $\mathfrak{M}^{(i)} = (f;\ f \in \mathfrak{M}(i),\ WBE(i)\ f \in \mathfrak{M}_i) = \mathfrak{M}(i) \cdot (f;\ WBE(i)f \in \mathfrak{M}_i) = \mathfrak{M}(i) \cdot (f;\ P_{\mathfrak{H}-\mathfrak{M}_i}\ WBE(i)f = 0)$. Now by Lemma 16.1.1,

$$D_{\mathbf{M}}(f;\ P_{\mathfrak{H}-\mathfrak{M}_i}\ WBE(i)f = 0) = D_{\mathbf{M}}(f;\ ((P_{\mathfrak{H}-\mathfrak{M}_i}\ WBE(i))^* f = 0)$$

$$= D_{\mathbf{M}}((f;\ (BE(i))^*\ W^*\ P_{\mathfrak{H}-\mathfrak{M}_i} f = 0))$$

and

$$(f;\ (BE(i))^*\ W^*\ P_{\mathfrak{H}-\mathfrak{M}_i} f = 0) \supset (f;\ P_{\mathfrak{H}-\mathfrak{M}_i} f = 0) = \mathfrak{M}_i\,.$$

So

$$D_{\mathbf{M}}((f;\ P_{\mathfrak{H}-\mathfrak{M}_i}\ WBE(i)\ f = 0)) \geqq D(\mathfrak{M}_i);$$

$$D_{\mathbf{M}}(\mathfrak{H} - (f;\ P_{\mathfrak{H}-\mathfrak{M}_i}\ WBE(i)f = 0)) \leqq D_{\mathbf{M}}(\mathfrak{H} - \mathfrak{M}_i)\,.$$

Now we obtain, using Lemma 14.2.3, or 14.2.5, this:

$$D_{\mathbf{M}}(\mathfrak{H} - \mathfrak{M}^{(i)}) = D_{\mathbf{M}}([\mathfrak{H} - \mathfrak{M}(i),\ \mathfrak{H} - (f;\ P_{\mathfrak{H}-\mathfrak{M}_i}\ WBE(i)f = 0)])$$

$$\leqq D_{\mathbf{M}}(\mathfrak{H} - \mathfrak{M}(i)) + D_{\mathbf{M}}(\mathfrak{H} - (f;\ P_{\mathfrak{H}-\mathfrak{M}_i}\ WBE(i)f = 0))$$

$$\leqq D_{\mathbf{M}}(\mathfrak{H} - \mathfrak{M}(i)) + D_{\mathbf{M}}(\mathfrak{H} - \mathfrak{M}_i)\,.$$

Thus

$$\lim_{i\to\infty} D_{\mathbf{M}}(\mathfrak{H} - \mathfrak{M}(i)) = 0\,, \qquad \lim_{i\to\infty} D_{\mathbf{M}}(\mathfrak{H} - \mathfrak{M}_i) = 0$$

give

$$\lim_{i\to\infty} (\mathfrak{H} - \mathfrak{M}^{(i)}) = 0\,.$$

By Lemma 8.3.4

$$D_{\mathbf{M}}(\mathfrak{H} - [\mathfrak{M}^{(1)}, \mathfrak{M}^{(2)}, \cdots]) = D_{\mathbf{M}}((\mathfrak{H} - \mathfrak{M}^{(1)}) \cdot (\mathfrak{H} - \mathfrak{M}^{(2)}) \cdot (\mathfrak{H} - \mathfrak{M}^{(3)}) \cdots)$$

$$= \lim_{i\to\infty} D_{\mathbf{M}}(\mathfrak{H} - \mathfrak{M}^{(i)}) = 0,$$

$$\mathfrak{H} - [\mathfrak{M}^{(1)}, \mathfrak{M}^{(2)}, \cdots] = (0)\,, \qquad [\mathfrak{M}^{(1)}, \mathfrak{M}^{(2)}, \cdots] = \mathfrak{H}\,.$$

Thus $\mathfrak{M}^{(1)}, \mathfrak{M}^{(2)}, \cdots$ fulfill the conditions (i)–(iii) of Definition 16.2.1 for $(f;\ f \in \text{Domain } X,\ Xf \in \mathfrak{A})$. Therefore this set is essentially dense, and the proof is completed.

Putting $\mathfrak{A} = \mathfrak{H}$ we see that Domain X itself is essentially dense. This of course, could have been easily proved directly too.

The considerations of this § are special cases of a generalisation of the notion of relative dimensionality (which was only defined for linear closed sets $\eta\,\mathbf{M}$ so far) to all linear sets. A theory very similar to that of Lebesgue's interior measure results. We do not go into the details of this subject here, but it will be dealt with in subsequent papers.

16.3. The Lemmas 16.2.1–16.2.3 have an interesting application to unbounded operators.

Definition 16.3.1. Let $x_1, y_1, x_2, y_2, \cdots$ be a (finite or infinite) sequence of

symbolic variables. By a (non-commutative) monomial we mean an expression of the form $z_1 \cdots z_n$ where $n = 0, 1, 2, \cdots$ and each z_ν is some x_i or y_i. If $n = 0$, we use the symbol 1. The order in which the factors $z_1, \cdots, z_n$ occur is essential; the number n is the degree of the monomial. By a (non-commutative) polynomial we mean a finite linear aggregate of non-commutative monomials, that is, an expression of the form

$$p(x_1, y_1, \cdots) = \sum_1^p a_\rho z_1^{(\rho)} \cdots z_{m_\rho}^{(\rho)}$$

where $p = 0, 1, 2, \cdots$; $a_1, \cdots a_\rho$, are complex numbers, and each $z_\nu^{(\rho)}$ is some x_i or y_i. If $p = 0$, we use the symbol 0. Monomial terms of the form $0 \cdot z_1 \cdots z_n$ cannot be omitted, but two terms $a \cdot z_1 \cdots z_n$ and $b \cdot z_1 \cdots z_n$ can be contracted to one and the order of the additive terms does not matter. We consider monomials as special cases of polynomials. If

$$p(x_1, y_1, \cdots) = \sum_1^p a_\rho z_1^{(\rho)} \cdots z_{n_\rho}^{(\rho)}$$

is a polynomial, then ${}^{(r)}p(x_1, y_1, \cdots)$ the reduced polynomial of $p(x_1, y_1, \cdots)$ obtains from it by omitting all terms with $a_\rho = 0$ (that is, of the form $0 \cdot z_1 \cdots z_n$). If $p(x_1, y_1, x_2, y_2, \cdots)$, $q(x_1 y_1, x_2, y_2, \cdots)$ are two polynomials, then

$$ap(x_1, y_1, x_2, y_2, \cdots),\ p(x_1, y_1, x_2, y_2, \cdots) + q(x_1, y_1, x_2, y_2, \cdots),$$

$$p(x_1, y_1, x_2, y_2, \cdots) \cdot q(x_1, y_1, x_2, y_2, \cdots)$$

are those which result by carrying out the indicated operations term by term, but without reducing (if the process ${}^{(r)}$ is not explicitly indicated).

The symbol $^+$ is defined as follows, in successive steps:

$$x_i^+ = y_i,\ y_i^+ = x_i;\ (z_1 \cdots z_n)^+ = z_n^+ \cdots z_1^+;\ (\sum_1^p a_\rho (z_1^{(\rho)} \cdots z_{n_\rho}^{(\rho)}))^+$$

$$= \sum_1^p \bar{a}_\rho (z_1^{(\rho)} \cdots z_{n_\rho}^{(\rho)})^+.$$

Obviously $({}^{(r)}p(x_1, y_1, \cdots))^+ = {}^{(r)}(p(x_1, y_1, x_2, y_2 \cdots)^+)$.

Definition 16.3.2. Let $x_1, y_1, x_2, y_2, \cdots$ be a (finite or infinite) sequence of symbolic variables; $X_1, X_2, \cdots$ a corresponding sequence of linear, closed operators, each one having an everywhere dense domain, so that to every pair of variables, x_i, y_i one operator X_i corresponds. If $p(x_1, y_1, x_2, y_2, \cdots) = \sum_1^p a_\rho z_1^{(\rho)} \cdots z_{n_\rho}^{(\rho)}$ is a polynomial, we mean by $p(X_1, X_1^*, X_2, X_2^*, \cdots)$ the operator which arises if we replace in $\sum_{\eta=1}^p a_\rho z_1^{(\rho)} \cdots \cdot z_{n_\rho}^{(\rho)}$ every x_i by X_i and every y_i by X_i^*. Here the domain and the values of $p(X_1, X_1^*, X_2, X_2^*, \cdots)$ are to be determined with the help of the following rules (cf. (18), p. 404):

$0f$ and $1f$ are everywhere defined and have the values 0 resp. f. $(aX)f$ is defined if and only if Xf is defined (even for $a = 0$), and its value is $a(Xf)$. $(X + Y)f$ is defined if and only if both Xf and Yf are defined and its value is $Xf + Yf$. $(XY)f$ is defined if and only if both Yf and $X(Yf)$ are defined, and its value is $X(Yf)$.

Variables x_i, y_i resp. the corresponding operators X_i, X_i^* which do not occur in the expression $p(x_1, y_1, x_2, y_2, \cdots) = \sum_1^p a_\rho z_1^{(\rho)} \cdots z_{n_\rho}^{(\rho)}$ can be omitted in the expression $p(x_1, y_1, x_2, y_2, \cdots)$ resp. $p(X_1, X_1^*, \cdots)$. (So we can write $p(x_1)$ for $p(x_1, y_1) = x_1$ but not for $p(x_1, y_1) = x_1 + 0 \cdot y_1$.). $p(X_1, X_1^*, \cdots)$ and ${}^{(r)}p(X_1, X_1^*, \cdots)$ are not identical in general. Thus if $p(x_1) = 0 \cdot x_1$, ${}^{(r)}p(x_1) = 0$ then $p(X_1) = 0 \cdot X_1$ and ${}^{(r)}p(X_1) = 0$ and these differ, because they have different domains, if X_1 is not everywhere defined. In this particular case however, we can obtain equality by forming the closure [ ] of all operators concerned (cf. (16), pp. 70–71, where a $\sim$ is used to denote closure): As X_1 has an everywhere dense domain, therefore $[0 \cdot X_1] = 0$ and so $[p(X_1)]$ $[{}^{(r)}p(X_1)]$. But in general this procedure too breaks down: Define $p(x_1, x_2) = 0 \cdot x_1 + 0 \cdot x_2$, ${}^{(r)}p(x_1, x_2) = 0$ and let X_1, X_2 be two operators for which Domain $X_1 \cdot$ Domain $X_2 = (0)$ (cf. (17), p. 230). Then $p(X_1, X_2)$ has the domain (0) and there of course the value 0. Thus the same is true for $[p(X_1, X_2)]$, while ${}^{(r)}p(X_1, X_2) = 0$, $[{}^{(r)}p(X_1, X_2)] = 0$. Thus $[p(X_1, X_2)]$ has the domain (0), while $[{}^{(r)}p(X_1, X_2)]$ has the domain $\mathfrak{H}$, and so they are different.

Another "pathological" possibility is that $(X_1 \cdot X_2)^*$ may not be $X_2^* \cdot X_1^*$. In fact it can happen that $X_1 X_2$ has no closure at all, and therefore no adjoint with an everywhere dense domain (cf. (21), pp. 296 and 300; our statement means that $X_1 X_2$ is not a **-operator, cf. loc. cit.); or again $(X_1 X_2)^*$ may exist and be a proper extension of $X_2^* \cdot X_1^*$. We will not go presently into the details of this matter.

For $X_1, X_2, \cdots \eta \mathbf{M}$ where $\mathbf{M}$ is a factor in one of the finite cases ((I_n), $n = 1, 2, \cdots$, or (II_1)), as we now always assume, nothing of this sort happens, and the algebra of the $[p(X_1, X_1^*, X_2, X_2^*, \cdots)]$ works perfectly smoothly. This is the main result of the considerations which follow. For the cases (I_n), $n = 1, 2, \cdots$, these things are obvious (then even the [ ] is superfluous, because every linear operator is closed), and the essential content of our results is that it is the same way in case (II_1). So while the usually considered case (I_∞), (including $\mathbf{M} = \mathbf{B}$ for a Hilbert space $\mathfrak{H}$) is highly pathological in this respect, (II_1) turns out to be the appropriate generalisation of the (I_n).

16.4. We prove first two Lemmas which have considerable interest of their own. In particular the first one proves that the "spectral problem" (the existence of a "resolution of unity" for every linear, closed Hermitian operator $X \eta \mathbf{M}$) has always a satisfactory solution. (Cf. (16), p. 92; observe the contrast with (I_∞).)

Lemma 16.4.1. *Every linear, closed, Hermitian operator* $X \eta \mathbf{M}$ *is maximal and self adjoint, (cf.* (16)*, p.* 88).

Proof: Let U be the Cayley transform of X, $\mathfrak{E}$, $\mathfrak{F}$ the connected linear, closed sets (cf. (16), p. 80). Then $U, \mathfrak{E}, \mathfrak{F}, E = P_{\mathfrak{E}}$ are invariant under every unitary transformation $U' \epsilon \mathbf{M}'$ along with X, so they are all $\eta \mathbf{M}$. Thus $UE \eta \mathbf{M}$ and it is obviously partially isometric, with the initial and final sets $\mathfrak{E}$ resp. $\mathfrak{F}$, so UE is bounded, and therefore $UE \epsilon \mathbf{M}$. Similarly $E \epsilon \mathbf{M}$.

We know that [Range $(U - 1)$] $= \mathfrak{H}$ and as Uf is only defined if $f \epsilon \mathfrak{E}$ we may

as well write [Range $(UE - E)$] = $\mathfrak{H}$. Now $UE - E = (UE - E)E$, $(UE - E)^* = E(UE - E)^*$ and so [Range $(UE - E)^*$] $\subset$ [Range E] = $\mathfrak{E}$. So by Lemma 6.2.1

$$D_{\mathbf{M}}(\mathfrak{E}) \geqq D_{\mathbf{M}}([\text{Range } (UE - E)^*]) = D_{\mathbf{M}}([\text{Range } (UE - E)]) = D_{\mathbf{M}}(\mathfrak{H}) = 1 .$$

Considering the properties of the partially isometric operator UE, we have further $D_{\mathbf{M}}(\mathfrak{E}) = D_{\mathbf{M}}(\mathfrak{F})$. Therefore $D_{\mathbf{M}}(\mathfrak{H} - \mathfrak{E}) = D_{\mathbf{M}}(\mathfrak{H} - \mathfrak{F}) \leqq 0$ that is 0 and so $\mathfrak{H} - \mathfrak{E} = \mathfrak{H} - \mathfrak{F} = (0)$. So X is maximal and self adjoint by (16) p. 88.

Lemma 16.4.2. *Let X, Y be linear, closed operators, with everywhere dense domains, X, Y η $\mathbf{M}$. If Y is an extension of X (that is: whenever Xf is defined, Yf is defined too, and $Xf = Yf$; cf.* (15), *p.* 70), *then $X = Y$.*

Proof: Let W, B be the canonical decomposition of Y: B self adjoint and (semi-) definite, W partially isometric, W η $\mathbf{M}$, B η $\mathbf{M}$ and $Y = WB$ (cf. Definition 4.4.1, and Lemma 4.4.1). At the same time $B = W^*Y$ (cf. (21), p. 306), so $Y = WW^*Y$ and as Y is a continuation of X, so $X = WW^*X$ too.

$W^*Y = B$ is self adjoint, so it is Hermitian, and so W^*X is too; besides it is linear, closed and has an everywhere dense domain along with X. As W^*X η $\mathbf{M}$ therefore Lemma 16.4.1 applies: W^*X is maximal. But W^*Y is a Hermitian continuation of W^*X so $W^*X = W^*Y$, $WW^*X = WW^*Y$, $X = Y$. Thus the proof is completed.

We pass now to the discussion of the (non-commutative) polynomials of operators.

Lemma 16.4.3. *Let $X_1, X_2, \cdots$ be a (finite or infinite) sequence of linear, closed operators, each one having an everywhere dense domain. Assume that all X_i η $\mathbf{M}$. Let $\mathfrak{DP} = \mathfrak{DP}(X_1, X_2, \cdots)$ be the common part of the domains of the operators $p(X_1, X_1^*, X_2, X_2^*, \cdots)$ where $p(x_1, y_1, x_2, y_2, \cdots)$ runs over all (non-commutative) polynomials of $x_1, y_1, x_2, y_2, \cdots$. Then $\mathfrak{DP}$ is essentially dense and everywhere dense.*

Proof: It suffices, by Lemma 16.2.1 to prove the essential density. It suffices furthermore obviously, to let $p(x_1, y_1, x_2, y_2, \cdots)$ run over all monomials only. These are countable in number, say $p^{(\sigma)}(x_1, y_1, \cdots)$, $\sigma = 1, 2, \cdots$, therefore $\mathfrak{DP} = \prod_{\sigma=1}^{\infty}$ Domain $p^{\sigma}(x_1, y_1, \cdots)$. By Lemma 16.2.2 we need only to prove that every Domain $p^{\sigma}(X_1, X_1^*, \cdots)$ is essentially dense. That is: That for every monomial $p(x_1, y_1, \cdots) = z_1 \cdots z_n$ (cf. Definition 16.3.1) Domain $p(X_1, X_1^*, \cdots)$ is essentially dense.

We prove this by induction on the degree n. For $n = 0$, the operator in question is 1, its domain is $\mathfrak{H}$ and so our statement is obvious. Assume now that it is already established for $n - 1$, $n = 1, 2, \cdots$ and let us consider n. Let $p(x_1, y_1, \cdots) = z_1 \cdots z_n$ be a monomial of degree n. Put $q(x_1, y_1, \cdots) = z_1 \cdots z_{n-1}$ and write $p(X_1, X_1^*, \cdots) = A$, $q(X_1, X_1^*, \cdots) = B$. z_n is a variable x_i or y_i; put correspondingly $Y = X_i$ resp. $= X_i^*$. Now $A = BY$ so Domain $A = (f; f \in \text{Domain } Y, Yf \in \text{Domain } B)$. Domain B is essentially dense by our induction assumption (the degree of $q(x_1, y_1, \cdots) = z_1 \cdots z_{n-1}$

is $n - 1$), Y is linear, closed and has an everywhere dense domain (for X_i this was assumed, for X_i^* it follows from (21), p. 301). Therefore Lemma 16.2.3 applies, and Domain A is essentially dense too. Thus the proof is completed.

Lemma 16.4.4. *Let $X_1, X_2, \cdots$ be a (finite or infinite) sequence of linear, closed operators, each one having an everywhere dense domain. Assume that all $X_i \eta \mathbf{M}$. Let $p(x_1, y_1, x_2, y_2, \cdots)$, $q(x_1, y_1, x_2, y_2, \cdots)$, $r(x_1, y_1, x_2, y_2, \cdots)$ be (non-commutative) polynomials of the symbolic variables x_i, y_i corresponding to X_i, $i = 1, 2, \cdots$.*

Then we have

(i) *$[p(X_1, X_1^*, X_2, X_2^*, \cdots)]$ can be formed, it is linear, closed, has an everywhere dense domain, and it is $\eta \mathbf{M}$ too.*

(ii) *If ${}^{(r)}p(x_1, y_1, x_2, y_2, \cdots) = {}^{(r)}q(x_1, y_1, x_2, y_2, \cdots)$ then $[p(X_1, X_1^*, X_2, X_2^*, \cdots)] = [q(X_1, X_1^*, X_2, X_2^*, \cdots)]$; that is $[p(X_1, X_1^*, X_2, X_2^*, \cdots)]$ depends on ${}^{(r)}p(x_1, y_1, x_2, y_2, \cdots)$ only.*

(iii) *If $p(x_1, y_1, x_2, y_2, \cdots)^+ = q(x_1, y_1, x_2, y_2, \cdots)$ then $[p(X_1, X_1^*, X_2, X_2^*, \cdots)]^* = [q(X_1, X_1^*, X_2, X_2^*, \cdots)]$.*

(iv) *If $ap(x_1, y_1, x_2, y_2, \cdots) = q(x_1, y_1, x_2, y_2, \cdots)$ then $[a[p(X_1, X_1^*, X_2, X_2^*, \cdots)]] = [q(X_1, X_1^*, X_2, X_2^*, \cdots)]$. (If $a \neq 0$ then the first [ ] is obviously unnecessary).*

(v) *If $p(x_1, y_1, x_2, y_2, \cdots) + q(x_1, y_1, x_2, y_2, \cdots) = r(x_1, y_1, x_2, y_2, \cdots)$ then $[[p(X_1, X_1^*, X_2, X_2^*, \cdots)] + [q(X_1, X_1^*, X_2, X_2^*, \cdots)]] = [r(X_1, X_1^*, X_2, X_2^*, \cdots)]$.*

(vi) *If $p(x_1, y_1, x_2, y_2, \cdots) \cdot q(x_1, y_1, x_2, y_2, \cdots) = r(x_1, y_1, x_2, y_2, \cdots)$ then $[[p(X_1, X_1^*, X_2, X_2^*, \cdots)] \cdot [q(X_1, X_1^*, X_2, X_2^*, \cdots)]] = [r(X_1, X_1^*, X_2, X_2^*, \cdots)]$.*

Proof: Ad (i): $p(X_1, X_1^*, \cdots)$ and $p(X_1, X_1^*, \cdots)^+$ have everywhere dense domains by Lemma 16.4.3. One verifies immediately that $(p(X_1, X_1^*, \cdots)f, g) = (f, p(X_1, X_1^*, \cdots)^+ g)$ for all $f \in$ Domain $p(X_1, X_1^*, \cdots)$, $g \in$ Domain $p(X_1, X_1^*, \cdots)^+$ and so $(p(X_1, X_1^*, \cdots))^*$ is an extension of $p(X_1, X_1^*, \cdots)^+$. Therefore $(p(X_1, X_1^*, \cdots))^*$ has an everywhere dense domain, and so we can form $[p(X_1, X_1^*, \cdots)]$ (cf. (21), p. 300). $[p(X_1, X_1^*, \cdots)]$ is invariant under every unitary transformation $U' \in \mathbf{M}'$ along with $X_1, X_2, \cdots$ so it is $\eta \mathbf{M}$.

Ad (ii): ${}^{(r)}p(X_1, X_1^*, X_2, X_2^*, \cdots)$ is obviously an extension of $p(X_1, X_1^*, X_2, X_2^*, \cdots)$ therefore $[{}^{(r)}p(X_1, X_1^*, \cdots)]$ is one of $[p(X_1, X_1^*, \cdots)]$. By (i) both operators are linear, closed, have everywhere dense domains, and are $\eta \mathbf{M}$. So by Lemma 16.4.2, $[{}^{(r)}p(X_1, X_1^*, \cdots)] = [p(X_1, X_1^*, X_2, X_2^*, \cdots)]$. Applying this to both $p(x_1, y_1, x_2, y_2, \cdots)$ and $q(x_1, y_1, x_2, y_2, \cdots)$ gives the statement of (ii).

Ad (iii): As we saw in the proof of (i), $(p(X_1, X_1^*, \cdots))^*$ is an extension of $p(X_1, X_1^*, \cdots)^+$, that is of $q(X_1, X_1^*, \cdots)$. The first operator is obviously equal to $[p(X_1, X_1^*, \cdots)]^*$ and as it is linear, closed, it must even be an extension of $[q(X_1, X_1^*, \cdots)]$. As these are both linear, closed operators with everywhere dense domains, Lemma 16.4.2 applies again: They must be equal.

Ad (iv)–(vi): All these statements are proved in the same way, therefore it suffices to consider one of them, say (vi):

Clearly $[[p(X_1, X_1^*, X_2, X_2^*, \cdots)] \cdot [q(X_1, X_1^*, X_2, X_2^*, \cdots)]]$ is an extension

of $[p(X_1, X_1^*, X_2, X_2^*, \cdots)]\cdot[q(X_1, X_1^*, X_2, X_2^*, \cdots)]$ and this again one of $p(X_1, X_1^*, X_2, X_2^*, \cdots)\cdot q(X_1, X_1^*, X_2, X_2^*, \cdots) = r(X_1, X_1^*, X_2, X_2^*, \cdots)$. The first mentioned operator is linear closed and therefore it is an extension of $[r(X_1, X_1^*, X_2, X_2^*, \cdots)]$, too. Now we can infer

$$[[p(X_1, X_1^*, X_2, X_2^*, \cdots)]\cdot[q(X_1, X_1^*, X_2, X_2^*, \cdots)]] = [r(X_1, X_1^*, X_2, X_2^*, \cdots)]$$

from Lemma 16.4.2, completing the proof.

Of course (iv) for $a = 0$ is trivial, even without the first [].

We restate the results of this §.

Theorem XV. *Let* $\mathbf{M}$ *be a factor of a finite class* ((I_n), $n = 1, 2, \cdots$ *or* (II_1)). *Denote by* $U(\mathbf{M})$ *the set of all linear, closed operators with an everywhere dense domain, which are* $\eta\,\mathbf{M}$. *Then for* $X, Y \in U(\mathbf{M})$ *the operators* $[aX]$ (*coinciding with* aX *for* $a \neq 0$), $[X + Y]$, $[XY]$ *can be formed, and are* $\in U(\mathbf{M})$ *again. These operations satisfy all customary laws of matrix algebra:*

$$[X + Y] = [Y + X];\ [[X + Y] + Z] = [X + [Y + Z]].$$

$$[a[X + Y]] = [a[X] + b[Y]],\ [(a + b)X] = [aX + bX].$$

$$[[XY]Z] = [X[YZ]],\ [[aX]Y] = [a[XY]],\ [a[bX]] = [(ab)X].$$

$$[[X + Y]Z] = [[XZ] + [YZ]];\ [X[Y + Z]] = [[XY] + [XZ]].$$

$$[aX]^* = [\bar{a}X^*];\ [X + Y]^* = [X^* + Y^*];\ [XY]^* = [Y^*X^*].$$

More generally: The (non-commutative) polynomials $[p(X_1, X_1^*, X_2, X_2^*, \cdots)]$ (*cf. Definitions* 16.3.1, *and* 16.3.2) *are* $\in U(\mathbf{M})$ *again;* $[p(X_1, X_1^*, X_2, X_2^*, \cdots)]$ *depends only on the reduced form* ${}^{(r)}p(x_1, y_1, x_2, y_2, \cdots)$ *of* $p(x_1, y_1, x_2, y_2, \cdots)$ *and the laws of matrix algebra hold for the* $[p(X_1, X_1^*, X_2, X_2^*, \cdots)]$ (*cf. Lemma* 16.4.4, (iii)–(vi)).

Every Hermitian $X \in U(\mathbf{M})$ *is maximal and self adjoint, and thus it has a unique resolution of unity* (*cf.* (15), *p.* 92).

For $X, Y, \in U(\mathbf{M})$ *whenever* Y *is an extension of* X (*cf.* (15), *p.* 70), *then* $X = Y$; *that is: proper extensions do not exist in* $U(\mathbf{M})$.

(Added in proof, Jan. 29, 1936.) The authors have since succeeded in establishing the following further facts concerning the finite cases:

(i) $T_{\mathbf{M}}(A + B) = T_{\mathbf{M}}(A) + T_{\mathbf{M}}(B)$ unrestrictedly. (Cf. Problem 11.)

(ii) $T_{\mathbf{M}}(A)$ is weakly continuous in A. (Cf. Lemma 15.5.5.)

(iii) If $C = 1$ then an f exists, for which identically $T_{\mathbf{M}}(A) = (Af, f)$.

(iv) In the same case certain isomorphisms between $\mathbf{M}$, $\mathbf{M}'$ and $\mathfrak{H}$ exist.

(v) Several examples of Case II, thus (α) – (γ) on page 208, are spacially isomorphic. (This answers the question at the end of §13.4.) The proofs will appear in a subsequent paper.

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ÜBER DIE ANALYTISCHE THEORIE DER QUADRATISCHEN FORMEN II

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Der im ersten Teile dieser Abhandlung bewiesene Hauptsatz lieferte einen Zusammenhang zwischen der Anzahl der ganzzahligen Darstellungen $\mathfrak{C}'\mathfrak{S}\mathfrak{C} = \mathfrak{T}$ einer definiten quadratischen Form $\mathfrak{y}'\mathfrak{T}\mathfrak{y}$ durch eine andere definite quadratische Form $\mathfrak{x}'\mathfrak{S}\mathfrak{x}$ und der Anzahl der Lösungen der entsprechenden Congruenz $\mathfrak{C}'\mathfrak{S}\mathfrak{C} \equiv \mathfrak{T} \pmod{q}$. Im vorliegenden zweiten Teil soll die Übertragung des Hauptsatzes auf indefinite quadratische Formen im rationalen Zahlkörper behandelt werden.

Zunächst sei an die Begriffe von Äquivalenz und Verwandtschaft erinnert. Zwei symmetrische Matrizen $\mathfrak{S}_1$ und $\mathfrak{S}_2$ mit rationalen Elementen heissen äquivalent, wenn die Gleichungen $\mathfrak{X}'\mathfrak{S}_1\mathfrak{X} = \mathfrak{S}_2$ und $\mathfrak{Y}'\mathfrak{S}_2\mathfrak{Y} = \mathfrak{S}_1$ ganzzahlig lösbar sind; sie heissen äquivalent mod. q, wenn die entsprechenden Congruenzen modulo q ganzzahlig lösbar sind; sie heissen reell äquivalent, wenn jene Gleichungen reell lösbar sind. Die beiden Matrizen $\mathfrak{S}_1$ und $\mathfrak{S}_2$ sind verwandt, wenn sie zugleich reell äquivalent und nach jedem Modul äquivalent sind. Alle miteinander äquivalenten Matrizen bilden eine Classe, alle verwandten ein Geschlecht. Jedes Geschlecht setzt sich aus vollen Classen zusammen und zwar, wie sich zeigen lässt, aus endlich vielen Classen.

Ist $\mathfrak{S}^{(m)}$ positiv, d.h. $\mathfrak{x}'\mathfrak{S}\mathfrak{x} > 0$ für jedes reelle $\mathfrak{x} \neq \mathfrak{n}$, so ist für jedes $\mathfrak{T}^{(n)}$ die Anzahl der ganzzahligen Lösungen von $\mathfrak{C}'\mathfrak{S}\mathfrak{C} = \mathfrak{T}$ endlich. Sie sei $A(\mathfrak{S}, \mathfrak{T})$, und man setze noch $A(\mathfrak{S}, \mathfrak{S}) = E(\mathfrak{S})$. Es seien $\mathfrak{S}_1, \cdots, \mathfrak{S}_h$ Repräsentanten der verschiedenen Classen des Geschlechtes von $\mathfrak{S}$. Der Hauptsatz bringt nun den Quotienten

$$\frac{A(\mathfrak{S}_1, \mathfrak{T})}{E(\mathfrak{S}_1)} + \cdots + \frac{A(\mathfrak{S}_h, \mathfrak{T})}{E(\mathfrak{S}_h)} : \frac{1}{E(\mathfrak{S}_1)} + \cdots + \frac{1}{E(\mathfrak{S}_h)} = A_0(\mathfrak{S}, \mathfrak{T})$$

in Zusammenhang mit der Lösungsanzahl $A_q(\mathfrak{S}, \mathfrak{T})$ der Congruenz $\mathfrak{C}'\mathfrak{S}\mathfrak{C} \equiv \mathfrak{T} \pmod{q}$. Man definiere noch $A_\infty(\mathfrak{S}, \mathfrak{T})$ als den Grenzwert des Ausdruckes $v(B') : v(B)$, wo $v(B)$ das Volumen eines $\mathfrak{T}$ enthaltenden Gebietes B im Raume der positiven n-reihigen symmetrischen Matrizen und $v(B')$ das Volumen des B vermöge der Gleichung $\mathfrak{X}'\mathfrak{S}\mathfrak{X} = \mathfrak{T}$ entsprechenden Gebietes B' im Raume der reellen Matrizen $\mathfrak{X}$ von m Zeilen und n Spalten bedeutet, falls B auf den Punkt $\mathfrak{T}$ zusammenschrumpft. Man setze $\eta = \frac{1}{2}$ für $m = n > 1$ und für $m = n + 1$, $\eta = 1$ in jedem andern Fall. Durchläuft dann q die Folge $2!, 3!, \cdots$, so ist

$$\frac{A_0(\mathfrak{S}, \mathfrak{T})}{A_\infty(\mathfrak{S}, \mathfrak{T})} = \eta \lim_{q \to \infty} \frac{A_q(\mathfrak{S}, \mathfrak{T})}{q^{mn - \frac{n(n+1)}{2}}}; \tag{1}$$

dabei ist für den Fall $m = n$ im Nenner auf der rechten Seite noch der Factor $2^{\omega(q)}$ hinzuzufügen, wo $\omega(q)$ die Anzahl der verschiedenen Primteiler von q bedeutet.

Im ersten Teil wurde nun gezeigt, dass der Grenzwert auf der rechten Seite von (1) für beliebige rationale $\mathfrak{S}^{(m)}$, $\mathfrak{T}^{(n)}$ mit $S = |\mathfrak{S}| \neq 0$, $T = |\mathfrak{T}| \neq 0$, $m \geqq n$ existiert, wenn nur nicht entweder $m = 2$ und $-S$ ein Quadrat oder $m - n = 2$ und $-ST$ ein Quadrat ist. Mit Ausnahme dieser beiden Fälle hat also die rechte Seite von (1) für beliebiges indefinites $\mathfrak{S}$ ebenfalls einen Sinn. Die linke Seite von (1) wird dagegen sinnlos für indefinites $\mathfrak{S}$, da das zur Definition von $A_\infty(\mathfrak{S}, \mathfrak{T})$ dienende Gebiet B' kein endliches Volumen besitzt und da ferner auch $E(\mathfrak{S})$ unendlich ist, falls nicht $m = 2$ und $-S$ ein Quadrat ist. Man kann nun aber den Ausdruck auf der linken Seite von (1) für positives $\mathfrak{S}$ so umformen, dass der abgeänderte Ausdruck auch für indefinites $\mathfrak{S}$ einen Sinn hat.

Man betrachte ein $\mathfrak{S}$ enthaltendes Gebiet B im Raum der symmetrischen m-reihigen Matrizen, so dass alle $\mathfrak{S}_0$ aus B mit $\mathfrak{S}$ reell äquivalent sind und $|\mathfrak{S}_0| \neq 0$ ist. Vermöge der Gleichung $\mathfrak{X}'\mathfrak{S}\mathfrak{X} = \mathfrak{S}_0$ entspricht dem $\mathfrak{S}_0$-Gebiete B ein Gebiet B_1 im m^2-dimensionalen $\mathfrak{X}$-Raume. Ist $\mathfrak{A}$ eine Einheit von $\mathfrak{S}$, also $\mathfrak{A}'\mathfrak{S}\mathfrak{A} = \mathfrak{S}$ und $\mathfrak{A}$ ganz, so gehört mit $\mathfrak{X}$ auch $\mathfrak{A}\mathfrak{X}$ zu B_1. Es gibt nun in B_1 einen Fundamentalbereich $\bar{B}$ gegenüber der Einheitengruppe, also einen Bereich von der Art, dass für jedes $\mathfrak{X}$ aus B_1 genau eine Einheit $\mathfrak{A}$ von $\mathfrak{S}$ mit $\mathfrak{A}\mathfrak{X}$ in $\bar{B}$ existiert. Man kann beweisen, dass bei geeigneter Festlegung von $\bar{B}$ dieser Bereich ein Volumen $v(\bar{B})$ besitzt, wenn B ein Volumen $v(B)$ hat; und zwar ist das Volumen $v(\bar{B})$ endlich, wenn nicht $m = 2$ und $-S$ ein Quadrat ist. Lässt man B auf den Punkt $\mathfrak{S}$ zusammenschrumpfen, so existiert ferner

$$\lim_{B \to \mathfrak{S}} \frac{v(\bar{B})}{v(B)} = \rho(\mathfrak{S}) .$$

Man setze

$$\rho(\mathfrak{S}_1) + \cdots + \rho(\mathfrak{S}_h) = \mu(\mathfrak{S}) .$$

Nun sei $n < m$ und $\mathfrak{C}'\mathfrak{S}\mathfrak{C} = \mathfrak{T}$ eine Darstellung von $\mathfrak{T}$ durch $\mathfrak{S}$. Es sei $\mathfrak{Q}$ eine reelle Matrix von n Zeilen und $m - n$ Spalten, $\mathfrak{R}$ eine reelle symmetrische Matrix von $m - n$ Reihen. Es mögen $\mathfrak{Q}$, $\mathfrak{R}$ in einem Gebiete G von $\frac{1}{2}(m - n)\cdot(m + n + 1)$ Dimensionen variieren, so dass die Matrix $\begin{pmatrix} \mathfrak{T} & \mathfrak{Q} \\ \mathfrak{Q}' & \mathfrak{R} \end{pmatrix} = \mathfrak{Z}$ mit $\mathfrak{S}$ reell äquivalent ist und eine von 0 verschiedene Determinante hat; ferner sei $|\mathfrak{Z}| = S$ für einen speciellen Punkt $\mathfrak{Q} = \mathfrak{Q}_0$, $\mathfrak{R} = \mathfrak{R}_0$ von G. Vermöge der Gleichung $(\mathfrak{C}\mathfrak{Y})'\,\mathfrak{S}\,(\mathfrak{C}\mathfrak{Y}) = \mathfrak{Z}$, also $\mathfrak{C}'\mathfrak{S}\mathfrak{Y} = \mathfrak{Q}$ und $\mathfrak{Y}'\mathfrak{S}\mathfrak{Y} = \mathfrak{R}$, entspricht dem Gebiete G ein $m(m-n)$-dimensionales $\mathfrak{Y}$-Gebiet G_1. Ist dann $\mathfrak{A}$ eine $\mathfrak{C}$ invariant lassende Einheit von $\mathfrak{S}$, also $\mathfrak{A}'\mathfrak{S}\mathfrak{A} = \mathfrak{S}$ und $\mathfrak{A}\mathfrak{C} = \mathfrak{C}$, so gehört mit $\mathfrak{Y}$ auch $\mathfrak{A}\mathfrak{Y}$ zu G_1. Es gibt in G_1 wieder einen Fundamentalbereich $\bar{G}$ gegenüber der $\mathfrak{C}$ invariant lassenden Einheitenuntergruppe, und zwar lässt sich zeigen, dass bei geeigneter Definition der Berandung von $\bar{G}$ mit dem Volumen $v(G)$ von G stets

auch zugleich das Volumen $v(\bar{G})$ von $\bar{G}$ vorhanden ist; die Zahl $v(\bar{G})$ ist endlich, wenn nicht $m - n = 2$ und $-ST$ ein Quadrat ist. Lässt man G auf den Punkt $\mathfrak{Q}_0$, $\mathfrak{R}_0$ zusammenschrumpfen, so existiert

$$\lim_{G \to \mathfrak{Q}_0, \mathfrak{R}_0} \frac{v(\bar{G})}{v(G)} = \rho(\mathfrak{S}, \mathfrak{C}) .$$

Für $n = m$ definiere man noch $\rho(\mathfrak{S}, \mathfrak{C}) = 1$.

Man lasse nun $\mathfrak{C}$ ein volles System solcher ganzzahligen Lösungen von $\mathfrak{C}' \mathfrak{S} \mathfrak{C} = \mathfrak{T}$ durchlaufen, dass keine zwei von ihnen durch linksseitige Multiplication mit einer Einheit auseinander hervorgehen, und setze

$$\alpha(\mathfrak{S}, \mathfrak{T}) = \sum_{\mathfrak{C}} \rho(\mathfrak{S}, \mathfrak{C}) .$$

Endlich sei noch

$$\alpha(\mathfrak{S}_1, \mathfrak{T}) + \cdots + \alpha(\mathfrak{S}_h, \mathfrak{T}) = \mu(\mathfrak{S}, \mathfrak{T}) .$$

Durch eine einfache Rechnung lässt sich nun zeigen, dass für positive $\mathfrak{S}$ die Formel

$$\frac{\mu(\mathfrak{S}, \mathfrak{T})}{\mu(\mathfrak{S})} = \frac{A_0(\mathfrak{S}, \mathfrak{T})}{A_\infty(\mathfrak{S}, \mathfrak{T})}$$

gilt. Also lässt sich der Hauptsatz des ersten Teils in der Gestalt

$$\frac{\mu(\mathfrak{S}, \mathfrak{T})}{\mu(\mathfrak{S})} = \eta \lim_{q \to \infty} \frac{A_q(\mathfrak{S}, \mathfrak{T})}{q^{mn - \frac{n(n+1)}{2}}} \tag{2}$$

schreiben, wo im Falle $m = n$ im Nenner der rechten Seite noch der Factor $2^{\omega(q)}$ hinzuzufügen ist und $\eta = \frac{1}{2}$ für $m = n > 1$ oder $m = n + 1$, $\eta = 1$ in jedem andern Fall zu setzen ist. Da nun die linke Seite von (2) auch für indefinite $\mathfrak{S}$ einen Sinn hat, falls nicht $m = 2$ und $-S$ ein Quadrat oder $m - n = 2$ und $-ST$ ein Quadrat ist, so wird man vermuten, dass die Formel (2) auch für indefinite $\mathfrak{S}$ richtig ist. Der Beweis dieser Vermutung wird im Folgenden geführt werden. Er erfolgt nach den Methoden des ersten Teils, erfordert aber eine sorgfältige Vorbereitung, da die Existenz der Volumina $v(\bar{B})$ und $v(\bar{G})$ sowie der Grenzwerte $\rho(\mathfrak{S})$ und $\rho(\mathfrak{S}, \mathfrak{C})$ für indefinite $\mathfrak{S}$ keineswegs so evident ist wie bei positivem $\mathfrak{S}$.

Im Falle $\mathfrak{S} = \mathfrak{T}$ ist $\mu(\mathfrak{S}, \mathfrak{T}) = 1$ und die Formel (2) liefert

$$\mu(\mathfrak{S}) = 2 \lim_{q \to \infty} \frac{2^{\omega(q)} q^{\frac{m(m-1)}{2}}}{E_q(\mathfrak{S})} \qquad (m > 1). \tag{3}$$

Für positives $\mathfrak{S}$ ist dies im wesentlichen Minkowskis Formel für das Mass des Geschlechtes von $\mathfrak{S}$, also die Verallgemeinerung der Dirichletschen Classenzahlformel bei definiten binären quadratischen Formen. Minkowski hatte eine Übertragung seiner Formel auf den indefiniten Fall in Aussicht gestellt, aber

offenbar niemals ausgeführt. Für $m = 3$ ist dies durch die letzten Untersuchungen von G. Humbert geschehen, der sein Ergebnis allerdings nicht in der transcendenten Form (3) schreibt, sondern nach Analogie der im definiten Fall $m = 3$ von Eisenstein angegebenen und von Smith bewiesenen Ausdrücke.

Der Hauptsatz gestattet auch im indefiniten Fall eine functionentheoretische Formulierung, analog zu den Entwicklungen des ersten Teiles. Es ist beachtenswert, dass auch die indefiniten ganzzahligen quadratischen Formen in Zusammenhang mit der Theorie der Modulfunctionen und der Thetanullwerte stehen. Dies ist für specielle Formen schon von Hecke bemerkt worden.

Erstes Capitel: Hilfssätze aus der Theorie der indefiniten quadratischen Formen

1. Reductionstheorie

Im Folgenden wird unter einer ganzen Matrix stets eine Matrix mit ganzen rationalen Elementen verstanden. Eine ganze quadratische Matrix, deren Determinante ± 1 ist, heisst unimodular. Zwei quadratische Matrizen $\mathfrak{A}$ und $\mathfrak{B}$ derselben Reihenzahl heissen rechtsseitig associiert, wenn $\mathfrak{A}^{-1}\mathfrak{B}$ unimodular ist.

Hilfssatz 1: *Es seien d und m natürliche Zahlen. Es gibt eine endliche Anzahl von m-reihigen quadratischen Matrizen, so dass jede ganze m-reihige quadratische Matrix der Determinante $\pm d$ mit einer von ihnen rechtsseitig associiert ist.*

Beweis: Die ganzen Matrizen $\mathfrak{A}^{(m)}$ zerfallen in d^{m^2} Restclassen modulo d. Ist nun $\mathfrak{A}_1$ ganz, $\mathfrak{A}_2$ ganz, $|\mathfrak{A}_1| = \pm d$, $|\mathfrak{A}_2| = \pm d$, $\mathfrak{A}_1 \equiv \mathfrak{A}_2 \pmod{d}$, so ist $d\mathfrak{A}_1^{-1}$ ganz, $d\mathfrak{A}_1^{-1}\mathfrak{A}_2$ ganz, $d\mathfrak{A}_1^{-1}\mathfrak{A}_2 \equiv d\mathfrak{E} \pmod{d}$, also $\mathfrak{A}_1^{-1}\mathfrak{A}_2$ ganz und ebenfalls $\mathfrak{A}_2^{-1}\mathfrak{A}_1$. Daher sind $\mathfrak{A}_1$ und $\mathfrak{A}_2$ rechtsseitig associiert. Folglich existieren höchstens d^{m^2} ganze m-reihige quadratische Matrizen der Determinante $\pm d$, von denen keine zwei rechtsseitig associiert sind.

Über die Reductionstheorie der definiten quadratischen Formen gilt der auf Hermite zurückgehende

Hilfssatz 2: *Zu jeder reellen, symmetrischen, positiv-definiten Matrix $\mathfrak{S}^{(m)}$ gibt es eine unimodulare Matrix $\mathfrak{U}$, eine Diagonalmatrix $\mathfrak{D}$, deren Diagonalelemente $\tau_1, \cdots, \tau_m$ positiv sind und den Ungleichungen*

$$\tau_k \leqq \tfrac{4}{3}\,\tau_{k+1} \qquad (k = 1, \cdots, m-1) \tag{4}$$

genügen, und eine Dreiecksmatrix $\mathfrak{C}^{(m)} = (c_{kl})$ mit $c_{kl} = 0$ $(k > l)$, $c_{kk} = 1$, $-\tfrac{1}{2} \leqq c_{kl} < \tfrac{1}{2}$ $(k < l)$, so dass

$$\mathfrak{U}'\mathfrak{S}\mathfrak{U} = \mathfrak{C}'\mathfrak{D}\mathfrak{C}$$

ist.

Beweis: Das Minimum der quadratischen Form $\mathfrak{x}'\mathfrak{S}\mathfrak{x}$ in den vom Nullpunkt verschiedenen Gitterpunkten werde für $\mathfrak{x} = \mathfrak{x}_0$ angenommen. Da die Elemente von $\mathfrak{x}_0$ teilerfremd sind, so existiert eine unimodulare Matrix $\mathfrak{U}_1$, deren erste

Spalte $\mathfrak{x}_0$ ist. Es sei $\mathfrak{x}_0'\mathfrak{S}\mathfrak{x}_0 = \tau_1$ und $\tau_1 a_l$ $(l = 2, \cdots, m)$ das l^{te} Element der ersten Zeile von $\mathfrak{U}_1'\mathfrak{S}\mathfrak{U}_1$. Dann ist

$$\mathfrak{x}'\mathfrak{U}_1'\mathfrak{S}\mathfrak{U}_1\mathfrak{x} - \tau_1 (x_1 + a_2x_2 + \cdots + a_mx_m)^2$$

eine positiv-definite quadratische Form der $m - 1$ Variabeln $x_2, \cdots, x_m$. Für diese habe τ_2 dieselbe Bedeutung wie τ_1 für $\mathfrak{x}'\mathfrak{S}\mathfrak{x}$. Wird τ_2 für

$$x_k = \xi_k \qquad (k = 2, \cdots, m)$$

angenommen, so bestimme man eine ganze Zahl $x_1 = \xi_1$ derart dass $-\frac{1}{2} \leqq \xi_1 + a_2\xi_2 + \cdots + a_m\xi_m < \frac{1}{2}$ und erhält als Wert von $\mathfrak{x}'\mathfrak{U}_1'\mathfrak{S}\mathfrak{U}_1\mathfrak{x}$ eine Zahl $\leqq \tau_2 + \frac{1}{4}\tau_1$, die nicht kleiner als das Minimum τ_1 sein kann. Also ist $\tau_1 \leqq \frac{4}{3}\tau_2$. Hieraus folgt durch vollständige Induction die Existenz einer unimodularen Matrix $\mathfrak{U}_2$, einer Diagonalmatrix $\mathfrak{D}$ mit positiven Diagonalelementen $\tau_1, \cdots, \tau_m$, welche den Ungleichungen (4) genügen, und einer Dreiecksmatrix $\mathfrak{B}^{(m)} = (b_{kl})$ mit $b_{kl} = 0$ $(k > l)$, $b_{kk} = 1$, so dass

$$\mathfrak{U}_2'\mathfrak{S}\mathfrak{U}_2 = \mathfrak{B}'\mathfrak{D}\mathfrak{B} \tag{5}$$

ist. Man kann eine unimodulare Dreiecksmatrix $\mathfrak{U}_3 = (u_{kl})$ mit

$$u_{kl} = 0 \ (k > l), \qquad u_{kk} = 1$$

eindeutig so bestimmen, dass die Elemente

$$c_{kl} = u_{kl} + \sum_{r=k+1}^{l} b_{kr}\, u_{rl} \qquad (k < l)$$

der Dreiecksmatrix $\mathfrak{B}\mathfrak{U}_3 = \mathfrak{C}$ den Ungleichungen $-\frac{1}{2} \leqq c_{kl} < \frac{1}{2}$ $(k < l)$ genügen. Setzt man noch $\mathfrak{U}_2\mathfrak{U}_3 = \mathfrak{U}$, so lehrt (5), dass $\mathfrak{U}$ und $\mathfrak{C}$ das Verlangte leisten.

Die Reductionstheorie der indefiniten quadratischen Formen ist bisher nur im binären und ternären Fall in befriedigender Weise behandelt worden. Für die weiterhin zu machende Anwendung der Reductionstheorie im allgemeinen Fall genügt ein Teilresultat, das gewisse von Stouff bewiesene Ungleichungen enthält, nämlich

HILFSSATZ 3: *Es sei $\mathfrak{S}^{(m)}$ symmetrisch und ganz, $|\mathfrak{S}| = S \neq 0$, $\mathfrak{X}^{(m)}$ reell, $|\mathfrak{X}| \neq 0$ und H eine obere Schranke der absoluten Beträge der Elemente von $\mathfrak{X}'\mathfrak{S}\mathfrak{X}$ und $(\mathfrak{X}'\mathfrak{S}\mathfrak{X})^{-1}$. Es gibt eine ganze Matrix $\mathfrak{Q}^{(m)}$, eine ganze Zahl g des Intervalles $0 \leqq g \leqq \frac{m}{2}$, ferner (im Falle $g > 0$) g Zahlen $\sigma_1, \cdots, \sigma_g$ und (im Falle $g < \frac{m}{2}$) eine ganze symmetrische Matrix $\mathfrak{H}^{(m-2g)}$, ausserdem eine nur vom m, S, H abhängige Zahl M, mit folgenden Eigenschaften:*

1) *Bedeutet das Zeichen $\mathfrak{N}$ eine Nullmatrix und $\mathfrak{E}$ die g-reihige Einheitsmatrix, so ist*

$$4\mathfrak{S} = \mathfrak{Q}'\begin{pmatrix} \mathfrak{N} & \mathfrak{E} & \mathfrak{N} \\ \mathfrak{E} & \mathfrak{N} & \mathfrak{N} \\ \mathfrak{N} & \mathfrak{N} & \mathfrak{H} \end{pmatrix}\mathfrak{Q}\,; \tag{6}$$

2) *es ist* $\sigma_1 \geqq \sigma_2 \geqq \cdots \geqq \sigma_g > 1$;

3) *alle Elemente von* $\mathfrak{H}$ *sind absolut kleiner als* M;

4) *setzt man noch* $\sigma_{g+k} = \sigma_k^{-1}$ $(k = 1, \cdots, g)$, $\sigma_{2g+k} = 1$ $(k = 1, \cdots, m - 2g)$, *so ist jedes Element der* k^{ten} *Zeile von* $\mathfrak{Q}\mathfrak{X}$ *absolut kleiner als* $M\sigma_k$ $(k = 1, \cdots, m)$.

Beweis: Man setze $\mathfrak{X}'\mathfrak{S}\mathfrak{X} = \mathfrak{T}$, $\mathfrak{X}^{-1} = \mathfrak{Y}$, also $\mathfrak{Y}'\mathfrak{T}\mathfrak{Y} = \mathfrak{S}$. Die Wurzeln $\lambda_1, \cdots, \lambda_m$ der Gleichung $|\lambda\mathfrak{E} - \mathfrak{T}| = 0$ liegen zwischen zwei nur von m und H abhängigen Schranken. Man wähle λ gleich dem Doppelten des Maximums der absoluten Beträge von $\lambda_1, \cdots, \lambda_m$; dann gilt

$$|\lambda_k| \leqq \lambda - \lambda_k, \qquad \frac{\lambda}{2} \leqq \lambda - \lambda_k \qquad (k = 1, \cdots, m), \tag{7}$$

ferner ist $\lambda\mathfrak{E} - \mathfrak{T}$ positiv-definit, also auch $\mathfrak{Y}'(\lambda\mathfrak{E} - \mathfrak{T})\mathfrak{Y} = \lambda\mathfrak{Y}'\mathfrak{Y} - \mathfrak{S}$. Nach Hilfssatz 2 gibt es eine Diagonalmatrix $\mathfrak{D}^{(m)}$ mit den positiven Diagonalelementen $\tau_1, \cdots, \tau_m$, die den Ungleichungen (4) genügen, eine Driecksmatrix $\mathfrak{C}^{(m)} = (c_{kl})$ mit $c_{kl} = 0$ $(k > l)$, $c_{kk} = 1$, $-\frac{1}{2} \leqq c_{kl} < \frac{1}{2}$ $(k < l)$ und eine unimodulare Matrix $\mathfrak{U}$, so dass

$$\mathfrak{U}'(\lambda\mathfrak{Y}'\mathfrak{Y} - \mathfrak{S})\mathfrak{U} = \mathfrak{C}'\mathfrak{D}\mathfrak{C}$$

wird. Setzt man noch

$$\mathfrak{U}'\mathfrak{S}\mathfrak{U} = \mathfrak{R} = (r_{kl}), \qquad \mathfrak{Y}\mathfrak{U} = \mathfrak{Z} = (z_{kl}), \tag{8}$$

so folgen aus (7) und der Schwarzschen Ungleichung die Formeln

$$\begin{gathered}(\mathfrak{x}'\mathfrak{R}\mathfrak{y})^2 \leqq (\mathfrak{C}\mathfrak{x})'\mathfrak{D}\mathfrak{C}\mathfrak{x}(\mathfrak{C}\mathfrak{y})'\mathfrak{D}\mathfrak{C}\mathfrak{y}, \\ \frac{\lambda}{2}(\mathfrak{Z}\mathfrak{x})'(\mathfrak{Z}\mathfrak{x}) \leqq (\mathfrak{C}\mathfrak{x})'\mathfrak{D}(\mathfrak{C}\mathfrak{x}).\end{gathered} \tag{9}$$

Zur Abkürzung sei

$$(\tfrac{4}{3})^{k-1}\tau_k = \rho_k^2, \qquad \rho_k > 0 \qquad (k = 1, \cdots, m), \tag{10}$$

so dass nach (4) die Ungleichung

$$0 < \rho_1 \leqq \rho_2 \leqq \cdots \leqq \rho_m \tag{11}$$

richtig ist. Da das l^{te} Diagonalelement von $\mathfrak{C}'\mathfrak{D}\mathfrak{C}$ den Wert

$$\sum_{k=1}^{l} \tau_k c_{kl}^2 \leqq \tfrac{1}{4}\tau_l\{(\tfrac{4}{3})^{l-1} + \cdots + \tfrac{4}{3}\} + \tau_l = (\tfrac{4}{3})^{l-1}\tau_l = \rho_l^2$$

hat, so liefern (9) und (10) die Abschätzungen

$$r_{kl}^2 \leqq \rho_k^2\rho_l^2 \qquad (k = 1, \cdots, m; l = 1, \cdots, m) \tag{12}$$

$$\frac{\lambda}{2}(z_{1l}^2 + \cdots + z_{ml}^2) \leqq \rho_l^2 \qquad (l = 1, \cdots, m). \tag{13}$$

Es ist nun

$$\tau_1 \cdots \tau_m = |\,\mathfrak{C}'\mathfrak{D}\mathfrak{C}\,| = |\,\mathfrak{U}'\mathfrak{Y}'(\lambda\mathfrak{E} - \mathfrak{T})\mathfrak{Y}\mathfrak{U}\,| \tag{14}$$
$$= |\,\mathfrak{S}\mathfrak{T}^{-1}\,|\,(\lambda - \lambda_1)\cdots(\lambda - \lambda_m) < c_1,$$

wo c_1, wie auch weiterhin $c_2, \cdots, c_g$, natürliche Zahlen bedeuten, die nur von m, S, H abhängen. Aus (10) und (14) folgt

$$\rho_1 \cdots \rho_m < c_2. \tag{15}$$

Da $\mathfrak{R}$ ganz und $|\,\mathfrak{R}\,| = S \neq 0$ ist, so gibt es unter den $m!$ Gliedern der entwickelten Determinante $|\,\mathfrak{R}\,|$ mindestens eins, das von 0 verschieden ist. Daher existiert wegen (12) eine Permutation $l_1, \cdots, l_m$ der Zahlen $1, \cdots, m$, so dass

$$\rho_k \rho_{l_k} \geqq 1 \qquad (k = 1, \cdots, m) \tag{16}$$

ist. Bedeutet a eine Zahl der Reihe $1, \cdots, m$, so ist unter den a Zahlen $l_1, \cdots, l_a$ mindestens eine $\leqq m - a + 1$; ist l_k diese Zahl, so gilt zufolge (11) die Ungleichung

$$\rho_{l_k} \leqq \rho_{m-a+1}$$

und auch

$$\rho_k \leqq \rho_a,$$

also nach (16) erst recht

$$\rho_a \rho_{m-a+1} \geqq 1 \qquad (a = 1, \cdots, m). \tag{17}$$

Hieraus folgt andererseits mit Hilfe von (15)

$$\rho_a \rho_{m-a+1} < c_2, \tag{18}$$

also erst recht

$$\rho_a \rho_b < c_2 \qquad (a + b \leqq m + 1). \tag{19}$$

Sind alle $m - 1$ Quotienten $\rho_{k+1} : \rho_k < c_2$ $(k = 1, \cdots, m - 1)$, so ist nach (15)

$$\rho_m^m < c_2^{1+\frac{m(m-1)}{2}},$$

also ρ_m beschränkt. In diesem Falle folgt aus (8), (11), (12), (13) die Behauptung von Hilfssatz 3 mit $g = 0$ und $\mathfrak{Q} = 2\mathfrak{U}^{-1}$, $\mathfrak{H} = \mathfrak{R}$.

Fortan liege nicht dieser Fall vor. Es sei $k = g$ der grösste Index $\leqq \frac{m}{2}$ mit $\rho_{k+1} : \rho_k \geqq c_2$, also $\rho_{g+1} : \rho_g \geqq c_2$ und $\rho_{k+1} : \rho_k < c_2$ für $g < k \leqq \frac{m}{2}$. Nach (17) und (18) ist dann

$$\rho_{k+1} : \rho_k = \frac{\rho_{k+1}\rho_{m-k}}{\rho_k \rho_{m-k+1}} \cdot \frac{\rho_{m-k+1}}{\rho_{m-k}} < c_2^2 \qquad \left(g < m - k \leqq \frac{m}{2}\right),$$

also

$$\rho_{k+1} : \rho_k < c_2^2 \qquad (g < k < m - g). \tag{20}$$

Ist nun $k \leqq g$, $b \leqq m-g$, so ist nach (11) und (19)

$$\rho_k \rho_b \leqq \rho_g \rho_b \leqq c_2^{-1} \rho_{g+1} \rho_b < 1 \qquad (k \leqq g;\ b \leqq m-g). \tag{21}$$

Ist andererseits $g < k \leqq m-g$, $g < b \leqq m-g$, so ist nach (11), (19) und (20)

$$\rho_k \rho_b \leqq c_2^{2(k-g-1)} \rho_{g+1} \rho_b < c_2^{2k-2g-1} \qquad (g < k \leqq m-g;\ g < b \leqq m-g),$$

also ρ_k beschränkt für $g < k \leqq m-g$ und nach (12) auch r_{kl} beschränkt für $g < k \leqq m-g$, $g < l \leqq m-g$. Ferner liefern (12) und (21) das Verschwinden von r_{kl} für $k \leqq g$, $l \leqq m-g$. Daher ist

$$\mathfrak{R} = \begin{pmatrix} \mathfrak{N} & \mathfrak{N} & \mathfrak{W}' \\ \mathfrak{N} & \mathfrak{H} & \mathfrak{A}' \\ \mathfrak{W} & \mathfrak{A} & \mathfrak{B} \end{pmatrix} \tag{22}$$

mit $\mathfrak{H} = \mathfrak{H}^{(m-2g)}$, $\mathfrak{W} = \mathfrak{W}^{(g)}$, $\mathfrak{A} = \mathfrak{A}^{(g,m-2g)}$, $\mathfrak{B} = \mathfrak{B}^{(g)}$, wo alle Elemente von $\mathfrak{H}$ absolut kleiner als c_3 sind. Im Falle $g = \frac{m}{2}$ fallen $\mathfrak{A}$ und $\mathfrak{H}$ fort.

Es sei $\mathfrak{J}$ die Matrix, die aus $\mathfrak{E}^{(g)}$ durch Umkehrung der Reihenfolge der Zeilen entsteht. Setzt man

$$\mathfrak{K} = \begin{pmatrix} \mathfrak{J}\mathfrak{W} & \mathfrak{J}\mathfrak{A} & \frac{1}{2}\mathfrak{J}\mathfrak{B} \\ \mathfrak{N} & \mathfrak{N} & \mathfrak{J} \\ \mathfrak{N} & \mathfrak{E} & \mathfrak{N} \end{pmatrix},$$

also

$$\mathfrak{K}^{-1} = \begin{pmatrix} \mathfrak{W}^{-1}\mathfrak{J} & -\frac{1}{2}\mathfrak{W}^{-1}\mathfrak{B}\mathfrak{J} & -\mathfrak{W}^{-1}\mathfrak{A} \\ \mathfrak{N} & \mathfrak{N} & \mathfrak{E} \\ \mathfrak{N} & \mathfrak{J} & \mathfrak{N} \end{pmatrix},$$

so wird nach (22)

$$\mathfrak{R} = \mathfrak{K}' \begin{pmatrix} \mathfrak{N} & \mathfrak{E} & \mathfrak{N} \\ \mathfrak{E} & \mathfrak{N} & \mathfrak{N} \\ \mathfrak{N} & \mathfrak{N} & \mathfrak{H} \end{pmatrix} \mathfrak{K}. \tag{23}$$

Es sei $\mathfrak{Z} = (\mathfrak{Z}_1 \mathfrak{Z}_2 \mathfrak{Z}_3)$ mit $\mathfrak{Z}_1 = \mathfrak{Z}_1^{(m,g)}$, $\mathfrak{Z}_2 = \mathfrak{Z}_2^{(m,m-2g)}$, $\mathfrak{Z}_3 = \mathfrak{Z}_3^{(m,g)}$; dann ist

$$\mathfrak{Z}\mathfrak{K}^{-1} = (\mathfrak{Z}_1\mathfrak{W}^{-1}\mathfrak{J},\ -\tfrac{1}{2}\mathfrak{Z}_1\mathfrak{W}^{-1}\mathfrak{B}\mathfrak{J} + \mathfrak{Z}_3\mathfrak{J},\ -\mathfrak{Z}_1\mathfrak{W}^{-1}\mathfrak{A} + \mathfrak{Z}_2). \tag{24}$$

Da $\mathfrak{J}\mathfrak{W}$ die Matrix $(r_{m+1-k,l})$ $(k, l = 1, \cdots, g)$ ist, so wird nach (12) und (18) der absolute Betrag des l^{ten} Elements der k^{ten} Zeile von $\mathfrak{W}^{-1}\mathfrak{J}$

$$\leqq (g-1)!\, \frac{\rho_1 \cdots \rho_g \cdot \rho_{m-g+1} \cdots \rho_m}{\rho_{m+1-l}\, \rho_k} < \frac{c_4}{\rho_k\, \rho_{m+1-l}}.$$

Andererseits ist nach (13)

$$\frac{\lambda}{2} z_{kl}^2 \leqq \rho_l^2 \qquad (k, l = 1, \cdots, m), \tag{25}$$

und folglich nach (17) jedes Element der l^{ten} Spalte von $\mathfrak{Z}_1\mathfrak{W}^{-1}\mathfrak{Z}$ absolut

$$< \sqrt{\frac{2}{\lambda}} \sum_{h=1}^{m} \rho_h \frac{c_4}{\rho_h \rho_{m+1-l}} < c_5 \rho_l \qquad (l = 1, \cdots, g).$$

Ferner ist $\mathfrak{Z}\mathfrak{B}\mathfrak{Z} = (r_{m+1-k,\, m+1-l})$ $(k, l = 1, \cdots, g)$, also nach (12) und (18) jedes Element der l^{ten} Spalte von $\mathfrak{Z}_1\mathfrak{W}^{-1}\mathfrak{B}\mathfrak{Z}$ absolut

$$< c_5 \sum_{h=1}^{g} \rho_k \rho_{m+1-h} \rho_{m+1-l} < c_6 \rho_{m+1-l} \qquad (l = 1, \cdots, g),$$

und jedes Element der l^{ten} Spalte von $\mathfrak{Z}_3\mathfrak{Z}$ nach (25) absolut $< c_7 \rho_{m+1-l}$. Endlich ist $\mathfrak{Z}\mathfrak{A} = (r_{m+1-k,\, g+l})$ $(k = 1, \cdots, g;\ l = 1, \cdots, m - 2g)$, also jedes Element der l^{ten} Spalte von $\mathfrak{Z}_1\mathfrak{W}^{-1}\mathfrak{A}$ absolut

$$< c_5 \sum_{h=1}^{g} \rho_h \rho_{m+1-h} \rho_{g+h} < c_8 \rho_{g+l} \qquad (l = 1, \cdots, m - 2g),$$

und jedes Element der l^{ten} Spalte von $\mathfrak{Z}_2$ absolut $< c_9 \rho_{g+l}$.

Nun sei $\rho_k^{-1} = \sigma_k$ $(k = 1, \cdots, g)$. Dann ist zunächst nach (11) und (21) die Behauptung 2) richtig. Setzt man $2\,\mathfrak{K}\mathfrak{U}^{-1} = \mathfrak{Q}$, so ist nach (8)

$$\mathfrak{Q}\mathfrak{X} = 2(\mathfrak{Z}\mathfrak{K}^{-1})^{-1},$$

und aus dem Ergebnis des vorigen Absatzes folgt mit Rücksicht auf (17), (21) und (24) die Richtigkeit der Behauptung 4). Aus (8), (23) und dem früher über $\mathfrak{H}$ Bewiesenen ergeben sich schliesslich die Aussagen 1) und 3).

HILFSSATZ 4: *Die Classenanzahl der symmetrischen ganzen m-reihigen Matrizen mit fester von* 0 *verschiedener Determinante ist endlich.*

BEWEIS: Es sei $\mathfrak{S}^{(m)}$ eine symmetrische ganze Matrix mit der Determinante $S \neq 0$. Es gibt eine reelle Matrix $\mathfrak{X}^{(m)}$, so dass $\mathfrak{X}'\mathfrak{S}\mathfrak{X}$ eine Diagonalmatrix ist, deren Diagonalelemente sämtlich ± 1 sind. Wendet man Hilfssatz 3 mit $H = 1$ an, so folgt die Existenz zweier ganzen Matrizen $\mathfrak{Q}$ und $\mathfrak{H}$, welche (6) erfüllen, und zwar gibt es dabei für $\mathfrak{H}$ nur eine endliche von m und S abhängige Anzahl von Möglichkeiten. Ferner ist

$$|\,4\,\mathfrak{S}\,| = \pm\,|\,\mathfrak{Q}\,|^2\,|\,\mathfrak{H}\,|,$$

also $|\,\mathfrak{Q}\,|$ ein Factor von $2^m S$. Nach Hilfssatz 1 ist demnach $\mathfrak{Q} = \mathfrak{Q}_1\mathfrak{U}$ mit unimodularem $\mathfrak{U}$, wobei die Anzahl der Möglichkeiten für $\mathfrak{Q}_1$ wieder durch m und S beschränkt ist. Aus (6) folgt jetzt die Behauptung.

Es sei wieder $\mathfrak{S}^{(m)}$ ganz, $|\,\mathfrak{S}\,| \neq 0$. Eine ganze Matrix $\mathfrak{A}$ heisst eine Einheit von $\mathfrak{S}$, wenn $\mathfrak{A}'\mathfrak{S}\mathfrak{A} = \mathfrak{S}$ ist. Zwei Matrizen $\mathfrak{X}^{(mn)}$ und $\mathfrak{Y}^{(mn)}$ mögen associiert in

bezug auf $\mathfrak{S}$ heissen, wenn $\mathfrak{Y} = \mathfrak{A}\mathfrak{X}$ ist mit einer geeigneten Einheit $\mathfrak{A}$ von $\mathfrak{S}$; diese Bezeichnung möge auch für die Punkte im mn-dimensionalen Raum verwendet werden, deren Coordinaten die Elemente von $\mathfrak{X}$ sind.

Man wähle nun eine Matrix $\mathfrak{Z}_0^{(m)}$ mit $|\mathfrak{Z}_0| \neq 0$, für welche die Gleichung $\mathfrak{X}'\mathfrak{S}\mathfrak{X} = \mathfrak{Z}_0$ eine reelle Lösung $\mathfrak{X}$ hat, und betrachte ein zusammenhängendes endliches Gebiet G im $\frac{m(m+1)}{2}$-dimensionalen $\mathfrak{Z}$-Raum, in dem $\mathfrak{Z}_0$ enthalten und $|\mathfrak{Z}| \neq 0$ ist. Vermöge der Gleichung $\mathfrak{X}'\mathfrak{S}\,\mathfrak{X} = \mathfrak{Z}$ wird das Gebiet G auf ein Gebiet G' im m^2-dimensionalen $\mathfrak{X}$-Raum abgebildet. Offenbar gehört mit $\mathfrak{X}$ auch jeder in bezug auf $\mathfrak{S}$ associierte Punkt zu G'. Wählt man aus allen mit $\mathfrak{X}$ associierten Punkten einen aus, so erhält man in G' eine Punktmenge $\bar{G}$, die reduciert genannt werden soll. Für das Folgende ist es wesentlich, dass $\bar{G}$ nicht zu compliciert ist; es ist also eine zweckmässige Auswahl der reducierten Punkte festzulegen.

Zunächst nehme man die endlich vielen orthogonalen Einheiten von $\mathfrak{S}$. Diese bilden eine Untergruppe Γ der Gruppe aller $2^m m!$ congruenten Abbildungen des m-dimensionalen Würfels auf sich. Es seien $x_1, \cdots, x_m$ die Elemente der ersten Spalte von $\mathfrak{X}$. Es heisse $\mathfrak{X}$ in bezug auf Γ reduciert, wenn der Ausdruck $x_1 + 2x_2 + \cdots + mx_m$ sich nicht verkleinert, falls $\mathfrak{X}$ durch $\mathfrak{O}\mathfrak{X}$ ersetzt wird, wo $\mathfrak{O}$ der Gruppe Γ angehört. Die reducierten Punkte erfüllen eine oder mehrere von Ebenen begrenzte Ecken, deren Spitze im Nullpunkt des $\mathfrak{X}$-Raumes liegt, und zwar gibt es zu jedem Punkt genau einen reducierten, falls er nicht auf gewissen endlich vielen Ebenen des $\mathfrak{X}$-Raumes liegt. Es sei F das von jenen Ecken erfüllte Gebiet.

Nun sei $\mathfrak{X}$ ein innerer Punkt von F, der nicht auf der Fläche $|\mathfrak{X}| = 0$ liegt. Man betrachte die Spur $\sigma(\mathfrak{X}'\mathfrak{A}'\mathfrak{A}\mathfrak{X})$ der Matrix $\mathfrak{X}'\mathfrak{A}'\mathfrak{A}\mathfrak{X}$ für alle Einheiten $\mathfrak{A}$ von $\mathfrak{S}$; ist sie beschränkt, so auch alle Elemente von $\mathfrak{A}\mathfrak{X}$ und daher von $\mathfrak{A}$. Folglich gibt es eine Einheit $\mathfrak{A}$, so dass jene Spur möglichst klein ist. Ferner ändert sich die Spur nicht, wenn $\mathfrak{A}$ durch $\mathfrak{O}\mathfrak{A}$ ersetzt wird, wo $\mathfrak{O}$ zu Γ gehört. Nennt man $\mathfrak{X}$ reduciert (in bezug auf die Einheitengruppe von $\mathfrak{S}$), wenn

$$\sigma(\mathfrak{X}'\mathfrak{X}) \leqq \sigma(\mathfrak{X}'\mathfrak{A}'\mathfrak{A}\mathfrak{X}) \tag{26}$$

für alle Einheiten von $\mathfrak{S}$ gilt und $\mathfrak{X}$ in F liegt, so gibt es also zu jedem Punkt $\mathfrak{Y}$ mit $|\mathfrak{Y}| \neq 0$ mindestens einen reducierten associierten Punkt $\mathfrak{X}$. Gäbe es zu $\mathfrak{Y}$ zwei reducierte associierte Punkte $\mathfrak{X}_1$ und $\mathfrak{X}_2$, so wäre nach (26)

$$\sigma(\mathfrak{X}_1'\mathfrak{X}_1) = \sigma(\mathfrak{X}_2'\mathfrak{X}_2) = \sigma(\mathfrak{X}_1'\mathfrak{B}'\mathfrak{B}\mathfrak{X}_1) \tag{27}$$

mit einer gewissen von $\mathfrak{E}$ verschiedenen Einheit $\mathfrak{B}$ von $\mathfrak{S}$. Gehörte nun $\mathfrak{B}$ zu Γ, so läge $\mathfrak{X}_1$ auf dem Rande von F; gehörte aber $\mathfrak{B}$ nicht zu Γ, so würde $\mathfrak{X}_1$ auf der durch (27) definierten Fläche zweiter Ordnung liegen.

Man betrachte jetzt die Menge $\bar{F}$ der reducierten Punkte. Es sei $\mathfrak{X}_0$ ein Randpunkt von $\bar{F}$, der weder auf dem Rande von F noch auf $|\mathfrak{X}| = 0$ liegt. In beliebiger Nähe von $\mathfrak{X}_0$ gibt es dann nicht zu $\bar{F}$ gehörige Punkte, also ein $\mathfrak{X}$

im Innern von F mit $|\mathfrak{X}| \neq 0$ und dazu eine Einheit $\mathfrak{B}$ von $\mathfrak{S}$, so dass auch $\mathfrak{B}\mathfrak{X}$ in F liegt und

$$\sigma(\mathfrak{X}'\mathfrak{B}'\mathfrak{B}\mathfrak{X}) < \sigma(\mathfrak{X}'\mathfrak{X}) \tag{28}$$

ist. Für $\mathfrak{X} \to \mathfrak{X}_0$ folgt aus (28) die Beschränktheit der zugehörigen $\mathfrak{B}$, also gilt auch

$$\sigma(\mathfrak{X}_0'\mathfrak{B}'\mathfrak{B}\mathfrak{X}_0) \leqq \sigma(\mathfrak{X}_0'\mathfrak{X}_0) \tag{29}$$

für eine geeignete nicht-orthogonale Einheit von $\mathfrak{S}$. Andererseits gibt es in beliebiger Nähe von $\mathfrak{X}_0$ zu $\bar{F}$ gehörige Punkte, also ein $\mathfrak{X}$ in F mit $|\mathfrak{X}| \neq 0$, so dass (26) für alle Einheiten $\mathfrak{A}$ gilt. Für $\mathfrak{X} \to \mathfrak{X}_0$ folgt dann

$$\sigma(\mathfrak{X}_0'\mathfrak{B}'\mathfrak{B}\mathfrak{X}_0) \geqq \sigma(\mathfrak{X}_0'\mathfrak{X}_0)\,. \tag{30}$$

Daher liegt $\mathfrak{X}_0$ auf der Fläche zweiter Ordnung $\sigma(\mathfrak{X}'\mathfrak{B}'\mathfrak{B}\mathfrak{X}) = \sigma(\mathfrak{X}'\mathfrak{X})$.

Ist K eine beliebig grosse positive Zahl, so gehen durch das durch die Ungleichungen $\sigma(\mathfrak{X}'\mathfrak{X}) \leqq K$, $-K \leqq |\mathfrak{X}|^{-1} \leqq +K$ definierte Raumstück nur endlich viele jener Flächen zweiter Ordnung. Der in diesem Raumstück gelegene Teil von $\bar{F}$ wird also nur von endlich vielen Flächen zweiter Ordnung begrenzt.

Wird jetzt wie früher das endliche $\mathfrak{Z}$-Gebiet G vermöge $\mathfrak{X}'\mathfrak{S}\mathfrak{X} = \mathfrak{Z}$ auf das $\mathfrak{X}$-Gebiet G' abgebildet, so definiere man als das zugehörige reducierte Gebiet $\bar{G}$ den Durchschnitt von G' und $\bar{F}$.

2. Integralsätze

Hilfssatz 5: *Hat G einen Inhalt, so hat auch $\bar{G}$ einen (endlichen) Inhalt, falls nicht $m = 2$ und zugleich $-|\mathfrak{S}|$ eine Quadratzahl ist.*

Beweis: Es sei $K > 0$ und $\bar{G}_K$ der durch die Ungleichung $\sigma(\mathfrak{X}'\mathfrak{X}) \leqq K$ definierte Teil von $\bar{G}$. Es ist $|\mathfrak{X}|^2 |\mathfrak{S}| = |\mathfrak{Z}|$, also $|\mathfrak{X}|^{-1}$ in G beschränkt. Der Rand von $\bar{G}_K$ besteht aus Teilen von endlich vielen Flächen zweiter Ordnung und aus einem Teil der Punktmenge, welche dem Rande von G vermöge $\mathfrak{X}'\mathfrak{S}\mathfrak{X} = \mathfrak{Z}$ im $\mathfrak{X}$-Raume entspricht. Folglich hat $\bar{G}_K$ einen Inhalt v_K, der offenbar eine monoton wachsende Funktion von K ist. Da $\bar{G}_K$ mit unendlich werdendem K gegen $\bar{G}$ convergiert, so hat man zum Beweise der Behauptung nur zu zeigen, dass v_K unter einer von K freien Schranke liegt.

Man wende Hilfssatz 3 an, für ein beliebiges $\mathfrak{X}$ aus $\bar{G}$. Da $|\mathfrak{Z}|$ und $|\mathfrak{Z}|^{-1}$ in G beschränkt sind, so kann die Zahl M des Hilfssatzes 3 constant auf $\bar{G}$ gewählt werden. Wie beim Beweise von Hilfssatz 4 folgt, dass $\mathfrak{Q} = \mathfrak{Q}_0\mathfrak{U}$ mit unimodularem $\mathfrak{U}$ ist, wo für $\mathfrak{Q}_0$ nur endlich viele Möglichkeiten in Betracht kommen. Daher ist $\mathfrak{U}'^{-1}\mathfrak{S}\mathfrak{U}^{-1}$ nach (6) eine von endlich vielen Matrizen $\mathfrak{S}_0$; und für jede von diesen gibt es eine feste unimodulare Matrix $\mathfrak{U}_0$, so dass $\mathfrak{S} = \mathfrak{U}_0'\mathfrak{S}_0\mathfrak{U}_0$ ist. Dann ist aber $\mathfrak{U}_0^{-1}\mathfrak{U} = \mathfrak{W}$ eine Einheit von $\mathfrak{S}$ und $\mathfrak{Q} = \mathfrak{Q}_0\mathfrak{U} = \mathfrak{Q}_0\mathfrak{U}_0\mathfrak{W} = \mathfrak{P}_0\mathfrak{W}$ mit endlich vielen $\mathfrak{P}_0$.

Der Wert der Zahl g des Hilfssatzes 3 kann von $\mathfrak{X}$ abhängen. Zunächst sei

$0 < g < \frac{m}{2}$, also $\mathfrak{H}$ vorhanden. Sind $\mathfrak{y}$, $\mathfrak{z}$, $\mathfrak{t}$ Spalten von g, g, $m - 2g$ Elementen, so geht die quadratische Form $2\mathfrak{y}'\mathfrak{z} + \mathfrak{t}'\mathfrak{H}\mathfrak{t}$ durch die Substitution

$$\mathfrak{y} = \mathfrak{y}_1 - 2\mathfrak{A}'\mathfrak{H}\mathfrak{A}\mathfrak{z}_1 - 2\mathfrak{A}'\mathfrak{H}\mathfrak{t}_1, \qquad \mathfrak{z} = \mathfrak{z}_1, \qquad \mathfrak{t} = \mathfrak{t}_1 + 2\mathfrak{A}\mathfrak{z}_1$$

in sich über, bei beliebigem $\mathfrak{A} = \mathfrak{A}^{(m-2g,\,g)}$. Setzt man

$$\mathfrak{F} = \begin{pmatrix} \mathfrak{E} & -2\mathfrak{A}'\mathfrak{H}\mathfrak{A} & -2\mathfrak{A}'\mathfrak{H} \\ \mathfrak{N} & \mathfrak{E} & \mathfrak{N} \\ \mathfrak{N} & 2\mathfrak{A} & \mathfrak{E} \end{pmatrix}, \tag{31}$$

so ist die Matrix $\mathfrak{Q}^{-1}\mathfrak{F}\mathfrak{Q}$ nach (6) eine Einheit von $\mathfrak{S}$, falls sie ganz ist. Dies ist sicher der Fall, wenn $\mathfrak{A}$ ganz und $\equiv \mathfrak{N} \pmod{S}$ gewählt wird. Dann gilt

$$\mathfrak{F}\mathfrak{Q}\mathfrak{X} = \mathfrak{P}_0\mathfrak{Y}, \tag{32}$$

wo $\mathfrak{Y}$ mit $\mathfrak{X}$ in bezug auf $\mathfrak{S}$ associiert ist. In $\mathfrak{Q}\mathfrak{X} = (b_{kl})$ sei $b_{g+1,\,h}$ das absolut grösste der Elemente der $(g+1)^{ten}$ Zeile. In $\mathfrak{A}$ wähle man alle Elemente gleich 0 bis auf das erste Element der ersten Zeile, dieses sei aS. Man kann ein ganzzahliges a so bestimmen, dass

$$|\, 2aS\, b_{g+1,\,h} + b_{2g+1,\,h}\,| \leqq |\, S\, b_{g+1,\,h}\,| \tag{33}$$

ist. Da nun $|\,\mathfrak{Q}\mathfrak{X}\,|^{-1}$ beschränkt ist, so folgt nach Aussage 4) von Hilfssatz 3 mit Rücksicht auf die Definition der σ_k, dass $\sigma_{g+1} : b_{g+1,\,h}$ beschränkt ist. Nach (33) ist daher $a : \sigma_1$ beschränkt. Aus (31) folgt dann, dass die Aussage 4) von Hilfssatz 3 auch für $\mathfrak{F}\mathfrak{Q}\mathfrak{X}$ statt $\mathfrak{Q}\mathfrak{X}$ gilt. Darüber hinaus besagt (33), dass das h^{te} Element der $(2g+1)^{ten}$ Zeile von $\mathfrak{F}\mathfrak{Q}\mathfrak{X}$ nicht nur beschränkt ist, sondern sogar absolut kleiner als $|\, S\,|M\,\sigma_1^{-1}$ ist.

Man setze $\mathfrak{P}_0\mathfrak{Y} = (z_{kl})$ und bezeichne mit μ_k das Maximum der absoluten Beträge der Zahlen $z_{kl}, \cdots, z_{km}$ $(k = 1, \cdots, m)$; ferner sei ν das Minimum der absoluten Beträge der Elemente der $(2g+1)^{ten}$ Zeile von $\mathfrak{P}_0\mathfrak{Y}$. Ist dann M irgend eine positive Zahl, so wird durch die Ungleichungen

$$\mu_k\mu_{g+k} \leqq M \quad (k = 1, \cdots, g), \qquad \mu_k \leqq M \quad (k = 2g+1, \cdots, m), \tag{34}$$

$$M^{-1} \leqq \mu_{k+1} \leqq M\mu_k \quad (k = 1, \cdots, g-1), \qquad \mu_1 \leqq M\nu^{-1} \tag{35}$$

bei variablem $\mathfrak{Y}$ ein Gebiet im $\mathfrak{Y}$-Raum definiert, und zwar hat dieses, wie eine leichte Rechnung zeigt, einen endlichen Inhalt. Andererseits ist die Anzahl der $\mathfrak{P}_0$ endlich und das durch (32) definierte $\mathfrak{Y}$ mit $\mathfrak{X}$ associiert. Betrachtet man nur denjenigen Teil von $\bar{G}_K$, für welchen die $\mathfrak{X}$ zugeordnete Zahl g dem Intervall $0 < g < \frac{m}{2}$ angehört, so ist also sein Inhalt beschränkt. Es ist noch zu zeigen, dass dies auch für $g = 0$ und $g = \frac{m}{2}$ gilt.

Im Fall $g = 0$ ist das trivial, da nach Hilfssatz 3 dann die Matrix $\mathfrak{Q}\mathfrak{X} = \mathfrak{P}_0\mathfrak{W}\mathfrak{X}$ beschränkt ist.

Im Fall $g = \frac{m}{2}$ ist $m > 2$; denn für $m = 2$ folgte aus (6), dass dann $-S$ eine Quadratzahl wäre, gegen die Voraussetzung. Es sei $\mathfrak{A} = \mathfrak{A}^{(g)}$ eine beliebige alternierende Matrix, also $\mathfrak{A}' = -\mathfrak{A}$. Setzt man dann

$$\mathfrak{F} = \begin{pmatrix} \mathfrak{E} & \mathfrak{A} \\ \mathfrak{N} & \mathfrak{E} \end{pmatrix},$$

so ist die Matrix $\mathfrak{Q}^{-1}\mathfrak{F}\mathfrak{Q}$ nach (6) eine Einheit von $\mathfrak{S}$, falls sie ganz ist. Dies ist sicher der Fall, wenn $\mathfrak{A}$ ganz und $\equiv \mathfrak{N} \pmod{S}$ gewählt wird. In $\mathfrak{A} = (a_{kl})$ seien nun alle Elemente 0 bis auf $a_{g1} = aS$ und $a_{1g} = -aS$. Es sei wieder $b_{g+1,h}$ das absolut grösste der Elemente der $(g+1)^{ten}$ Zeile von $\mathfrak{Q}\mathfrak{X} = (b_{kl})$. Da das h^{te} Element der g^{ten} Zeile von $\mathfrak{F}\mathfrak{Q}\mathfrak{X}$ den Wert $b_{gh} + aS\, b_{g+1,h}$ besitzt, so kann es durch geeignete ganzzahlige Wahl von a absolut kleiner als $|\, S\, b_{g+1,h}\,|$ gemacht werden. Nach Aussage 4) von Hilfssatz 3 ist dabei $a : \sigma_1\sigma_g$ beschränkt. Hieraus folgt leicht, dass jene Aussage auch für $\mathfrak{F}\mathfrak{Q}\mathfrak{X}$ statt $\mathfrak{Q}\mathfrak{X}$ gilt.

Nun sei ν das Minimum der absoluten Beträge der Elemente der g^{ten} Zeile von $\mathfrak{P}_0\mathfrak{Y}$, während μ_k die frühere Bedeutung habe. Durch die Ungleichungen

$$\mu_k\mu_{g+k} \leqq M \quad (k = 1, \cdots, g), \qquad M^{-1} \leqq \mu_{k+1} \leqq M\mu_k \quad (k = 1, \cdots, g-1), \tag{36}$$
$$\mu_1 \leqq M\nu^{-1}$$

wird wieder ein endliches Volumen im $\mathfrak{Y}$-Raum definiert. Dies zeigt vermöge (32), dass auch der zu $g = \frac{m}{2}$ gehörige Teil von $\bar{G}_K$ beschränkten Inhalt besitzt. Damit ist Hilfssatz 5 vollständig bewiesen.

Fortan sei der Ausnahmefall $m = 2$, $-|\mathfrak{S}|$ gleich einer Quadratzahl ausgeschlossen.

Läge das reducierte Gebiet $\bar{G}$ ganz im Endlichen, so folgt aus der Definition des Inhalts, dass das Volumen von $\bar{G}$ gleich $\lim t^{-m^2} a_t$ für $t \to \infty$ wäre, wo a_t die Anzahl der in $\bar{G}$ gelegenen Gitterpunkte eines Würfelgitters der Breite t^{-1} bedeutet. Hat aber die Gleichung $\mathfrak{x}'\mathfrak{S}\mathfrak{x} = 0$ eine nicht-triviale ganzzahlige Lösung $\mathfrak{x}$, so reicht $\bar{G}$ ins Unendliche, wie sich durch eine genauere Untersuchung zeigen lässt; und zwar tritt dies im Falle $m \geqq 5$ für jedes indefinite $\mathfrak{S}$ ein. Obwohl sich jene arithmetische Approximation des Inhalts bei beliebigen unendlichen Gebieten nicht stets übertragen lässt, so gilt doch

HILFSSATZ 6: *Es habe G einen Inhalt. Es sei $a(t)$ die Anzahl der ganzen $\mathfrak{X}$, für welche die Matrix $t^{-1}\mathfrak{X}$ im reducierten Gebiet $\bar{G}$ liegt. Dann strebt $t^{-m^2}a(t)$ für unendlich werdendes t gegen den Inhalt von $\bar{G}$.*

BEWEIS: Es sei $a_K(t)$ die Anzahl der ganzen $\mathfrak{X}$, für welche $t^{-1}\mathfrak{X}$ in $\bar{G}_K$ liegt. Da $\bar{G}_K$ ein endliches Gebiet ist, so ist $\lim t^{-m^2} a_K(t)$ für jedes feste K gleich dem Inhalt von $\bar{G}_K$. Zum Beweise der Behauptung genügt es daher, zu zeigen, dass der Ausdruck $t^{-m^2}(a(t) - a_K(t))$ für $K \to \infty$ gleichmässig in t den Grenzwert 0 hat.

Man wende nun dieselbe Überlegung an wie beim Beweise von Hilfssatz 5,

mit $t^{-1}\mathfrak{x}$ statt $\mathfrak{x}$. Es gibt endlich viele Matrizen $\mathfrak{P}_0$ und zu jedem $\mathfrak{x}$ ein associiertes $\mathfrak{y}$, so dass für $t^{-1}\mathfrak{P}_0\mathfrak{y}$ im Falle $0 < g < \frac{m}{2}$ die Ungleichungen (34), (35), im Falle $g = \frac{m}{2}$ die Ungleichungen (36) gelten, während $t^{-1}\mathfrak{P}_0\mathfrak{y}$ im Falle $g = 0$ beschränkt ist. Liegt nun $t^{-1}\mathfrak{x}$ in $\bar{G}$, aber nicht in $\bar{G}_K$, so ist $\sigma(\mathfrak{x}'\mathfrak{x}) > Kt^2$, also für das associierte $\mathfrak{y}$ erst recht $\sigma(\mathfrak{y}'\mathfrak{y}) > Kt^2$; und daher ist mindestens ein Element von $t^{-1}\mathfrak{P}_0\,\mathfrak{y}$ absolut grösser als $M^{-1}K^{\frac{1}{2}}$, bei geeignetem constanten $M > 0$. Der Fall $g = 0$ kann also bei hinreichend grossem K nicht vorkommen.

Es sei $0 < g < \frac{m}{2}$. Es bedeute μ_k das Maximum der absoluten Werte der Elemente in der k^{ten} Zeile von $t^{-1}\mathfrak{P}_0\mathfrak{y}$ und ν das Minimum der absoluten Werte für die $(2g+1)^{te}$ Zeile. Es genügt, die Anzahl der ganzen $\mathfrak{y}$ mit $|\mathfrak{y}| \neq 0$ abzuschätzen, für welche die Ungleichungen

$$\mu_k\mu_{g+k} \leqq M \quad (k = 1, \cdots, g), \qquad \mu_k \leqq M \quad (k = 2g+1, \cdots, m),$$

$$M^{-1} \leqq \mu_{k+1} \leqq M\mu_k \quad (k = 1, \cdots, g-1), \qquad M^{-1}K^{\frac{1}{2}} \leqq \mu_1 \leqq M\nu^{-1}$$

sämtlich erfüllt sind. Durch eine einfache Abzählung folgt, dass jene Anzahl kleiner als $ct^{m^2}K^{-\frac{1}{2}}\log^{g-1}K$ ist, wo c von K und t frei ist.

Im Falle $g = \frac{m}{2}$ definiere man analog wie früher ν als das Minimum der absoluten Beträge der Elemente der g^{ten} Zeile von $t^{-1}\mathfrak{P}_0\mathfrak{y}$ und findet für die Anzahl der ganzen $\mathfrak{y}$ mit $|\mathfrak{y}| \neq 0$ und

$$\mu_k\mu_{g+k} \leqq M \quad (k = 1, \cdots, g), \qquad M^{-1} \leqq \mu_{k+1} \leqq M\mu_k \quad (k = 1, \cdots, g-1),$$

$$M^{-1}K^{\frac{1}{2}} \leqq \mu_1 \leqq M\nu^{-1}$$

ebenfalls die im vorigen Absatz angegebene Schranke.

Folglich ist

$$a(t) - a_K(t) < ct^{m^2}K^{-\frac{1}{2}}\log^{g-1}K$$

und damit die Behauptung bewiesen.

Ebenso beweist man den allgemeineren

HILFSSATZ 7: *Es sei $f(\mathfrak{Z})$ eine Function der Elemente der Matrix $\mathfrak{Z}$, die im $\mathfrak{Z}$-Gebiet G integrierbar ist. Es sei $\mathfrak{C}_1^{(m)}$ eine ganze Matrix und q eine natürliche Zahl. Durchläuft dann $\mathfrak{C}$ alle ganzen Matrizen $\equiv \mathfrak{C}_1 \pmod q$, für welche die Matrix $t^{-1}\mathfrak{C}$ im reducierten Gebiet $\bar{G}$ liegt, so gilt*

$$\int_{\bar{G}} f(\mathfrak{x}'\mathfrak{S}\mathfrak{x})\,d\mathfrak{x} = \lim_{t\to\infty}\left(\frac{q}{t}\right)^{m^2} \sum_{\substack{\mathfrak{C}\equiv\mathfrak{C}_1 \pmod q \\ t^{-1}\mathfrak{C} \text{ in } \bar{G}}} f(t^{-2}\,\mathfrak{C}'\mathfrak{S}\mathfrak{C}). \tag{37}$$

Das Integral in (37) lässt sich in ein Integral über G transformieren. Um dies zu zeigen, benötigt man

HILFSSATZ 8: *Die Gleichung* $\mathfrak{X}'\mathfrak{S}\mathfrak{X} = \mathfrak{Z}$ *hat für hinreichend nahe bei* $\mathfrak{S}$ *gelegene* $\mathfrak{Z}$ *eine in* $\mathfrak{Z}$ *regulär analytische Lösung* $\mathfrak{X}$, *die für* $\mathfrak{Z} \to \mathfrak{S}$ *den Grenzwert* $\mathfrak{E}$ *hat.*

BEWEIS: Es genügt, eine Lösung $\mathfrak{X}$ mit symmetrischem $\mathfrak{S}\mathfrak{X}$ zu finden. Ist nun $\mathfrak{X}'\mathfrak{S} = \mathfrak{S}\mathfrak{X}$, so geht die Gleichung $\mathfrak{X}'\mathfrak{S}\mathfrak{X} = \mathfrak{Z}$ über in

$$\mathfrak{X}^2 = \mathfrak{S}^{-1}\mathfrak{Z}\,. \tag{38}$$

Setzt man $\mathfrak{S}^{-1}\mathfrak{Z} = \mathfrak{E} - 4\mathfrak{H}$, so ist

$$\mathfrak{X} = (\mathfrak{E} - 4\mathfrak{H})^{\frac{1}{2}} = \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} (-4)^k \mathfrak{H}^k = \mathfrak{E} - 2\sum_{k=1}^{\infty} \frac{(2k-2)!}{(k-1)!\,k!}\,\mathfrak{H}^k \tag{39}$$

eine Lösung von (38), falls die Reihe absolut convergiert. Dies ist sicher der Fall, wenn alle Elemente von $\mathfrak{H}$ hinreichend dicht bei 0 liegen, und zwar genügt es, wie man leicht erkennt, dass alle Wurzeln der Gleichung $|\lambda\mathfrak{S} - \mathfrak{Z}| = 0$ dem Intervall $0 < \lambda < 2$ angehören. Nun ist aber $\mathfrak{S}\mathfrak{H}^k$ symmetrisch, also nach (39) auch $\mathfrak{S}\mathfrak{X}$, so dass die durch (39) definierte Matrix $\mathfrak{X}$ auch der Gleichung $\mathfrak{X}'\mathfrak{S}\mathfrak{X} = \mathfrak{Z}$ genügt. Wegen $\mathfrak{H} = 1/4\,(\mathfrak{E} - \mathfrak{S}^{-1}\mathfrak{Z})$ ist die gefundene Lösung in $\mathfrak{Z}$ analytisch, und es gilt $\mathfrak{X} \to \mathfrak{E}$ für $\mathfrak{Z} \to \mathfrak{S}$.

HILFSSATZ 9: *Es sei* $v(G)$ *der Inhalt des Gebietes* G *im* $\mathfrak{Z}$*-Raum und* $v(\bar{G})$ *der Inhalt des vermöge* $\mathfrak{X}'\mathfrak{S}\mathfrak{X} = \mathfrak{Z}$ *zugehörigen reducierten Gebietes* $\bar{G}$ *im* $\mathfrak{X}$*-Raum. Lässt man* G *auf einen Punkt* $\mathfrak{Z}$ *zusammenschrumpfen, so existiert gleichmässig in* $\mathfrak{Z}$ *der Grenzwert*

$$\lim_{G \to \mathfrak{Z}} \frac{v(\bar{G})}{v(G)} = \rho(\mathfrak{S})\,|\,\mathfrak{S}\mathfrak{Z}^{-1}\,|^{\frac{1}{2}}, \tag{40}$$

und zwar hängt $\rho(\mathfrak{S})$ *nur von der Classe von* $\mathfrak{S}$ *ab.*

BEWEIS: Die Gleichung $\mathfrak{X}'\mathfrak{S}\mathfrak{X} = \mathfrak{Z}$ geht durch die Substitution $\mathfrak{X} = \mathfrak{X}_1\mathfrak{C}^{(m)}$, $\mathfrak{Z} = \mathfrak{C}'\mathfrak{Z}_1\mathfrak{C}$ in $\mathfrak{X}_1'\mathfrak{S}\mathfrak{X}_1 = \mathfrak{Z}_1$ über. Den Gebieten G und $\bar{G}$ entsprechen zwei Gebiete G_1 und G_1' in den Räumen von $\mathfrak{Z}_1$ und $\mathfrak{X}_1$, und es ist

$$v(\bar{G}) = |\,\mathfrak{C}\,|^m\, v(G_1'), \qquad v(G) = |\,\mathfrak{C}\,|^{m+1}\, v(G_1), \qquad |\,\mathfrak{Z}\,| = |\,\mathfrak{C}\,|^2\,|\,\mathfrak{Z}_1\,|\,. \tag{41}$$

Nun braucht G_1' nicht das zu G_1 gehörige reducierte Gebiet $\bar{G}_1$ zu sein. Es enthält aber, wenn von Randpunkten abgesehen wird, zu jeder Lösung $\mathfrak{X}_1$ von $\mathfrak{X}_1'\mathfrak{S}\mathfrak{X}_1 = \mathfrak{Z}_1$ genau einen associierten Punkt. Daraus folgt leicht, dass $v(G_1') = v(\bar{G}_1)$ ist. Also gilt

$$|\,\mathfrak{Z}_1\,|^{\frac{1}{2}}\,\frac{v(\bar{G}_1)}{v(G_1)} = |\,\mathfrak{Z}\,|^{\frac{1}{2}}\,\frac{v(\bar{G})}{v(G)}\,. \tag{42}$$

Setzt man ferner $\mathfrak{X} = \mathfrak{U}\mathfrak{X}_0$ mit unimodularem $\mathfrak{U}$ und $\mathfrak{U}'\mathfrak{S}\mathfrak{U} = \mathfrak{S}_0$, so geht $\mathfrak{X}'\mathfrak{S}\mathfrak{X} = \mathfrak{Z}$ in $\mathfrak{X}_0'\mathfrak{S}_0\mathfrak{X}_0 = \mathfrak{Z}$ über. Dabei entsprechen in bezug auf $\mathfrak{S}$ associierten Punkten im $\mathfrak{X}$-Raum stets in bezug auf $\mathfrak{S}_0$ associierte Punkte im $\mathfrak{X}_0$-Raum. Folglich hängt $v(\bar{G})$ bei festem G nur von der Classe von $\mathfrak{S}$ ab. Mit Rücksicht auf (42) hat man zum Beweise der Behauptung nur noch zu zeigen, dass der Grenzwert in (40) existiert.

Es sei $\epsilon > 0$. Man betrachte eine so kleine Umgebung G_ϵ von $\mathfrak{Z}_0$, so dass für alle $\mathfrak{Z}_1$ aus G_ϵ bei der nach Hilfssatz 8 existierenden Lösung $\mathfrak{X}$ von $\mathfrak{X}'\mathfrak{Z}_0\mathfrak{X} = \mathfrak{Z}_1$ alle Elemente der Matrix $\mathfrak{X} - \mathfrak{E}$ absolut kleiner als ϵ sind. Es sei W_0 ein Würfel in G_ϵ mit der Kantenlänge w_0 und dem Mittelpunkt $\mathfrak{Z}_0$ und W_1 ein dazu parallel orientierter Würfel in G_ϵ mit der Kantenlänge w_1 und dem Mittelpunkt $\mathfrak{Z}_1$. Durchläuft $\mathfrak{Z}$ den Würfel W_0, so beschreibt $\mathfrak{X}'\mathfrak{Z}\mathfrak{X}$ bei festem $\mathfrak{X}$ ein Parallelepiped G. Es ist nun

$$\mathfrak{X}'\mathfrak{Z}\mathfrak{X} - \mathfrak{Z}_1 = \mathfrak{X}'(\mathfrak{Z} - \mathfrak{Z}_0)\mathfrak{X} = \mathfrak{Z} - \mathfrak{Z}_0 + (\mathfrak{X}' - \mathfrak{E})(\mathfrak{Z} - \mathfrak{Z}_0) + (\mathfrak{Z} - \mathfrak{Z}_0)(\mathfrak{X} - \mathfrak{E}) + (\mathfrak{X}' - \mathfrak{E})(\mathfrak{Z} - \mathfrak{Z}_0)(\mathfrak{X} - \mathfrak{E}),$$

und folglich unterscheiden sich entsprechende Elemente von $\mathfrak{X}'\mathfrak{Z}\mathfrak{X} - \mathfrak{Z}_1$ und $\mathfrak{Z} - \mathfrak{Z}_0$ höchstens um die Zahl $(m\epsilon + \frac{1}{2}m^2\epsilon^2)w_0$. Bedeutet ϵ_1, wie auch weiterhin $\epsilon_2, \cdots, \epsilon_5$, eine mit ϵ gegen 0 strebende Grösse, so liegt daher für

$$w_0 = w_1(1 + \epsilon_1) \tag{43}$$

der Würfel W_1 ganz in G. Es seien $\overline{W}_0$, $\overline{W}_1$, $\bar{G}$ die zu W_0, W_1, G gehörigen reducierten Gebiete. Nach (41) ist

$$v(\overline{W}_1) < v(\bar{G}) = |\mathfrak{X}|^m v(\overline{W}_0), \qquad v(W_1) = \left(\frac{w_1}{w_0}\right)^{\frac{m(m+1)}{2}} v(W_0)$$

und demnach

$$\frac{v(\overline{W}_1)}{v(W_1)} < (1 + \epsilon_2)\frac{v(\overline{W}_0)}{v(W_0)}. \tag{44}$$

Zerteilt man jetzt W_0 und W_1 in je $t^{\frac{m(m+1)}{2}}$ parallel orientierte Teilwürfel der Kanten $\frac{w_0}{t}$ und $\frac{w_1}{t}$, so folgt aus (43) und (44) für zwei beliebige dieser Teilwürfel W_0' und W_1' die Ungleichung

$$\frac{v(\overline{W}_1')}{v(W_1')} < (1 + \epsilon_2)\frac{v(W_0')}{v(W_0')}. \tag{45}$$

Es seien W_1 und W_2 die aus W_0 durch Verkürzung in den Verhältnissen $1:1 + \epsilon_1$ und $1:(1 + \epsilon_1)^2$ hervorgehenden Würfel mit dem Mittelpunkt $\mathfrak{Z}_0$. Dann ist auch

$$\frac{v(\overline{W}_2)}{v(W_2)} < (1 + \epsilon_2)\frac{v(\overline{W}_1)}{v(W_1)}.$$

Für hinreichend grosses t kann man mit den Würfeln W_0' den Würfel W_2 überdecken, ohne dabei aus W_1 herauszukommen. Daher ist zufolge (45)

$$\frac{v(\overline{W}_1')}{v(W_1')} < (1 + \epsilon_2)\frac{v(\overline{W}_1)}{v(W_2)} = (1 + \epsilon_3)\frac{v(\overline{W}_1)}{v(W_1)}. \tag{46}$$

Zerlegt man auch W_2 in $t^{\frac{m(m+1)}{2}}$ gleiche Teilwürfel W_2', so kann man für hinreichend grosses t mit den W_2' den Würfel W_1 überdecken, ohne dabei W_0 zu verlassen. Folglich ist auch

$$(47) \qquad (1-\epsilon_4)\frac{v(\overline{W}_1)}{v(W_1)} < \frac{v(\overline{W}_1)}{v(W_0)} < (1+\epsilon_2)\frac{v(\overline{W}_1')}{v(W_1')}.$$

Aus (46) und (47) ergibt sich, dass für alle W_1' die Beziehung

$$\frac{v(\overline{W}_1')}{v(W_1')} = (1+\eta)\frac{v(\overline{W}_1)}{v(W_1)}$$

mit $|\eta| \leqq \epsilon_5$ gilt. Ist also G irgend ein Gebiet in G_ϵ, das einen Inhalt $v(G)$ besitzt, so ist auch

$$\frac{v(\bar{G})}{v(G)} = (1+\eta)\frac{v(\overline{W}_1)}{v(W_1)}.$$

Hieraus folgt die Existenz des Grenzwertes in (40).

Eine unmittelbare Folge vom Hilfssatz 9 ist

HILFSSATZ 10: *Es sei $f(\mathfrak{Z})$ in G integrierbar. Dann ist*

$$\int_{\bar{G}} f(\mathfrak{X}'\mathfrak{S}\mathfrak{X})\,d\mathfrak{X} = \rho(\mathfrak{S})\int_G f(\mathfrak{Z})\,|\mathfrak{S}\mathfrak{Z}^{-1}|^{\frac{1}{2}}\,d\mathfrak{Z}.$$

Weiterhin bedeute $\mathfrak{T} = \mathfrak{T}^{(n)}$ eine ganze symmetrische Matrix, die durch $\mathfrak{S}^{(m)}$ ganzzahlig darstellbar ist; es sei $|\mathfrak{T}| \neq 0$ und $n < m$. Zu jeder Darstellung $\mathfrak{C}'\mathfrak{S}\mathfrak{C} = \mathfrak{T}$ gehört eine gewisse endliche oder unendliche Gruppe, nämlich die Gruppe derjenigen Einheiten $\mathfrak{A}$ von $\mathfrak{S}$, welche $\mathfrak{C}$ invariant lassen, also den Gleichungen $\mathfrak{A}'\mathfrak{S}\mathfrak{A} = \mathfrak{S}$ und $\mathfrak{A}\mathfrak{C} = \mathfrak{C}$ genügen.

Es sei $\mathfrak{X}_0^{(m,\,m-n)}$ eine reelle Matrix, so dass die aus $\mathfrak{C}$ und $\mathfrak{X}_0$ zusammengesetzte Matrix $(\mathfrak{C}\mathfrak{X}_0)$ eine von 0 verschiedene Determinante besitzt. Man setze

$$(\mathfrak{C}\mathfrak{X}_0)'\mathfrak{S}(\mathfrak{C}\mathfrak{X}_0) = \begin{pmatrix} \mathfrak{T} & \mathfrak{Q}_0 \\ \mathfrak{Q}_0' & \mathfrak{R}_0 \end{pmatrix},$$

also $\mathfrak{C}'\mathfrak{S}\mathfrak{X}_0 = \mathfrak{Q}_0$, $\mathfrak{X}_0'\mathfrak{S}\mathfrak{X}_0 = \mathfrak{R}_0$. Es soll nun gezeigt werden, dass die Gleichung

$$(48) \qquad (\mathfrak{C}\mathfrak{X})'\mathfrak{S}(\mathfrak{C}\mathfrak{X}) = \begin{pmatrix} \mathfrak{T} & \mathfrak{Q} \\ \mathfrak{Q}' & \mathfrak{R} \end{pmatrix}$$

eine reelle Lösung $\mathfrak{X}$ besitzt, falls $\mathfrak{Q}$ und $\mathfrak{R} = \mathfrak{R}'$ genügend nahe bei $\mathfrak{Q}_0$ und $\mathfrak{R}_0$ liegen. Um nämlich die Gleichungen $\mathfrak{C}'\mathfrak{S}\mathfrak{X} = \mathfrak{Q}$, $\mathfrak{X}'\mathfrak{S}\mathfrak{X} = \mathfrak{R}$ zu lösen, setze man $\mathfrak{X} = \mathfrak{C}\mathfrak{F} + \mathfrak{X}_0\mathfrak{W}$ mit unbekannten $\mathfrak{F}^{(n,\,m-n)}$ und $\mathfrak{W}^{(m-n)}$. Führt man noch die Abkürzungen $\mathfrak{R}_0 - \mathfrak{Q}_0'\mathfrak{T}^{-1}\mathfrak{Q}_0 = \mathfrak{H}_0$ und $\mathfrak{R} - \mathfrak{Q}'\mathfrak{T}^{-1}\mathfrak{Q} = \mathfrak{H}$ ein, so ist

$$\begin{pmatrix} \mathfrak{T} & \mathfrak{Q}_0 \\ \mathfrak{Q}_0' & \mathfrak{R}_0 \end{pmatrix} = \begin{pmatrix} \mathfrak{T} & \mathfrak{R} \\ \mathfrak{Q}_0' & \mathfrak{E} \end{pmatrix}\begin{pmatrix} \mathfrak{T}^{-1} & \mathfrak{R} \\ \mathfrak{R} & \mathfrak{H}_0 \end{pmatrix}\begin{pmatrix} \mathfrak{T} & \mathfrak{Q}_0 \\ \mathfrak{R} & \mathfrak{E} \end{pmatrix},$$

also $|\mathfrak{H}_0| \neq 0$, und die Gleichungen gehen über in $\mathfrak{T}\mathfrak{F} + \mathfrak{Q}_0\mathfrak{W} = \mathfrak{Q}$, $\mathfrak{W}'\mathfrak{H}_0\mathfrak{W} = \mathfrak{H}$. Von diesen ist die zweite auf grund von Hilfssatz 8 lösbar, falls $\mathfrak{H}$ genügend dicht bei $\mathfrak{H}_0$ liegt, und aus der ersten bestimmt sich dann $\mathfrak{F}$. Man wähle im $(m-n)\,(m+n+1)/2$-dimensionalen Raum der Paare $\mathfrak{Q}$, $\mathfrak{R}$ ein Gebiet B, in welchem überall (48) lösbar ist. Vermöge (48) wird dann B auf ein Gebiet B' in $m(m-n)$-dimensionalen $\mathfrak{x}$-Raum abgebildet. Nennt man zwei Punkte $\mathfrak{x}_1$, $\mathfrak{x}_2$ des $\mathfrak{x}$-Raumes associiert in bezug auf $\mathfrak{S}$, $\mathfrak{C}$, wenn für eine geeignete $\mathfrak{C}$ invariant lassende Einheit $\mathfrak{A}$ von $\mathfrak{S}$ die Gleichung $\mathfrak{x}_2 = \mathfrak{A}\mathfrak{x}_1$ gilt, so hat man wieder in B' einen reducierten Bereich $\bar{B}$ derart zu definieren, dass abgesehen von Randpunkten jeder Punkt von B' genau einen ihm associierten in $\bar{B}$ besitzt. Man kann den reducierten Bereich ganz analog wie früher so festlegen, dass der Ausdruck $\sigma(\mathfrak{A}(\mathfrak{C}\mathfrak{x})\,(\mathfrak{C}\mathfrak{x})'\mathfrak{A}')$ für $\mathfrak{A} = \mathfrak{E}$ sein Minimum hat, falls $\mathfrak{A}$ alle $\mathfrak{C}$ invariant lassenden Einheiten von $\mathfrak{S}$ durchläuft. Für den Nachweis der Existenz der Inhaltes von $\bar{B}$ braucht man den auch weiterhin wichtigen

HILFSSATZ 11: *Es sei $\mathfrak{C}'\mathfrak{S}\mathfrak{C} = \mathfrak{T}$ eine primitive Darstellung und $\mathfrak{Y}_0$ ein Complement von $\mathfrak{C}$, also $(\mathfrak{C}\mathfrak{Y}_0) = \mathfrak{U}$ unimodular. Man setze*

$$\mathfrak{C}'\mathfrak{S}\mathfrak{Y}_0 = \mathfrak{Q}_0, \qquad \mathfrak{Y}_0'\mathfrak{S}\mathfrak{Y}_0 - \mathfrak{Q}_0'\mathfrak{T}^{-1}\mathfrak{Q}_0 = \mathfrak{H}_0.$$

Ist dann $\mathfrak{A}$ eine $\mathfrak{C}$ invariant lassende Einheit von $\mathfrak{S}$, so wird durch

$$\mathfrak{U}^{-1}\mathfrak{A}\mathfrak{U} = \begin{pmatrix} \mathfrak{E} & \mathfrak{F}_0 \\ \mathfrak{N} & \mathfrak{W}_0 \end{pmatrix} \tag{49}$$

eine Einheit $\mathfrak{W}_0$ von $\mathfrak{H}_0$ definiert, für welche

$$\mathfrak{F}_0 = \mathfrak{T}^{-1}\mathfrak{Q}_0\,(\mathfrak{E} - \mathfrak{W}_0) \tag{50}$$

ganz ist. Ist umgekehrt $\mathfrak{W}_0$ eine derartige Einheit von $\mathfrak{H}_0$, dass die durch (50) *definierte Matrix $\mathfrak{F}_0$ ganz ist, so liefert* (49) *eine $\mathfrak{C}$ invariant lassende Einheit $\mathfrak{A}$ von $\mathfrak{S}$.*

BEWEIS: Ist $\mathfrak{A}\mathfrak{C} = \mathfrak{C}$, so ist $\mathfrak{U}^{-1}\mathfrak{A}\mathfrak{U}$ von der Form (49) mit ganzen $\mathfrak{F}_0$, $\mathfrak{W}_0$. Setzt man noch

$$\mathfrak{T}\mathfrak{F}_0 + \mathfrak{Q}_0\mathfrak{W}_0 = \mathfrak{Q}, \qquad \mathfrak{W}_0'\mathfrak{H}_0\mathfrak{W}_0 = \mathfrak{H}, \tag{51}$$

so ist

$$\begin{pmatrix} \mathfrak{T} & \mathfrak{Q}_0 \\ \mathfrak{N} & \mathfrak{E} \end{pmatrix}\begin{pmatrix} \mathfrak{E} & \mathfrak{F}_0 \\ \mathfrak{N} & \mathfrak{W}_0 \end{pmatrix} = \begin{pmatrix} \mathfrak{E} & \mathfrak{N} \\ \mathfrak{N} & \mathfrak{W}_0 \end{pmatrix}\begin{pmatrix} \mathfrak{T} & \mathfrak{Q} \\ \mathfrak{N} & \mathfrak{E} \end{pmatrix}. \tag{52}$$

Da nun $\mathfrak{U}^{-1}\mathfrak{A}\mathfrak{U}$ ein Einheit von

$$\mathfrak{U}'\mathfrak{S}\mathfrak{U} = \begin{pmatrix} \mathfrak{T} & \mathfrak{Q}_0 \\ \mathfrak{N} & \mathfrak{E} \end{pmatrix}'\begin{pmatrix} \mathfrak{T}^{-1} & \mathfrak{N} \\ \mathfrak{N} & \mathfrak{H}_0 \end{pmatrix}\begin{pmatrix} \mathfrak{T} & \mathfrak{Q}_0 \\ \mathfrak{N} & \mathfrak{E} \end{pmatrix} = \begin{pmatrix} \mathfrak{T} & \mathfrak{Q}_0 \\ \mathfrak{Q}_0' & \mathfrak{H}_0 + \mathfrak{Q}_0'\mathfrak{T}^{-1}\mathfrak{Q}_0 \end{pmatrix} \tag{53}$$

ist, so liefern (49) und (52)

$$\begin{pmatrix} \mathfrak{T} & \mathfrak{Q}_0 \\ \mathfrak{Q}_0' & \mathfrak{H}_0 + \mathfrak{Q}_0'\mathfrak{T}^{-1}\mathfrak{Q}_0 \end{pmatrix} = \begin{pmatrix} \mathfrak{T} & \mathfrak{Q} \\ \mathfrak{Q}' & \mathfrak{H} + \mathfrak{Q}'\mathfrak{T}^{-1}\mathfrak{Q} \end{pmatrix},$$

also $\mathfrak{Q} = \mathfrak{Q}_0$, $\mathfrak{H} = \mathfrak{H}_0$, so dass $\mathfrak{W}_0$ auf grund von (51) eine (50) genügende Einheit von $\mathfrak{H}_0$ ist.

Ist umgekehrt $\mathfrak{W}_0$ eine Einheit von $\mathfrak{H}_0$, für welche die durch (50) festgelegte Matrix $\mathfrak{F}_0$ ganz ist, so gilt (52) mit $\mathfrak{Q} = \mathfrak{Q}_0$, und (53) zeigt, dass die Matrix $\begin{pmatrix} \mathfrak{E} & \mathfrak{F}_0 \\ \mathfrak{N} & \mathfrak{W}_0 \end{pmatrix}$ eine Einheit von $\mathfrak{U}'\mathfrak{S}\mathfrak{U}$ ist. Dann ist aber die durch (49) definierte Matrix $\mathfrak{A}$ eine Einheit von $\mathfrak{S}$, die offenbar $\mathfrak{C}$ invariant lässt.

Jetzt lassen sich die Hilfssätze 5, 9, 10 übertragen.

HILFSSATZ 12: *Es sei nicht zugleich* $m - n = 2$ *und* $-|\mathfrak{S}||\mathfrak{T}|$ *eine Quadratzahl.*

Hat das Gebiet B im Raume der Paare $\mathfrak{Q}, \mathfrak{R}$ *einen Inhalt, so hat auch das durch die Gleichung* $(\mathfrak{C}\mathfrak{Y})'\mathfrak{S}(\mathfrak{C}\mathfrak{Y}) = \begin{pmatrix} \mathfrak{T} & \mathfrak{Q} \\ \mathfrak{Q}' & \mathfrak{R} \end{pmatrix}$ *bestimmte* $\mathfrak{Y}$*-Gebiet* $\bar{B}$*, das in bezug auf die Gruppe der* $\mathfrak{C}$ *invariant lassenden Einheiten von* $\mathfrak{S}$ *reduciert ist, einen endlichen Inhalt.*

BEWEIS: Es gibt eine primitive Matrix $\mathfrak{C}_1^{(m,n)}$ und eine ganze Matrix $\mathfrak{B}^{(n)}$, so dass $\mathfrak{C} = \mathfrak{C}_1\mathfrak{B}$ ist. Durch die Substitutionen $\mathfrak{T} = \mathfrak{B}'\mathfrak{T}_1\mathfrak{B}$, $\mathfrak{Q} = \mathfrak{B}'\mathfrak{Q}_1$ geht dann die gegebene Gleichung in $(\mathfrak{C}_1\mathfrak{Y})'\mathfrak{S}(\mathfrak{C}_1\mathfrak{Y}) = \begin{pmatrix} \mathfrak{T}_1 & \mathfrak{Q}_1 \\ \mathfrak{Q}_1' & \mathfrak{R} \end{pmatrix}$ über, mit primitivem $\mathfrak{C}_1$. Also genügt zum Beweise der Behauptung die Untersuchung des Falles eines primitiven $\mathfrak{C}$. Es sei $\mathfrak{Y}_0$ ein Complement von $\mathfrak{C}$ und $(\mathfrak{C}\mathfrak{Y}_0) = \mathfrak{U}$. Man setze wieder $\mathfrak{U}'\mathfrak{S}\mathfrak{U} = \begin{pmatrix} \mathfrak{T} & \mathfrak{Q}_0 \\ \mathfrak{Q}_0' & \mathfrak{R}_0 \end{pmatrix}$, $\mathfrak{H}_0 = \mathfrak{R}_0 - \mathfrak{Q}_0'\mathfrak{T}^{-1}\mathfrak{Q}_0$. Durch die Substitution

$$(54) \qquad (\mathfrak{C}\mathfrak{Y}) = \mathfrak{U}\begin{pmatrix} \mathfrak{E} & \mathfrak{F} \\ \mathfrak{N} & \mathfrak{W} \end{pmatrix},$$

d.h. $\mathfrak{Y} = \mathfrak{C}\mathfrak{F} + \mathfrak{Y}_0\mathfrak{W}$, ist der $\mathfrak{Y}$-Raum auf den $\mathfrak{F}, \mathfrak{W}$-Raum abgebildet, und dieser geht wegen (52) und (53) durch die Substitutionen

$$(55) \qquad \mathfrak{T}\mathfrak{F} + \mathfrak{Q}_0\mathfrak{W} = \mathfrak{Q}, \qquad \mathfrak{W}'\mathfrak{H}_0\mathfrak{W} = \mathfrak{H}, \qquad \mathfrak{H} + \mathfrak{Q}'\mathfrak{T}^{-1}\mathfrak{Q} = \mathfrak{R}$$

in den $\mathfrak{Q}, \mathfrak{R}$-Raum über. Sind nun $\mathfrak{Y}_1$, $\mathfrak{Y}_2$ zwei associierte Punkte des $\mathfrak{Y}$-Raumes, also $\mathfrak{Y}_2 = \mathfrak{A}\mathfrak{Y}_1$, wo $\mathfrak{A}$ eine $\mathfrak{C}$ invariant lassende Einheit von $\mathfrak{S}$ bedeutet, so gilt für die zugehörigen Wertepaare $\mathfrak{F}_1, \mathfrak{W}_1$ und $\mathfrak{F}_2, \mathfrak{W}_2$ von $\mathfrak{F}, \mathfrak{W}$ nach (54) die Formel

$$\mathfrak{A}\mathfrak{U}\begin{pmatrix} \mathfrak{E} & \mathfrak{F}_1 \\ \mathfrak{N} & \mathfrak{W}_1 \end{pmatrix} = \mathfrak{U}\begin{pmatrix} \mathfrak{E} & \mathfrak{F}_2 \\ \mathfrak{N} & \mathfrak{W}_2 \end{pmatrix},$$

aus der vermöge (49) die Beziehungen

$$\mathfrak{F}_2 = \mathfrak{F}_1 + \mathfrak{F}_0\mathfrak{W}_1, \qquad \mathfrak{W}_2 = \mathfrak{W}_0\mathfrak{W}_1$$

folgen; dabei ist $\mathfrak{W}_0$ nach Hilfssatz 11 eine Einheit von $\mathfrak{H}_0$ und

$$\mathfrak{T}^{-1}\mathfrak{Q}_0(\mathfrak{E} - \mathfrak{W}_0) = \mathfrak{F}_0$$

ganz. Nun bilden die $\mathfrak{W}_0$ mit ganzem $\mathfrak{F}_0$ eine Untergruppe von endlichem Index in bezug auf die volle Einheitengruppe von $\mathfrak{H}_0$. Ferner ist $|\mathfrak{S}| = |\mathfrak{H}_0||\mathfrak{T}|$ nach (53), also nicht zugleich $m - n = 2$ und $-|\mathfrak{H}_0|$ ein Quadrat. Durchläuft $\mathfrak{H}$ ein quadrierbares Gebiet, so ist auf grund von Hilfssatz 5 das durch $\mathfrak{W}'\mathfrak{H}_0\mathfrak{W} = \mathfrak{H}$ bestimmte bezüglich jener Untergruppe reducierte $\mathfrak{W}$-Gebiet quadrierbar. Endlich ist nach (55) das $\mathfrak{Q}, \mathfrak{H}$-Gebiet zugleich mit dem $\mathfrak{Q}, \mathfrak{R}$-Gebiet quadrierbar, und das $\mathfrak{F}, \mathfrak{W}$-Gebiet zugleich mit dem $\mathfrak{Q}, \mathfrak{W}$-Gebiet. Hieraus folgt die Behauptung.

Fortan sei der Fall ausgeschlossen, dass $m - n = 2$ und $-|\mathfrak{S}||\mathfrak{T}|$ ein Quadrat ist.

Hilfssatz 13: *Es sei $v(B)$ der Inhalt des Gebietes B im $\mathfrak{Q}, \mathfrak{R}$-Raum und $v(\bar{B})$ der Inhalt des vermöge* $(\mathfrak{C}\mathfrak{Y})'\mathfrak{S}(\mathfrak{C}\mathfrak{Y}) = \begin{pmatrix} \mathfrak{T} & \mathfrak{Q} \\ \mathfrak{Q}' & \mathfrak{R} \end{pmatrix}$ *zugehörigen reducierten Gebietes $\bar{B}$ im $\mathfrak{Y}$-Raum. Lässt man dann B auf einen Punkt $\mathfrak{Q}, \mathfrak{R}$ zusammenschrumpfen, so existiert gleichmässig in $\mathfrak{Q}, \mathfrak{R}$ der Grenzwert*

$$\lim_{B \to \mathfrak{Q}, \mathfrak{R}} \frac{v(\bar{B})}{v(B)} = \rho(\mathfrak{S}, \mathfrak{C}) \left| \mathfrak{S} \begin{pmatrix} \mathfrak{T} & \mathfrak{Q} \\ \mathfrak{Q}' & \mathfrak{R} \end{pmatrix}^{-1} \right|^{\frac{1}{2}}, \tag{56}$$

wo $\rho(\mathfrak{S}, \mathfrak{C})$ nur von $\mathfrak{S}$ und $\mathfrak{C}$ abhängt.

Beweis: Man setze $(\mathfrak{C}\mathfrak{Y}_1)'\mathfrak{S}(\mathfrak{C}\mathfrak{Y}_1) = \begin{pmatrix} \mathfrak{T} & \mathfrak{Q}_1 \\ \mathfrak{Q}_1' & \mathfrak{R}_1 \end{pmatrix}$, $(\mathfrak{C}\mathfrak{Y}) = (\mathfrak{C}\mathfrak{Y}_1)\begin{pmatrix} \mathfrak{E} & \mathfrak{F} \\ \mathfrak{R} & \mathfrak{W} \end{pmatrix}$, also $\mathfrak{Y} = \mathfrak{C}\mathfrak{F} + \mathfrak{Y}_1\mathfrak{W}$, $\mathfrak{Q} = \mathfrak{T}\mathfrak{F} + \mathfrak{Q}_1\mathfrak{W}$, $\mathfrak{R} = \mathfrak{W}'\mathfrak{R}_1\mathfrak{W} + \mathfrak{F}'\mathfrak{Q}_1\mathfrak{W} + \mathfrak{W}'\mathfrak{Q}_1'\mathfrak{F} + \mathfrak{F}'\mathfrak{T}\mathfrak{F}$. Es bedeute B_1 das B entsprechende $\mathfrak{Q}_1, \mathfrak{R}_1$-Gebiet. Da die Functionaldeterminanten von $\mathfrak{Y}$ nach $\mathfrak{Y}_1$, $\mathfrak{Q}$ nach $\mathfrak{Q}_1$, $\mathfrak{R}$ nach $\mathfrak{R}_1$ die Werte $|\mathfrak{W}|^m$, $|\mathfrak{W}|^n$, $|\mathfrak{W}|^{m-n+1}$ haben, so ist

$$v(\bar{B}) = |\mathfrak{W}|^m v(\bar{B}_1), \qquad v(B) = |\mathfrak{W}|^{m+1} v(B_1).$$

Ferner ist

$$|(\mathfrak{C}\mathfrak{Y})| = |(\mathfrak{C}\mathfrak{Y}_1)||\mathfrak{W}|$$

$$|(\mathfrak{C}\mathfrak{Y})|^2 |\mathfrak{S}| = \begin{vmatrix} \mathfrak{T} & \mathfrak{Q} \\ \mathfrak{Q}' & \mathfrak{R} \end{vmatrix}$$

und folglich

$$\left| \mathfrak{S}^{-1} \begin{pmatrix} \mathfrak{T} & \mathfrak{Q} \\ \mathfrak{Q}' & \mathfrak{R} \end{pmatrix} \right|^{\frac{1}{2}} \frac{v(\bar{B})}{v(B)} = \left| \mathfrak{S}^{-1} \begin{pmatrix} \mathfrak{T} & \mathfrak{Q}_1 \\ \mathfrak{Q}_1' & \mathfrak{R}_1 \end{pmatrix} \right|^{\frac{1}{2}} \frac{v(\bar{B}_1)}{v(B_1)}.$$

Die Behauptung ist also bewiesen, wenn man die Existenz des Grenzwertes in (56) gezeigt hat. Hierzu kann man entweder die Gedanken des Beweises von

Hilfssatz 9 wiederholen oder die beim Beweise von Hilfssatz 12 benutzte Transformation anwenden.

Aus Hilfssatz 13 folgt

HILFSSATZ 14: *Es sei $f(\mathfrak{Q}, \mathfrak{R})$ in B integrierbar. Dann ist*

$$\int_{\bar{B}} f(\mathfrak{C}'\mathfrak{S}\mathfrak{Y}, \mathfrak{Y}'\mathfrak{S}\mathfrak{Y})\, d\mathfrak{Y} = \rho(\mathfrak{S}, \mathfrak{C}) \int_B f(\mathfrak{Q}, \mathfrak{R}) \left| \mathfrak{S} \begin{pmatrix} \mathfrak{C}'\mathfrak{S}\mathfrak{C} & \mathfrak{Q} \\ \mathfrak{Q}' & \mathfrak{R} \end{pmatrix}^{-1} \right|^{\frac{1}{2}} d\mathfrak{Q}\, d\mathfrak{R}.$$

Ist für ein reelles $\mathfrak{C}$ die Matrix $\mathfrak{C}'\mathfrak{S}\mathfrak{C} = \mathfrak{D}$ eine Diagonalmatrix, so hängt die Anzahl μ der positiven Diagonalelemente nur von $\mathfrak{S}$ und nicht von $\mathfrak{C}$ *ab*, ebenso die Anzahl ν der negativen Diagonalelemente. Es heisse $\mu - \nu$ die Signatur von $\mathfrak{S}$.

HILFSSATZ 15: *Es sei G ein Gebiet im Raume der Matrizen*

$$\mathfrak{Z}^{(m)} = \begin{pmatrix} \mathfrak{T}^{(n)} & \mathfrak{Q} \\ \mathfrak{Q}' & \mathfrak{R} \end{pmatrix},$$

in welchem die Gleichung $\mathfrak{X}'\mathfrak{S}\mathfrak{X} = \mathfrak{Z}$ eine reelle Lösung $\mathfrak{X}$ mit $|\mathfrak{X}| \neq 0$ hat, und es sei dort die Differenz der Signaturen von $\mathfrak{S}$ und $\mathfrak{T}$ constant gleich $m - n$ oder $n - m$. Es sei $\bar{G}$ das in bezug auf die Einheitengruppe von $\mathfrak{S}$ reducierte $\mathfrak{X}$-Gebiet. Bedeutet $\mathfrak{C}_0^{(m,n)}$ eine ganze Matrix und q eine natürliche Zahl, so gilt

$$\int_{\bar{G}} f(\mathfrak{X}'\mathfrak{S}\mathfrak{X})\, d\mathfrak{X} = \lim_{t\to\infty} \left(\frac{q}{t}\right)^{mn} \sum_{\substack{\mathfrak{C} \equiv \mathfrak{C}_0 \pmod q \\ (t^{-1}\mathfrak{C},\, \mathfrak{Y}) \text{ in } \bar{G}}} \int f((t^{-1}\mathfrak{C}, \mathfrak{Y})' \mathfrak{S} (t^{-1}\mathfrak{C}, \mathfrak{Y}))\, d\mathfrak{Y}.$$

BEWEIS: Bei jedem festen $\mathfrak{T}$ wird aus G ein $\mathfrak{Q}$, $\mathfrak{R}$-Gebiet B ausgeschnitten. Ist $t^{-2}\, \mathfrak{C}'\mathfrak{S}\mathfrak{C} = \mathfrak{T}$ bei ganzem $\mathfrak{C}$, so sei $\bar{B}$ das in bezug auf die Gruppe der $\mathfrak{C}$ invariant lassenden Einheiten von $\mathfrak{S}$ reducierte $\mathfrak{Y}$-Gebiet, welches durch $t^{-1}\, \mathfrak{C}'\mathfrak{S}\mathfrak{Y} = \mathfrak{Q}$, $\mathfrak{Y}'\mathfrak{S}\mathfrak{Y} = \mathfrak{R}$ festgelegt wird. Auf grund von Hilfssatz 7 hat man zum Beweise von Hilfssatz 15 nur zu zeigen, dass

$$\lim_{u\to\infty} \left(\frac{q}{u}\right)^{m(m-n)} \sum_{\substack{\mathfrak{Y} \equiv \mathfrak{Y}_0 \left(\mathrm{mod}\, \frac{q}{u}\right) \\ \mathfrak{Y} \text{ in } \bar{B}}} f((t^{-1}\mathfrak{C}, \mathfrak{Y})' \mathfrak{S} (t^{-1}\mathfrak{C}, \mathfrak{Y})) = \int_{\bar{B}} f((t^{-1}\mathfrak{C}, \mathfrak{Y})' \mathfrak{S} (t^{-1}\mathfrak{C}, \mathfrak{Y}))\, d\mathfrak{Y} \tag{57}$$

ist, und zwar gleichmässig in $\mathfrak{C}$.

Sind nun σ und τ die Signaturen von $\mathfrak{S}$ und $\mathfrak{T}$, so hat nach (53) die in Hilfssatz 11 mit $\mathfrak{H}_0$ bezeichnete Matrix die Signatur $\sigma - \tau$, ist also definit unter der Voraussetzung von Hilfssatz 15. Wie Minkowski bewiesen hat, ist nun aber die Ordnung der Einheitengruppe einer definiten quadratischen Form mit fester Variabelnzahl beschränkt. Aus dem Beweise von Hilfssatz 12 folgt jetzt ohne Mühe, dass (57) gleichmässig in $\mathfrak{C}$ richtig ist.

HILFSSATZ 16: *Es sei H ein Gebiet im $\mathfrak{T}$-Raum, in dem die Differenz der Signaturen von $\mathfrak{S}$ und $\mathfrak{T}$ constant gleich $m - n$ oder $n - m$ ist. Für jede in H integrierbare Function $f(\mathfrak{T})$ gilt dann*

$$(58)\quad \rho(\mathfrak{S})\int_H f(\mathfrak{T})\,d\mathfrak{T} = \lim_{t\to\infty} q^{mn}t^{-\frac{n(n+1)}{2}} \sum_{\substack{\mathfrak{C}=\mathfrak{C}_0 \pmod q \\ t^{-1}\mathfrak{C}'\mathfrak{S}\mathfrak{C} \text{ in } H}} \rho(\mathfrak{S},\mathfrak{C})\,f(t^{-1}\mathfrak{C}'\mathfrak{S}\mathfrak{C}),$$

wobei $\mathfrak{C}$ ein volles System nicht-associierter Matrizen $\equiv \mathfrak{C}_0$ (mod q) durchläuft, für welche $t^{-1}\mathfrak{C}'\mathfrak{S}\mathfrak{C}$ in H liegt.

Beweis: Schreibt man für $\mathfrak{Q}$ und $\mathfrak{R}$ geeignete von $\mathfrak{T}$ unabhängige Bereiche vor, so wird H in ein Gebiet G des Raumes der Matrizen $\mathfrak{Z} = \begin{pmatrix} \mathfrak{T} & \mathfrak{Q} \\ \mathfrak{Q}' & \mathfrak{R} \end{pmatrix}$ eingebettet. Die Behauptung folgt jetzt aus den Hilfssätzen 10, 14, 15, indem man unter $f(\mathfrak{Z})$ die specielle Function $f(\mathfrak{T})\,|\,\mathfrak{S}^{-1}\mathfrak{Z}\,|^{\frac{1}{2}}$ versteht.

Hilfssatz 17: *Es sei $\mathfrak{C}^{(m,n)}$ primitiv, $\mathfrak{B}$ ein Complement zu $\mathfrak{C}$, $\mathfrak{C}'\mathfrak{S}\mathfrak{C} = \mathfrak{T}$, $\mathfrak{C}'\mathfrak{S}\mathfrak{B} = \mathfrak{Q}$, $\mathfrak{B}'\mathfrak{S}\mathfrak{B} - \mathfrak{Q}'\mathfrak{T}^{-1}\mathfrak{Q} = \mathfrak{H}$ und λ die Anzahl der modulo 1 incongruenten Matrizen $\mathfrak{T}^{-1}\mathfrak{Q}\mathfrak{W}$, falls $\mathfrak{W}$ alle Einheiten von $\mathfrak{H}$ durchläuft. Bedeutet T den absoluten Betrag der Determinante von $\mathfrak{T}$, so ist*

$$\rho(\mathfrak{S},\mathfrak{C}) = \lambda T^{n-m}\rho(\mathfrak{H}).$$

Beweis: Die Zahl λ ist zugleich der Index der Untergruppe derjenigen Einheiten $\mathfrak{W}$, für welche $\mathfrak{T}^{-1}\mathfrak{Q}(\mathfrak{E}-\mathfrak{W})$ ganz ist. Setzt man nun analog wie beim Beweise von Hilfssatz 12

$$\mathfrak{X} = (\mathfrak{C}\mathfrak{Y}) = (\mathfrak{C}\mathfrak{B})\begin{pmatrix} \mathfrak{E} & \mathfrak{F} \\ \mathfrak{N} & \mathfrak{W}_1 \end{pmatrix}, \quad \mathfrak{X}'\mathfrak{S}\mathfrak{X} = \begin{pmatrix} \mathfrak{T} & \mathfrak{Q}_1 \\ \mathfrak{Q}_1' & \mathfrak{R}_1 \end{pmatrix}, \quad \mathfrak{H}_1 = \mathfrak{R}_1 - \mathfrak{Q}_1'\mathfrak{T}^{-1}\mathfrak{Q}_1,$$

also $\mathfrak{Y} = \mathfrak{C}\mathfrak{F} + \mathfrak{B}\mathfrak{W}_1$, $\mathfrak{Q}_1 = \mathfrak{T}\mathfrak{F} + \mathfrak{Q}\mathfrak{W}_1$, $\mathfrak{H}_1 = \mathfrak{W}_1'\mathfrak{H}\mathfrak{W}_1$, so haben die Functionaldeterminanten von $\mathfrak{H}_1$ nach $\mathfrak{R}_1$, $\mathfrak{F}$ nach $\mathfrak{Q}_1$, $\mathfrak{Y}$ nach $\mathfrak{F}$, $\mathfrak{W}_1$ die Werte 1, $|\mathfrak{T}|^{n-m}$, $|(\mathfrak{C}\mathfrak{B})|^{m-n} = 1$. Ist B ein $\mathfrak{Q}_1, \mathfrak{R}_1$-Gebiet, $\bar{B}$ das zugehörige reducierte $\mathfrak{Y}$-Gebiet und B_1 das durch $\bar{B}$ eindeutig fixierte $\mathfrak{Q}_1, \mathfrak{W}_1$-Gebiet, so ist B_1 reduciert in bezug auf die Untergruppe der Einheiten $\mathfrak{W}$ von $\mathfrak{H}$ mit ganzem $\mathfrak{T}^{-1}\mathfrak{Q}(\mathfrak{E}-\mathfrak{W})$. Folglich ist nach Hilfssatz 9

$$(59)\quad \lim_{B\to\mathfrak{Q},\mathfrak{R}} \frac{v(B_1)}{v(B)} = \lambda\rho(\mathfrak{H}),$$

wo $\mathfrak{R} = \mathfrak{B}'\mathfrak{S}\mathfrak{B}$ gesetzt ist. Ferner gilt

$$(60)\quad \frac{v(\bar{B})}{v(B_1)} = T^{n-m}$$

und nach Hilfssatz 13

$$(61)\quad \lim_{B\to\mathfrak{Q},\mathfrak{R}} \frac{v(\bar{B})}{v(B)} = \rho(\mathfrak{S},\mathfrak{C}).$$

Aus (59), (60), (61) folgt die Behauptung.

Hilfssatz 18: *Es sei $\mathfrak{A}^{(n)}$ grösster rechtsseitiger Teiler von $\mathfrak{C}^{(m,n)}$, also $\mathfrak{A}$ ganz, $\mathfrak{C}\mathfrak{A}^{-1} = \mathfrak{C}_1$ primitiv. Es bedeute A den absoluten Betrag von $|\mathfrak{A}|$. Dann ist*

$$\rho(\mathfrak{S},\mathfrak{C}) = A^{n-m+1}\rho(\mathfrak{S},\mathfrak{C}_1).$$

BEWEIS: Setzt man $\mathfrak{Q} = \mathfrak{A}'\mathfrak{Q}_1$, $\mathfrak{T} = \mathfrak{A}'\mathfrak{T}_1\mathfrak{A}$, $(\mathfrak{C}\mathfrak{Y})'\mathfrak{S}(\mathfrak{C}\mathfrak{Y}) = \begin{pmatrix} \mathfrak{T} & \mathfrak{Q} \\ \mathfrak{Q}' & \mathfrak{R} \end{pmatrix}$, so ist $(\mathfrak{C}_1\mathfrak{Y})'\mathfrak{S}(\mathfrak{C}_1\mathfrak{Y}) = \begin{pmatrix} \mathfrak{T}_1 & \mathfrak{Q}_1 \\ \mathfrak{Q}_1' & \mathfrak{R} \end{pmatrix}$. Die Functionaldeterminante von $\mathfrak{Q}$ nach $\mathfrak{Q}_1$ ist $|\mathfrak{A}|^{m-n}$. Entspricht dem $\mathfrak{Q}$, $\mathfrak{R}$-Gebiete B das $\mathfrak{Q}_1$, $\mathfrak{R}$-Gebiet B_1, so ist

$$\frac{v(B)}{v(B_1)} = A^{m-n}.$$

Andererseits gilt nach Hilfssatz 13

$$\lim_{B\to\mathfrak{Q},\mathfrak{R}} \frac{v(B)}{v(B)} = \rho(\mathfrak{S}, \mathfrak{C}) \left| \mathfrak{S} \begin{pmatrix} \mathfrak{T} & \mathfrak{Q} \\ \mathfrak{Q}' & \mathfrak{R} \end{pmatrix}^{-1} \right|^{\frac{1}{2}}$$

$$\lim_{B\to\mathfrak{Q}_1,\mathfrak{R}} \frac{v(B)}{v(B_1)} = \rho(\mathfrak{S}, \mathfrak{C}_1) \left| \mathfrak{S} \begin{pmatrix} \mathfrak{T}_1 & \mathfrak{Q}_1 \\ \mathfrak{Q}_1' & \mathfrak{R} \end{pmatrix}^{-1} \right|^{\frac{1}{2}}.$$

Die Behauptung ergibt sich mit Hilfe von

$$\begin{vmatrix} \mathfrak{T} & \mathfrak{Q} \\ \mathfrak{Q}' & \mathfrak{R} \end{vmatrix} = A^2 \begin{vmatrix} \mathfrak{T}_1 & \mathfrak{Q}_1 \\ \mathfrak{Q}_1' & \mathfrak{R} \end{vmatrix}.$$

ZWEITES CAPITEL: BEWEIS DES HAUPTSATZES

Der Beweis erfolgt nach dem Vorbild des im ersten Teil untersuchten Falles positiv-definiter $\mathfrak{S}$. Zunächst überzeuge man sich durch eine triviale Überlegung, dass der Hauptsatz richtig ist im Specialfalle $m = 1$, $\mathfrak{S} = \mathfrak{T}$. Ferner zeigt man leicht durch dieselbe Betrachtung wie im ersten Teile, dass aus der Richtigkeit im Falle $m = n$, $\mathfrak{S} = \mathfrak{T}$ die Richtigkeit für $m = n$ und beliebiges $\mathfrak{T}$ folgt. Ausführlicher müssen nur die beiden andern Schritte des Beweises verfolgt werden: Erstens wird arithmetisch bewiesen, dass aus der Richtigkeit für $m = n < m_0$, $\mathfrak{S} = \mathfrak{T}$ die Richtigkeit für $m = m_0 > n$ und für $m = n = m_0$, $\mathfrak{S} = \mathfrak{T}$ folgt, bis auf einen nur von $\mathfrak{S}$ und nicht von $\mathfrak{T}$ abhängigen Factor $\gamma(\mathfrak{S})$ auf der rechten Seite der Formel des Hauptsatzes; zweitens wird analytisch gezeigt, dass $\gamma(\mathfrak{S}) = 1$ ist.

3. Arithmetischer Teil des Beweises

Es sei $\mathfrak{C}$ eine primitive Lösung von $\mathfrak{C}'\mathfrak{S}\mathfrak{C} = \mathfrak{T}$ und $\mathfrak{B}$ ein Complement von $\mathfrak{C}$. Setzt man $\mathfrak{C}'\mathfrak{S}\mathfrak{B} = \mathfrak{Q}$, $\mathfrak{B}'\mathfrak{S}\mathfrak{B} - \mathfrak{Q}'\mathfrak{T}^{-1}\mathfrak{Q} = \mathfrak{H}$, so ist nach Hilfssatz 17

$$\rho(\mathfrak{S}, \mathfrak{C}) = \lambda T^{n-m} \rho(\mathfrak{H}), \tag{62}$$

wo T den absoluten Betrag der Determinante von $\mathfrak{T}$ und λ die Anzahl der modulo 1 incongruenten Matrizen $\mathfrak{T}^{-1}\mathfrak{Q}\mathfrak{W}$ bedeutet, falls $\mathfrak{W}$ alle Einheiten von $\mathfrak{H}$ durchläuft. Da das allgemeine Complement von $\mathfrak{C}$ die Form $\mathfrak{C}\mathfrak{F} + \mathfrak{B}\mathfrak{W}$ mit ganzem $\mathfrak{F}$ und unimodularem $\mathfrak{W}$ hat, so sind $\mathfrak{T}\mathfrak{F} + \mathfrak{Q}\mathfrak{W}$ und $\mathfrak{W}'\mathfrak{H}\mathfrak{W}$ die zugehörigen

Werte von $\mathfrak{Q}$ und $\mathfrak{H}$. Folglich ist die Classe von $\mathfrak{H}$ durch $\mathfrak{C}$ eindeutig festgelegt. Wählt man für $\mathfrak{H}$ einen festen Classenrepräsentanten, so liefern die zu $\mathfrak{C}$ gehörigen $\mathfrak{Q}$ genau λ Restclassen von $\mathfrak{T}^{-1}\mathfrak{Q}$ modulo 1.

Gehören ferner zu zwei primitiven Lösungen $\mathfrak{C}_1$ und $\mathfrak{C}_2$ dieselben Paare $\mathfrak{Q}$, $\mathfrak{H}$, so folgt aus

$$(\mathfrak{C}\mathfrak{B})'\mathfrak{S}(\mathfrak{C}\mathfrak{B}) = \begin{pmatrix} \mathfrak{T} & \mathfrak{Q} \\ \mathfrak{Q}' & \mathfrak{H} + \mathfrak{Q}'\mathfrak{T}^{-1}\mathfrak{Q} \end{pmatrix}, \tag{63}$$

dass $(\mathfrak{C}_2\mathfrak{B}_2)\,(\mathfrak{C}_1\mathfrak{B}_1)^{-1}$ eine Einheit von $\mathfrak{S}$ ist; es sind also $\mathfrak{C}_1$ und $\mathfrak{C}_2$ associiert.

Man betrachte nun die Repräsentanten $\mathfrak{S}_1, \mathfrak{S}_2, \cdots$ der verschiedenen Classen des Geschlechtes von $\mathfrak{S}$. Nach Hilfssatz 4 sind es endlich viele. Für jedes $\mathfrak{S}_k$ summiere man (62) über ein volles System nicht-associierter primitiver $\mathfrak{C}$, die zu demselben $\mathfrak{H}$ gehören, und dann summiere man noch über die einzelnen $\mathfrak{S}_k$. Nach dem in den vorigen beiden Absätzen Bewiesenen ist dann

$$\sum_{\mathfrak{S}_k,\,\mathfrak{C}} \rho(\mathfrak{S}, \mathfrak{C}) = T^{n-m} F(\mathfrak{H}, \mathfrak{S})\, \rho(\mathfrak{H}), \tag{64}$$

wo $F(\mathfrak{H}, \mathfrak{S})$ die Anzahl derjenigen $\mathfrak{Q}$ bedeutet, für welche die Werte $\mathfrak{T}^{-1}\mathfrak{Q}$ modulo 1 incongruent sind und die rechte Seite von (63) dem Geschlechte von $\mathfrak{S}$ angehört. Genau wie im ersten Teile folgt, dass $F(\mathfrak{H}, \mathfrak{S})$ nur von dem Geschlechte von $\mathfrak{H}$ abhängt.

Man setze nun

$$\beta(\mathfrak{S}, \mathfrak{T}) = \sum_{\mathfrak{C}'\mathfrak{S}\mathfrak{C} = \mathfrak{T}} \rho(\mathfrak{S}, \mathfrak{C}),$$

wo $\mathfrak{C}$ alle primitiven nicht-associierten Lösungen durchläuft, und

$$\nu(\mathfrak{S}, \mathfrak{T}) = \beta(\mathfrak{S}_1, \mathfrak{T}) + \beta(\mathfrak{S}_2, \mathfrak{T}) + \cdots$$

$$\mu(\mathfrak{S}) = \rho(\mathfrak{S}_1) + \rho(\mathfrak{S}_2) + \cdots.$$

Aus (64) ergibt sich dann

$$\nu(\mathfrak{S}, \mathfrak{T}) = T^{n-m} \sum_{\{\mathfrak{H}\}} F(\mathfrak{H}, \mathfrak{S})\, \mu(\mathfrak{H}), \tag{65}$$

wo $\mathfrak{H}^{(m-n)}$ alle Geschlechtsrepräsentanten durchläuft.

Andererseits sei $B_q(\mathfrak{S}, \mathfrak{T})$ die Anzahl der modulo q incongruenten primitiven Lösungen $\mathfrak{C}$ von $\mathfrak{C}'\mathfrak{S}\mathfrak{C} \equiv \mathfrak{T} \pmod q$ und $E_q(\mathfrak{S})$ die Anzahl der Lösungen von $\mathfrak{X}'\mathfrak{S}\mathfrak{X} \equiv \mathfrak{S} \pmod q$. Ist dann $(2ST^m)^4$ ein Teiler von q, so folgt genau wie im ersten Teile die Gleichung

$$\frac{B_q(\mathfrak{S}, \mathfrak{T})}{E_q(\mathfrak{S})} = T^{\frac{1}{2}(m-n)(m-n-1)} \sum_{\{\mathfrak{H}\}} \frac{F(\mathfrak{H}, \mathfrak{S})}{E_q(T\mathfrak{H})}. \tag{66}$$

Nach Hilfssatz 9 ist ferner

$$\mu(\mathfrak{H}) = T^{\frac{1}{2}(m-n)(m-n+1)}\, \mu(T\mathfrak{H}). \tag{67}$$

Der Hauptsatz sei nun richtig für $m = n < m_0$, $\mathfrak{S} = \mathfrak{T}$. Es sei $m = m_0 > n$, also $m - n < m_0$. Es sei nicht zugleich $m - n = 2$ und $-|\mathfrak{S}||\mathfrak{T}|$ eine Quadratzahl; dann ist auch nicht zugleich $m - n = 2$ und $-|T\mathfrak{H}|$ eine Quadratzahl. Wendet man den Hauptsatz an auf $T\mathfrak{H}$, $T\mathfrak{H}$ statt $\mathfrak{S}$, $\mathfrak{T}$, so folgt

$$\mu(T\mathfrak{H}) = \eta_{m-n}^{-1} \lim_{q\to\infty} \frac{2^{\omega(q)}\, q^{\frac{1}{2}(m-n)(m-n-1)}}{E_q(T\mathfrak{H})} \tag{68}$$

mit $\eta_k = \frac{1}{2}$ für $k > 1$ und $\eta_k = 1$ für $k = 1$.

Nach (65), (66), (67), (68) wird

$$\nu(\mathfrak{S}, \mathfrak{T}) = \eta_{m-n}^{-1} \lim_{q\to\infty} \frac{B_q(\mathfrak{S}, \mathfrak{T})}{E_q(\mathfrak{S})}\, 2^{\omega(q)}\, q^{\frac{1}{2}(m-n)(m-n-1)}\,. \tag{69}$$

Bedeutet $A_q(\mathfrak{S}, \mathfrak{T}, \mathfrak{A})$ die Anzahl der Lösungen von $\mathfrak{C}'\mathfrak{S}\mathfrak{C} \equiv \mathfrak{T} \pmod q$ mit primitivem $\mathfrak{C}\mathfrak{A}^{-1}$, so gilt nach Hilfssatz 17 des ersten Teils die Beziehung

$$A_q(\mathfrak{S}, \mathfrak{T}, \mathfrak{A}) = A^{n-m+1} B_q(\mathfrak{S}, \mathfrak{T}_1)\,, \tag{70}$$

falls $(2T)^3$ in q aufgeht, $\mathfrak{T} = \mathfrak{A}'\mathfrak{T}_1\mathfrak{A}$ ist und A den absoluten Betrag von $|\mathfrak{A}|$ bedeutet. Ist nun

$$\alpha(\mathfrak{S}, \mathfrak{T}) = \sum_{\mathfrak{C}'\mathfrak{S}\mathfrak{C}=\mathfrak{T}} \rho(\mathfrak{S}, \mathfrak{C})\,,$$

wo jetzt $\mathfrak{C}$ alle nicht-associierten Lösungen von $\mathfrak{C}'\mathfrak{S}\,\mathfrak{C} = \mathfrak{T}$ durchläuft, also nicht nur die primitiven, und

$$\mu(\mathfrak{S}, \mathfrak{T}) = \alpha(\mathfrak{S}_1, \mathfrak{T}) + \alpha(\mathfrak{S}_2, \mathfrak{T}) + \cdots,$$

so folgt aus (69), (70) in Verbindung mit Hilfssatz 18 die Gleichung

$$\mu(\mathfrak{S}, \mathfrak{T}) = \eta_{m-n}^{-1} \lim_{q\to\infty} \frac{A_q(\mathfrak{S}, \mathfrak{T})}{E_q(\mathfrak{S})}\, 2^{\omega(q)}\, q^{\frac{1}{2}(m-n)(m-n-1)}\,. \tag{71}$$

Falls nicht zugleich $m = 2$ und $-|\mathfrak{S}|$ ein Quadrat ist, so ist nach Hilfssatz 25 des ersten Teiles der Grenzwert $\lim_{q\to\infty} E_q(\mathfrak{S}) 2^{-\omega(q)}\, q^{-\frac{1}{2}m(m-1)}$ vorhanden und von 0 verschieden. Setzt man

$$\lim_{q\to\infty} \frac{2^{\omega(q)}\, q^{\frac{1}{2}m(m-1)}}{E_q(\mathfrak{S})} = \eta_m\, \gamma(\mathfrak{S})\, \mu(\mathfrak{S})\,, \tag{72}$$

so geht (71) über in

$$\frac{\mu(\mathfrak{S}, \mathfrak{T})}{\mu(\mathfrak{S})} = \frac{\eta_m}{\eta_{m-n}}\, \gamma(\mathfrak{S}) \lim_{q\to\infty} \frac{A_q(\mathfrak{S}, \mathfrak{T})}{q^{mn-\frac{1}{2}n(n+1)}}\,. \tag{73}$$

Da der Quotient $\eta_m : \eta_{m-n} = \frac{1}{2}$ für $m - n = 1$ und $= 1$ für $m - n > 1$ ist, so ist für den Beweis des Hauptsatzes im Falle $m = m_0 > n$ nur noch der Beweis der Gleichung $\gamma(\mathfrak{S}) = 1$ notwendig; ist dies erledigt, so zeigt (72) die Richtigkeit des Hauptsatzes für $m = n = m_0$, $\mathfrak{S} = \mathfrak{T}$.

4. Analytischer Teil des Beweises

Es sei $\mathfrak{T}_0^{(n)}$ durch $\mathfrak{S}$ ganzzahlig darstellbar und Q eine durch $8\,|\,\mathfrak{T}_0\,|^3\,|\,\mathfrak{S}\,|$ teilbare natürliche Zahl. Definiert man

$$\delta_p(\mathfrak{S}, \mathfrak{T}) = \lim_{a\to\infty} \frac{A_q(\mathfrak{S}, \mathfrak{T})}{q^{mn-\frac{n(n+1)}{2}}} \qquad (q = p^a),$$

so gilt nach Hilfssatz 13 des ersten Teils für alle Primfactoren p von Q im Falle $\mathfrak{T} \equiv \mathfrak{T}_0 \pmod{Q}$ die Gleichung $\delta_p(\mathfrak{S}, \mathfrak{T}) = \delta_p(\mathfrak{S}, \mathfrak{T}_0)$ und

$$\prod_{p\,|\,Q} \delta_p(\mathfrak{S}, \mathfrak{T}) = \frac{A_Q(\mathfrak{S}, \mathfrak{T}_0)}{Q^{mn-\frac{n(n+1)}{2}}} \neq 0\,. \tag{74}$$

Man erhält sämtliche Lösungen von $\mathfrak{C}'\mathfrak{S}\mathfrak{C} = \mathfrak{T}$, indem man zunächst alle Darstellungen mit der Nebenbedingung $\mathfrak{C} \equiv \mathfrak{C}_0 \pmod{Q}$ bestimmt und dann $\mathfrak{C}_0$ über die $A_Q(\mathfrak{S}, \mathfrak{T}_0)$ Lösungen von $\mathfrak{C}_0'\mathfrak{S}\mathfrak{C}_0 \equiv \mathfrak{T}_0 \pmod{Q}$ laufen lässt. Es bedeute H ein Gebiet im $\mathfrak{T}$-Raum, in dem die Signaturen von $\mathfrak{S}$ und $\mathfrak{T}$ die constante Differenz $m - n$ oder $n - m$ haben, und es sei $f(\mathfrak{T})$ eine in H integrierbare Function von $\mathfrak{T}$. Summiert man die Gleichung (58) von Hilfssatz 16 über die $A_Q(\mathfrak{S}, \mathfrak{T}_0)$ Werte von $\mathfrak{C}_0$ und alle Classenrepräsentanten des Geschlechts von $\mathfrak{S}$, so folgt

$$\frac{A_Q(\mathfrak{S}, \mathfrak{T}_0)}{Q^{mn-\frac{n(n+1)}{2}}}\,\mu(\mathfrak{S}) \int_H f(\mathfrak{T})d\mathfrak{T} = \lim_{t\to\infty} \left(\frac{Q}{t}\right)^{\frac{n(n+1)}{2}} \sum_{\substack{\mathfrak{T}\equiv\mathfrak{T}_0 \pmod{Q} \\ t^{-1}\mathfrak{T} \text{ in } H}} \mu(\mathfrak{S}, \mathfrak{T})\, f(t^{-1}\mathfrak{T})\,. \tag{75}$$

Man setze noch

$$\prod_{(p,\,Q)=1} \delta_p(\mathfrak{S}, \mathfrak{T}) = \begin{cases} \Theta_Q(\mathfrak{S}, \mathfrak{T}) & (m - n > 1) \\ 2\Theta_Q(\mathfrak{S}, \mathfrak{T}) & (m - n = 1); \end{cases}$$

dann liefern (73), (74), (75) die Formel

$$\int_H f(\mathfrak{T})d\mathfrak{T} = \gamma(\mathfrak{S}) \lim_{t\to\infty} \left(\frac{Q}{t}\right)^{\frac{n(n+1)}{2}} \sum_{\substack{\mathfrak{T}\equiv\mathfrak{T}_0 \pmod{Q} \\ t^{-1}\mathfrak{T} \text{ in } H}} \Theta_Q(\mathfrak{S}, \mathfrak{T})\, f(t^{-1}\mathfrak{T})\,. \tag{76}$$

Nun ist $\mathfrak{S}$ nach jedem zu $2\,|\,\mathfrak{S}\,|$ teilerfremden Modul äquivalent mit der Diagonalmatrix, deren Diagonalelemente die Werte $|\,\mathfrak{S}\,|, 1, \cdots, 1$ sind. Für $|\,\mathfrak{S}_1^{(m)}\,| = |\,\mathfrak{S}\,|$ ist daher $\Theta_Q(\mathfrak{S}_1, \mathfrak{T}) = \Theta_Q(\mathfrak{S}, \mathfrak{T})$.

Gibt es nun ein durch $\mathfrak{S}_1$ und durch $\mathfrak{S}$ ganzzahlig darstellbares $\mathfrak{T}_0$ von der Art, dass die Differenz der Signaturen von $\mathfrak{S}_1$ und $\mathfrak{T}_0$ und ebenfalls die Differenz der Signaturen von $\mathfrak{S}$ und $\mathfrak{T}_0$ den Wert $\pm\,(m - n)$ hat, so ist $\gamma(\mathfrak{S}_1) = \gamma(\mathfrak{S})$ auf grund von (76). Es gibt für jedes $n \leqq \frac{m}{2}$ eine rationale Matrix $\mathfrak{P}^{(m)}$ mit der

Determinante 1, so dass

$$\mathfrak{P}'\mathfrak{S}\mathfrak{P} = \begin{pmatrix} \mathfrak{D}_1^{(n)} & \mathfrak{N} \\ \mathfrak{N} & \mathfrak{D}_2 \end{pmatrix} \tag{77}$$

eine Diagonalmatrix ist, bei der die Diagonalelemente von $\mathfrak{D}_2^{(m-n)}$ alle das gleiche Vorzeichen haben. Es sei g der Hauptnenner der Elemente von $\mathfrak{P}$. Setzt man

$$g^2 \begin{pmatrix} \mathfrak{D}_1 & \mathfrak{N} \\ \mathfrak{N} & -\mathfrak{D}_2 \end{pmatrix} = \mathfrak{S}_1 ,$$

so stellen $g^2\mathfrak{S}$ und $\mathfrak{S}_1$ beide $g^2\mathfrak{D}_1$ dar und die Bedingung über die Differenz der Signaturen ist auch erfüllt. Ferner ist $|\, g^2\mathfrak{S}\,| = (-1)^{m-n}\,|\,\mathfrak{S}_1\,|$. Da nun nach (72) die Beziehung $\gamma(g^2\mathfrak{S}) = \gamma(\mathfrak{S})$ gilt, so ist $\gamma(\mathfrak{S}) = \gamma(\mathfrak{S}_1)$ im Falle eines geraden $m - n$.

Nach dem Hauptsatze des ersten Teiles ist $\gamma(\mathfrak{S}) = 1$ für definites $\mathfrak{S}$. Also sei jetzt $\mathfrak{S}$ indefinit, und es seien μ und ν die Anzahlen der positiven und negativen Diagonalelemente in (77). Ist ν gerade, so wähle man $m - n = \nu$ und die Elemente von $\mathfrak{D}_2$ negativ, also die von $\mathfrak{D}_1$ positiv. Da dann $\mathfrak{S}_1$ positiv definit ist, so folgt $\gamma(\mathfrak{S}) = 1$. Ebenso erledigt man den Fall eines geraden μ. Nun seien μ und ν beide ungerade, also $\mu + \nu = m$ gerade.

Zunächst sei $\nu = 1$. Dann ist es zulässig, in (76) für n den Wert 1 zu nehmen und H als Intervall negativer Zahlen. Indem man sich der expliciten Werte der $\Theta_Q(\mathfrak{S}, \mathfrak{T})$ bedient, folgt genau wie im ersten Teil, dass $\gamma(\mathfrak{S}) = 1$ ist. Nun sei das ungerade $\nu > 1$. Dann wähle man $m - n = \nu - 1$ in (77) und die Elemente von $\mathfrak{D}_2$ negativ, also $n - 1$ Elemente von $\mathfrak{D}_1$ positiv und eines negativ. Die zu $\mathfrak{S}_1$ gehörige Zahl ν ist dann 1, also ist nach dem soeben Bewiesenen $\gamma(\mathfrak{S}_1) = 1$. Da aber $m - n$ gerade ist, so folgt auch $\gamma(\mathfrak{S}) = 1$.

Drittes Capitel: Anwendungen

5. Nichteuklidische Volumina

Es sei $\mathfrak{S}^{(m)}$ von der Signatur $2 - m$, also die quadratische Form $\mathfrak{x}'\mathfrak{S}\mathfrak{x}$ reell in $-(y_1^2 + \cdots + y_{m-1}^2) + y_m^2$ transformierbar. Deutet man die Elemente von $\mathfrak{x}$ als projective Coordinaten im $(m-1)$-dimensionalen Raum und bezeichnet als Bewegungen diejenigen Collineationen $\mathfrak{x} = \mathfrak{C}\mathfrak{x}_1$, bei denen das Gebilde $\mathfrak{x}'\mathfrak{S}\mathfrak{x} = 0$ in sich übergeht, so erhält man das Cayley-Kleinsche Modell der nichteuklidischen Geometrie. Hierin definieren nun die Einheiten $\mathfrak{C}$ von $\mathfrak{S}$ eine im Gebiete $\mathfrak{x}'\mathfrak{S}\mathfrak{x} > 0$ discontinuierliche Bewegungsgruppe; dabei ist aber zu beachten, dass den beiden Einheiten $\pm\mathfrak{C}$ dieselbe Bewegung entspricht. Das einzige invariante Bestimmungsstück des Fundamentalbereiches der Bewegungsgruppe ist sein nichteuklidisches Volumen. Setzt man $x_k : x_m = z_k$ $(k = 1, \cdots, m)$ und bezeichnet mit F einen Fundamentalbereich im Gebiete der $z_1, \cdots, z_{m-1}$, so ist

$$v(\mathfrak{S}) = S^{\frac{1}{2}} \int_F (\mathfrak{z}'\mathfrak{S}\mathfrak{z})^{-\frac{m}{2}}\, dz_1 \cdots dz_{m-1} \tag{78}$$

das nichteuklidische Volumen von F. Es soll nun der Zusammenhang zwischen $v(\mathfrak{S})$ und der Grösse $\rho(\mathfrak{S})$ von Hilfssatz 9 hergeleitet werden.

Zunächst sei $\mathfrak{S}^{(m)}$ beliebig, also nicht notwendigerweise von der Signatur $2-m$. Es liege $\mathfrak{T}^{(n)}$ in einem Gebiet B, für welches die Signaturen von $\mathfrak{S}$ und $\mathfrak{T}$ constante Differenz $\pm(m-n)$ haben. Ist $\mathfrak{C}$ eine Einheit von $\mathfrak{S}$, so ist mit $\mathfrak{x}$ auch $\mathfrak{C}\mathfrak{x}$ eine Lösung von $\mathfrak{x}'\mathfrak{S}\mathfrak{x} = \mathfrak{T}$. Im $\mathfrak{x}$-Gebiet ist die Einheitengruppe wieder discontinuierlich. Es sei $\bar{B}$ ein B entsprechender Fundamentalbereich im $\mathfrak{x}$-Raum. Indem man die Rechnung von Hilfssatz 17 wiederholt, erhält man nach Hilfssatz 10

$$\rho(\mathfrak{S})S^{\frac{m-n+1}{2}}\int_B T^{\frac{m-n-1}{2}}\,d\mathfrak{T} = \lambda_{m-n}\int_{\bar{B}} d\mathfrak{X}, \tag{79}$$

wo S und T die absoluten Beträge der Determinanten von $\mathfrak{S}$ und $\mathfrak{T}$ bedeuten und

$$\lambda_{m-n} = \prod_{k=1}^{m-n}\frac{\pi^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right)}$$

gesetzt ist.

Im Falle $n = 1$ setze man nun $\mathfrak{X} = \mathfrak{x} = x_m\mathfrak{z}$, also $T = \mathfrak{x}'\mathfrak{S}\mathfrak{x} = x_m^2\,\mathfrak{z}'\mathfrak{S}\mathfrak{z}$. Die Functionaldeterminante von $\mathfrak{x}$ nach $z_1, \cdots, z_{m-1}, T$ ist $\frac{1}{2}T^{\frac{m}{2}-1}(\mathfrak{z}'\mathfrak{S}\mathfrak{z})^{-\frac{m}{2}}$, und folglich gilt

$$\int_{\bar{B}} d\mathfrak{X} = \tfrac{1}{2}\int_F (\mathfrak{z}'\mathfrak{S}\mathfrak{z})^{-\frac{m}{2}}\,dz_1\cdots dz_{m-1}\int_B T^{\frac{m}{2}-1}\,dT. \tag{80}$$

Aus (78), (79), (80) erhält man die gesuchte Beziehung

$$v(\mathfrak{S}) = 2\prod_{k=1}^{m-1}\pi^{-\frac{k}{2}}\Gamma\left(\frac{k}{2}\right)S^{\frac{m+1}{2}}\rho(\mathfrak{S}). \tag{81}$$

Ist nun $m > 2$, $\mathfrak{S}$ indefinit und S ungerade, so besteht nach einem wichtigen Satz von A. Meyer das Geschlecht von $\mathfrak{S}$ aus einer einzigen Classe. Dann ist aber $\rho(\mathfrak{S}) = \mu(\mathfrak{S})$, und der Hauptsatz liefert für diesen Fall

$$\rho(\mathfrak{S}) = 2\lim_{q\to\infty}\frac{2^{\omega(q)}q^{\frac{m(m-1)}{2}}}{E_q(\mathfrak{S})}$$

und

$$v(\mathfrak{S}) = 4\prod_{k=1}^{m-1}\pi^{-\frac{k}{2}}\Gamma\left(\frac{k}{2}\right)S^{\frac{m+1}{2}}\lim_{q\to\infty}\frac{2^{\omega(q)}q^{\frac{m(m-1)}{2}}}{E_q(\mathfrak{S})}. \tag{82}$$

Ist S gerade, so kann das Geschlecht von $\mathfrak{S}$ mehr als eine Classe enthalten. Sind in diesem Falle $\mathfrak{S}_1, \cdots, \mathfrak{S}_h$ die einzelnen Classenrepräsentanten des Geschlechts von $\mathfrak{S}$, so bleibt (82) auch für gerades S richtig, wenn die linke Seite durch $v(\mathfrak{S}_1) + \cdots + v(\mathfrak{S}_h)$ ersetzt wird.

Unter Benutzung von Hilfssatz 12 des erstes Teiles lässt sich der Grenzwert auf der rechten Seite von (82) durch elementare arithmetische Functionen von $\mathfrak{S}$ ausdrücken, was aber im Einzelnen mühsame Fallunterscheidungen nötig macht. Für $m = 3$ wurde eine mit (82) gleichwertige Formel bereits von G. Humbert gefunden. Für beliebiges ungerades m folgt aus (82), dass die Zahl $v(\mathfrak{S})\pi^{-\frac{m-1}{2}}$ rational ist. Dies ist in Uebereinstimmung mit dem von Dehn und Poincaré bewiesenen Satze, welcher den nichteuklidischen Inhalt eines Polyeders bei gerader Dimensionenzahl mit der Summe der räumlichen Winkel in Zusammenhang bringt; der Fundamentalbereich kann nämlich als ein von Ebenen begrenztes Polyeder gewählt werden.

6. Beispiele

1) Sei $\mathfrak{x}'\mathfrak{S}\mathfrak{x} = x_2^2 - Sx_1^2$ und S keine Quadratzahl. Ist t, u die Lösung von $t^2 - Su^2 = 1$ mit kleinstem $t + u\sqrt{S} > 1$, so ist der Sector $0 \leqq x_1 < \frac{u}{t} x_2$ ein Fundamentalbereich gegenüber den Einheiten der Determinante $+1$, also das Doppelte eines Fundamentalbereiches gegenüber der vollen Einheitengruppe. Nach (78) ist daher

$$v(\mathfrak{S}) = \frac{S^{\frac{1}{2}}}{2} \int_0^{\frac{u}{t}} \frac{dz}{1 - Sz^2} = \frac{1}{2} \log (t + u\sqrt{S}),$$

und nach (81)

$$\rho(\mathfrak{S}) = \tfrac{1}{4} S^{-\frac{1}{2}} \log (t + u\sqrt{S}).$$

Nimmt man speciell $S = 5$, so ist die Classenzahl 1 und $\rho(\mathfrak{S}) = \mu(\mathfrak{S})$. Nach dem Hauptsatz wird dann andererseits

$$\rho(\mathfrak{S}) = 2 \cdot \frac{1}{2} \cdot \frac{1}{10} \prod_{p \neq 2,5} \left(1 - \left(\frac{5}{p}\right)\frac{1}{p}\right)^{-1} = \frac{3}{20} \sum_{k=1}^{\infty} \left(\frac{5}{k}\right) k^{-1} = \frac{3}{10\sqrt{5}} \log \frac{1+\sqrt{5}}{2}.$$

In der Tat liefert

$$\left(\frac{1+\sqrt{5}}{2}\right)^6 = t + u\sqrt{5}$$

die Fundamentallösung $t = 9, u = 4$ der Pellschen Gleichung $t^2 - 5u^2 = 1$.

2) $\mathfrak{x}'\mathfrak{S}\mathfrak{x} = 2(x_1x_3 - x_2^2)$. Die Einheiten sind die Substitutionen

$$\begin{pmatrix} y_1 & y_2 \\ y_2 & y_3 \end{pmatrix} = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}' \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{83}$$

mit unimodularem $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Der Fundamentalbereich ist also der Raum der re-

ducierten positiv-definiten binären quadratischen Formen, nämlich $0 \leqq 2x_2 \leqq x_1 \leqq x_3$. Folglich wird

$$v(\mathfrak{S}) = \sqrt{2}\int_0^{\frac{1}{2}}\left\{\int_{2z_2}^{1}(2z_1 - 2z_2^2)^{-\frac{1}{2}}\,dz_1\right\}dz_2 = \frac{\pi}{6}$$

$$\rho(\mathfrak{S}) = \frac{\pi^2}{48}.$$

Die Classenzahl ist wieder 1, und der Hauptsatz ergibt ebenfalls

$$\rho(\mathfrak{S}) = 2\cdot\frac{1}{12}\prod_{p>2}\left(1 - \frac{1}{p^2}\right)^{-1} = \frac{\pi^2}{48}.$$

Hierdurch ist bewiesen, dass erstens (83) sämtliche Einheiten liefert und dass zweitens nicht schon ein Teil von $0 \leqq 2\,x_2 \leqq x_1 \leqq x_3$ sämtliche reducierten positiv-definiten binären quadratischen Formen enthält.

3) $\mathfrak{x}'\mathfrak{S}\mathfrak{x} = 2(x_1x_4 - x_2x_3)$. Die Einheitengruppe besteht aus den Substitutionen

$$\begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix} \tag{84}$$

und

$$\begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix}, \tag{85}$$

wo $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ und $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ unimodulare Matrizen gleicher Determinante sind. Die Paare $\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ und $-\begin{pmatrix} a & b \\ c & d \end{pmatrix}, -\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ liefern dieselben Substitutionen. Die Gruppe ist nicht discontinuierlich im Raume der $x_1, \cdots, x_4$. Man erhält aber einen achtdimensionalen Fundamentalbereich, indem man noch vier Variable $v_1, \cdots, v_4$ einführt, die cogredient zu $x_1, \cdots, x_4$ transformiert werden. Man setze $2(x_1x_4 - x_2x_3) = t_1$, $2(v_1v_4 - v_2v_3) = t_3$, $x_1v_4 + v_1x_4 - x_2v_3 - v_2x_3 = t_2$ und definiere noch zwei complexe Variable $\xi = \kappa + i\lambda$, $\eta = \mu - i\nu$ durch die Gleichungen

$$x_1\xi\eta + x_2\xi + x_3\eta + x_4 = 0, \qquad v_1\xi\eta + v_2\xi + v_3\eta + v_4 = 0.$$

Die Substitutionen (84) und (85) sind dann gleichwertig mit

$$\xi = \frac{a\,\xi_1 + b}{c\,\xi_1 + d}, \qquad \eta = \frac{p\,\eta_1 + q}{r\,\eta_1 + s} \tag{86}$$

und

$$\xi = \frac{a\,\eta_1 + b}{c\,\eta_1 + d}, \qquad \eta = \frac{p\,\xi_1 + q}{r\,\xi_1 + s}. \tag{87}$$

Ist $t_1 > 0$ und $t_1 t_3 - t_2^2 > 0$, so ist auch $\lambda\nu > 0$. Es sei B ein Gebiet im Raume der t_1, t_2, t_3 und $\bar{B}$ ein zugehöriger Fundamentalbereich im Raume der $x_1, \cdots, v_4$. Die Functionaldeterminante von $x_1, \cdots, v_4$ als Functionen von $x_1, t_1, t_2, t_3, \kappa, \lambda, \mu, \nu$ hat den Wert

$$\frac{\sqrt{t_1 t_3 - t_2^2}}{16\,(\lambda\nu)^2 \sqrt{\frac{t_1}{2\lambda\nu} - x_1^2}},$$

und es entspricht je 4 Systemen $x_1, \cdots, v_4$ ein System $x_1, t_1, \cdots, \nu$. Ferner ist das Gebiet $|\xi| \geqq 1$, $0 \leqq \kappa \leqq \frac{1}{2}$ ein Fundamentalbereich für $\xi_1 = \dfrac{a\,\xi_1 + b}{c\,\xi_1 + d}$ gegenüber unimodularen Substitutionen und $|\eta| \geqq 1$, $-\frac{1}{2} \leqq \mu \leqq +\frac{1}{2}$ ein Fundamentalbereich für $\eta = \dfrac{p\,\eta_1 + q}{r\,\eta_1 + s}$ gegenüber Modulsubstitutionen. Das hierdurch definierte vierdimensionale Gebiet der $\kappa, \lambda, \mu, \nu$ ist nun ein doppelter Fundamentalbereich für die durch (86) und (87) definierte Gruppe, da nämlich in dieser durch die Substitutionen (86) eine Untergruppe vom Index 2 gebildet wird. Daher erhält man

$$\int_{\bar{B}} d\mathfrak{X} = 4 \cdot \tfrac{1}{2} \cdot \tfrac{1}{16} \int_B d\mathfrak{T} \sqrt{t_1 t_3 - t_2^2} \int_0^{\frac{1}{2}} d\kappa \int_{\sqrt{1-\kappa^2}}^{\infty} \frac{d\lambda}{\lambda^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} d\mu$$

$$\int_{\sqrt{1-\mu^2}}^{\infty} \frac{d\nu}{\nu^2} \int_0^{\sqrt{\frac{t_1}{2\lambda\nu}}} \frac{dx_1}{\sqrt{\frac{t_1}{2\lambda\nu} - x_1^2}}$$

mit $\mathfrak{T} = \begin{pmatrix} t_1 & t_2 \\ t_2 & t_3 \end{pmatrix}$, also mit Rücksicht auf (79)

$$\rho(\mathfrak{S}) = \frac{\pi^4}{288}. \tag{88}$$

Andererseits ist die Classenzahl 1, und der Hauptsatz ergibt ebenfalls

$$\rho(\mathfrak{S}) = 2 \cdot \tfrac{1}{9} \prod_{p>2} \left(1 - \frac{1}{p^2}\right)^{-2} = \frac{\pi^4}{288}.$$

4) Es sei $l > 0$. Jede Lösung von $2(x_1 x_4 - x_2 x_3) = 2l$ ist vermöge (84) associiert mit einer Lösung, bei der $0 < x_1 \mid x_4$, $x_2 = x_3 = 0$ ist. Bedeutet $\mathfrak{C} = (h\ 0\ 0\ hk)'$ eine solche Lösung, so ist $\mathfrak{C}$ nur bei solchen Einheiten von $\mathfrak{S}$ invariant, bei denen in der Darstellung durch (84) oder (85) die Zahl c durch k teilbar ist. Unter Benutzung von Hilfssatz 14 folgt dann, dass $\dfrac{24\,l}{\pi^2}\,\rho(\mathfrak{S}, \mathfrak{C})$ gleich dem Index der Untergruppe der Modulsubstitutionen $\dfrac{a\,\xi + b}{c\,\xi + d}$ mit $k \mid c$

ist. Dieser Index hat aber den Wert $k \prod_{p|k} (1 + p^{-1})$. Da nun k alle Lösungen von $h^2k = l$ durchläuft, so wird

$$\mu(\mathfrak{S}, 2l) = \frac{\pi^2}{24l} \sum_{t|l} t\,, \tag{89}$$

wo t alle positiven Teiler von l durchläuft. Andererseits ist

$$q^{-3} A_q(\mathfrak{S}, 2l) = (1 - p^{-2}) \sum_{t|(l,\,q)} t^{-1}\,,$$

falls q eine genügend hohe Potenz der ungeraden Primzahl p ist, und

$$q^{-3} A_q(\mathfrak{S}, 2l) = 2(1 - 2^{-2}) \sum_{t|(l,\,q)} t^{-1}\,,$$

falls q eine genügend hohe Potenz von 2 ist. Der Hauptsatz ergibt daher

$$\mu(\mathfrak{S}, 2l) = \mu(\mathfrak{S})\, 2 \prod_{p} (1 - p^{-2}) \sum_{t|l} t^{-1}\,, \tag{90}$$

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7. Modulfunctionen

Im ersten Teil wurde gezeigt, wie sich für den definiten Fall der Hauptsatz in eine der Theorie der Modulformen angehörige Identität überführen lässt. Es ist bemerkenswert, dass dies für indefinite $\mathfrak{S}$ ein Analogon besitzt.

Ist $\mathfrak{S}$ positiv-definit, so hat die Einheitengruppe von $\mathfrak{S}$ eine endliche Ordnung $E(\mathfrak{S})$. Setzt man $|\mathfrak{S}| = S$ und

$$\lambda_m = \prod_{k=1}^{m} \frac{\pi^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right)}\,,$$

so folgt aus der Definition von $\rho(\mathfrak{S})$ die Beziehung

$$\rho(\mathfrak{S}) = \frac{\lambda_m}{E(\mathfrak{S})}\, S^{-\frac{m+1}{2}}\,. \tag{91}$$

Ist ferner $\mathfrak{C}'\mathfrak{S}\mathfrak{C} = \mathfrak{T}^{(n)}$, $|\mathfrak{T}| = T > 0$ und $E(\mathfrak{S}, \mathfrak{C})$ die Anzahl der mit $\mathfrak{C}$ in bezug auf die Einheitengruppe von $\mathfrak{S}$ linksseitig Associierten, so gilt

$$E(\mathfrak{S}, \mathfrak{C}) = \frac{\lambda_m}{\lambda_{m-n}}\, S^{-\frac{n}{2}}\, T^{\frac{m-n-1}{2}}\, \frac{\rho(\mathfrak{S}, \mathfrak{C})}{\rho(\mathfrak{S})}\,. \tag{92}$$

Es sei der imaginäre Teil der symmetrischen Matrix $\mathfrak{X}^{(n)}$ positiv definit. Ordnet man in der analytischen Classeninvariante

$$\sum_{\mathfrak{C}} e^{\pi i \sigma(\mathfrak{C}'\mathfrak{S}\mathfrak{C}\mathfrak{X})} = f(\mathfrak{S}, \mathfrak{X})\,, \tag{93}$$

Ist $t_1 > 0$ und $t_1 t_3 - t_2^2 > 0$, so ist auch $\lambda\nu > 0$. Es sei B ein Gebiet im Raume der t_1, t_2, t_3 und $\bar{B}$ ein zugehöriger Fundamentalbereich im Raume der $x_1, \cdots, v_4$. Die Functionaldeterminante von $x_1, \cdots, v_4$ als Functionen von $x_1, t_1, t_2, t_3, \kappa, \lambda, \mu, \nu$ hat den Wert

$$\frac{\sqrt{t_1 t_3 - t_2^2}}{16\,(\lambda\nu)^2 \sqrt{\frac{t_1}{2\lambda\nu} - x_1^2}},$$

und es entspricht je 4 Systemen $x_1, \cdots, v_4$ ein System $x_1, t_1, \cdots, \nu$. Ferner ist das Gebiet $|\xi| \geqq 1$, $0 \leqq \kappa \leqq \frac{1}{2}$ ein Fundamentalbereich für $\xi = \frac{a\,\xi_1 + b}{c\,\xi_1 + d}$ gegenüber unimodularen Substitutionen und $|\eta| \geqq 1$, $-\frac{1}{2} \leqq \mu \leqq +\frac{1}{2}$ ein Fundamentalbereich für $\eta = \frac{p\,\eta_1 + q}{r\,\eta_1 + s}$ gegenüber Modulsubstitutionen. Das hierdurch definierte vierdimensionale Gebiet der $\kappa, \lambda, \mu, \nu$ ist nun ein doppelter Fundamentalbereich für die durch (86) und (87) definierte Gruppe, da nämlich in dieser durch die Substitutionen (86) eine Untergruppe vom Index 2 gebildet wird. Daher erhält man

$$\int_{\bar{B}} d\mathfrak{X} = 4 \cdot \tfrac{1}{2} \cdot \tfrac{1}{16} \int_B d\mathfrak{T} \sqrt{t_1 t_3 - t_2^2} \int_0^{\frac{1}{2}} d\kappa \int_{\sqrt{1-\kappa^2}}^{\infty} \frac{d\lambda}{\lambda^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} d\mu$$

$$\int_{\sqrt{1-\mu^2}}^{\infty} \frac{d\nu}{\nu^2} \int_0^{\sqrt{\frac{t_1}{2\lambda\nu}}} \frac{dx_1}{\sqrt{\frac{t_1}{2\lambda\nu} - x_1^2}}$$

mit $\mathfrak{T} = \begin{pmatrix} t_1 & t_2 \\ t_2 & t_3 \end{pmatrix}$, also mit Rücksicht auf (79)

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Es sei der imaginäre Teil der symmetrischen Matrix $\mathfrak{X}^{(n)}$ positiv definit. Ordnet man in der analytischen Classeninvariante

$$\sum_{\mathfrak{C}} e^{\pi i \sigma(\mathfrak{C}'\mathfrak{S}\mathfrak{C}\mathfrak{X})} = f(\mathfrak{S}, \mathfrak{X})\,, \tag{93}$$

wo $\mathfrak{C}^{(mn)}$ alle ganzen Matrizen durchläuft, die $\mathfrak{C}$ nach ihrem Range, so erhält man

$$f(\mathfrak{S}, \mathfrak{X}) = 1 + \sum_{r=1}^{n} f_r(\mathfrak{X}) \tag{94}$$

mit

$$f_r(\mathfrak{X}) = \sum_{\mathfrak{B}, \mathfrak{Q}} e^{\pi i \sigma(\mathfrak{B}'\mathfrak{S}\mathfrak{B}\mathfrak{Q}\mathfrak{X}\mathfrak{Q}')},$$

wo $\mathfrak{B}^{(mr)}$ alle ganzen Matrizen vom Range r und $\mathfrak{Q}^{(rn)}$ ein volles System primitiver in bezug auf die Gruppe der unimodularen Matrizen linksseitig nicht associierter Matrizen durchlaufen. Mit Rücksicht auf (92) ist dann

$$\rho(\mathfrak{S}) f_r(\mathfrak{X}) = \frac{\lambda_m}{\lambda_{m-r}} S^{-\frac{r}{2}} \sum_{\mathfrak{B}, \mathfrak{Q}} |\mathfrak{B}'\mathfrak{S}\mathfrak{B}|^{\frac{m-r-1}{2}} \rho(\mathfrak{S}, \mathfrak{B})\, e^{\pi i \sigma(\mathfrak{B}'\mathfrak{S}\mathfrak{B}\mathfrak{Q}\mathfrak{X}\mathfrak{Q}')}, \tag{95}$$

wo $\mathfrak{B}^{(mr)}$ nur ein volles System in bezug auf die Einheitengruppe von $\mathfrak{S}$ linksseitig nicht associierter Matrizen durchläuft.

Bedeuten $\mathfrak{S}_1, \cdots, \mathfrak{S}_h$ Classenrepräsentanten des Geschlechtes von $\mathfrak{S}$, so war die analytische Geschlechtsinvariante definiert durch den Ausdruck

$$\left(\frac{f(\mathfrak{S}_1, \mathfrak{X})}{E(\mathfrak{S}_1)} + \cdots + \frac{f(\mathfrak{S}_h, \mathfrak{X})}{E(\mathfrak{S}_h)}\right) : \left(\frac{1}{E(\mathfrak{S}_1)} + \cdots + \frac{1}{E(\mathfrak{S}_h)}\right) = F(\mathfrak{S}, \mathfrak{X}).$$

Nach (91) ist dann auch

$$F(\mathfrak{S}, \mathfrak{X}) = \frac{\rho(\mathfrak{S}_1) f(\mathfrak{S}_1, \mathfrak{X}) + \cdots + \rho(\mathfrak{S}_h) f(\mathfrak{S}_h, \mathfrak{X})}{\rho(\mathfrak{S}_1) + \cdots + \rho(\mathfrak{S}_h)}, \tag{96}$$

und nach (94) und (95)

$$F(\mathfrak{S}, \mathfrak{X}) = 1 + \sum_{r=1}^{n} F_r(\mathfrak{X}) \tag{97}$$

mit

$$F_r(\mathfrak{X}) = \frac{\lambda_m}{\lambda_{m-r}} S^{-\frac{r}{2}} \sum_{\mathfrak{T}, \mathfrak{Q}} \frac{\mu(\mathfrak{S}, \mathfrak{T})}{\mu(\mathfrak{S})} |\mathfrak{T}|^{\frac{m-r-1}{2}} e^{\pi i \sigma(\mathfrak{T}\mathfrak{Q}\mathfrak{X}\mathfrak{Q}')}; \tag{98}$$

dabei durchläuft $\mathfrak{Q}$ dieselben Matrizen wie oben und $\mathfrak{T}^{(r)}$ alle ganzen positiven symmetrischen Matrizen.

Die durch (94), (95) und (96) gegebene Definition von $f(\mathfrak{S}, \mathfrak{X})$ und $F(\mathfrak{S}, \mathfrak{X})$ lässt sich auf den Fall eines indefiniten $\mathfrak{S}$ übertragen. Man hat nur in (95) die Summation auf die $\mathfrak{B}$ mit positivem $\mathfrak{B}'\mathfrak{S}\mathfrak{B}$ zu beschränken. Die Definition von $f(\mathfrak{S}, \mathfrak{X})$ lässt sich dann auch folgendermassen fassen. Man setze $E(\mathfrak{S}, \mathfrak{C}) = 0$, falls nicht zugleich $\mathfrak{C}'\mathfrak{S}\mathfrak{C}$ definit oder semidefinit und von demselben Range wie $\mathfrak{C}$ ist. Ist $\mathfrak{C}$ vom Range r und $\mathfrak{C} = \mathfrak{B}\mathfrak{Q}$ mit ganzem $\mathfrak{B}^{(mr)}$ und primitivem $\mathfrak{Q}^{(rn)}$, so definiere man für semidefinites $\mathfrak{C}'\mathfrak{S}\mathfrak{C}$ vom Range r in Anlehnung an (92)

$$E(\mathfrak{S}, \mathfrak{C}) = \frac{\lambda_m}{\lambda_{m-r}} S^{-\frac{r}{2}} |\mathfrak{B}'\mathfrak{S}\mathfrak{B}|^{\frac{m-r-1}{2}} \frac{\rho(\mathfrak{S}, \mathfrak{B})}{\rho(\mathfrak{S})}$$

und $E(\mathfrak{S}, \mathfrak{N}) = 1$. Dann ist

$$f(\mathfrak{S}, \mathfrak{X}) = \sum_{\mathfrak{C}} E(\mathfrak{S}, \mathfrak{C}) e^{\pi i \sigma(\mathfrak{C}'\mathfrak{S}\mathfrak{C}\mathfrak{X})}, \tag{99}$$

wo $\mathfrak{C}^{(mn)}$ ein volles System in bezug auf die Einheitengruppe von $\mathfrak{S}$ nicht associierter ganzer Matrizen durchläuft.

Wendet man auf den in (98) vorkommenden Ausdruck $\frac{\mu(\mathfrak{S}, \mathfrak{T})}{\mu(\mathfrak{S})}$ den Hauptsatz an, so ergibt sich genau wie im ersten Teil für die Geschlechtsinvariante $F(\mathfrak{S}, \mathfrak{X})$ eine Partialbruchzerlegung der Gestalt

$$F(\mathfrak{S}, \mathfrak{X}) = \sum_{\mathfrak{A}, \mathfrak{B}} H(\mathfrak{S}, \mathfrak{A}, \mathfrak{B}) |\mathfrak{A}\mathfrak{X} + \mathfrak{B}|^{-\frac{m}{2}} \quad (m > n^2 + n + 2). \tag{100}$$

Dabei durchlaufen $\mathfrak{A}$, $\mathfrak{B}$ alle nicht-associierten primitiven symmetrischen Matrizenpaare, während $H(\mathfrak{S}, \mathfrak{A}, \mathfrak{B})$ nicht von $\mathfrak{X}$ abhängt und durch verallgemeinerte Gausssche Summen ausgedrückt werden kann. Hierdurch wird die Stellung von $F(\mathfrak{S}, \mathfrak{X})$ in der Theorie der Modulfunctionen n^{ten} Grades klargelegt. In einigen speciellen Fällen hat bereits Hecke indefinite quadratische Formen zur Construction von Modulformen herangezogen. Aus (96), (99) und (100) folgt, dass man mit jedem indefiniten ganzen $\mathfrak{S}$ Thetareihen bilden kann, die zur Theorie der Modulfunctionen in enger Beziehung stehen. Es wäre von Interesse, das Verhalten der durch (99) definierten Function $f(\mathfrak{S}, \mathfrak{X})$ bei den Modulsubstitutionen zu untersuchen; im Falle eines definiten $\mathfrak{S}$ folgt vermöge (93) die Transformationstheorie von $f(\mathfrak{S}, \mathfrak{X})$ aus den Eigenschaften der Thetafunctionen, und so ist zu vermuten, dass auch für indefinites $\mathfrak{S}$ die Classeninvariante $f(\mathfrak{S}, \mathfrak{X})$ einfache Transformationseigenschaften besitzt.

Es sei als Beispiel noch die zur quaternären Form $2(y_1y_4 - y_2y_3)$ gehörige Function $F(\mathfrak{S}, x)$ angegeben. Nach (90) ist in diesem Fall

$$\frac{\mu(\mathfrak{S}, 2l)}{\mu(\mathfrak{S})} = \frac{12}{\pi^2} \sum_{t|l} t^{-1},$$

also nach (97) und (98)

$$F(\mathfrak{S}, x) = 1 + 24 \sum_{a,b=1}^{\infty} a e^{2\pi i a b x}.$$

Da die Dedekindsche η-Function durch die Formel

$$\eta(x) = e^{-\frac{\pi i x}{}} \prod_{a=1}^{\infty} (1 - e^{2\pi i a x})$$

definiert wird, so ist

$$F(\mathfrak{S}, x) = \frac{12i}{\pi} \frac{\eta'(x)}{\eta(x)}.$$

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